Network Algorithms

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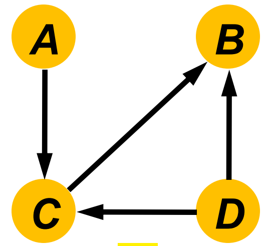
Some slides adapted from:
Networked Life (NETS) 112, Univ. of Penn., 2018, Prof. Michael Kearns
Determining the Diameter of Small World Networks, Frank W. Takes & Walter A. Kosters, Leiden University, The Netherlands
Structure and models of real-world graphs and networks, Jure Leskovec, Carnegie Mellon University
Complex (Biological) Networks, by Elhanan Borenstein, Roded Sharan, and Tomer Shlomi

Traditional Computational Representations of Networks



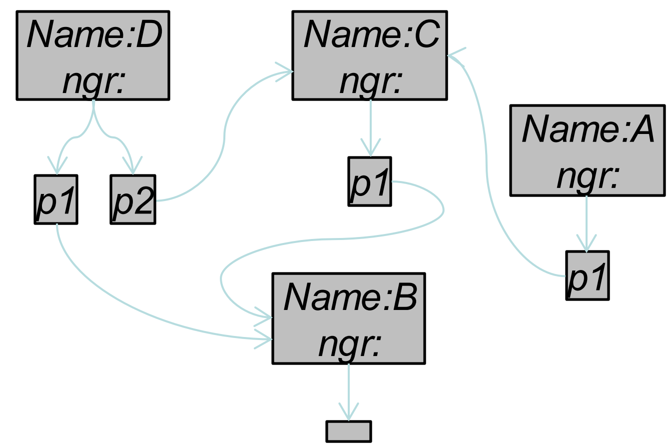
List or hash table of edges: (ordered) pairs of nodes

{ (A,C) , (C,B) , (D,B) , (D,C) }

Adjacency Matrix

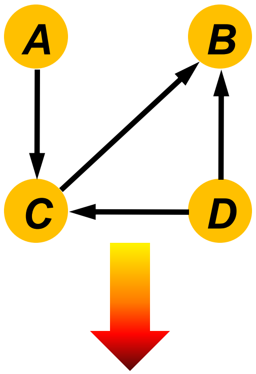
	A	B	C	D
A	0	0	1	0
B	0	0	0	0
C	0	1	0	0
D	0	1	1	0

Adjacency List



- **Which is the most useful representation?**
- **Should you use them in combination?**

A Unified Computational Representation of Networks



Adjacency Set

A: Adjacencies: {C}, Edges: {(A,C)}
B: Adjacencies: {}, Edges: {}
C: Adjacencies: {B}, Edges: {(C,B)}
D: Adjacencies: {B,C}, Edges: {(D,B), (D,C)}

- **Time for listAdjacent(v) is O(degree(v))**
- **Time for areAdjacent(v,w) is O(1) if sets have hash tables, like in Python**

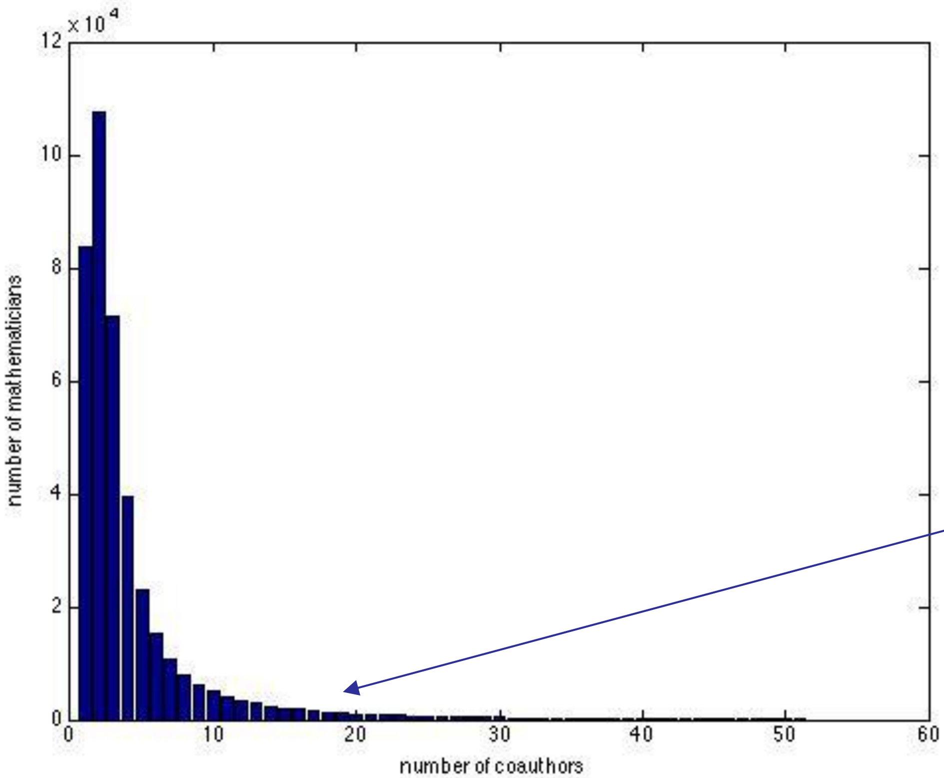
Network Structures

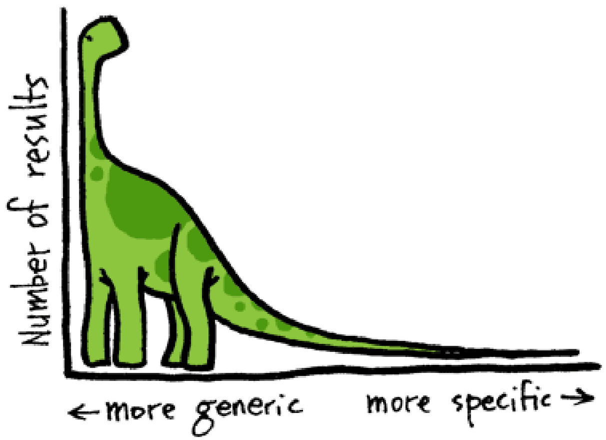
- Network structures characterize how networks "look":
  - Large or small diameter?
  - Number of edges: sparse or dense?
  - Degree distributions: heavy/long tail with a power law?
  - Clustering coefficient: high or low?
- These are empirical phenomena
- How do you compute them?

Image from *https://matrix.berkeley.edu/research/social-networks-history*

Degree Distribution

- x axis: number of neighbors (degree)
- y axis: number of vertices with that degree





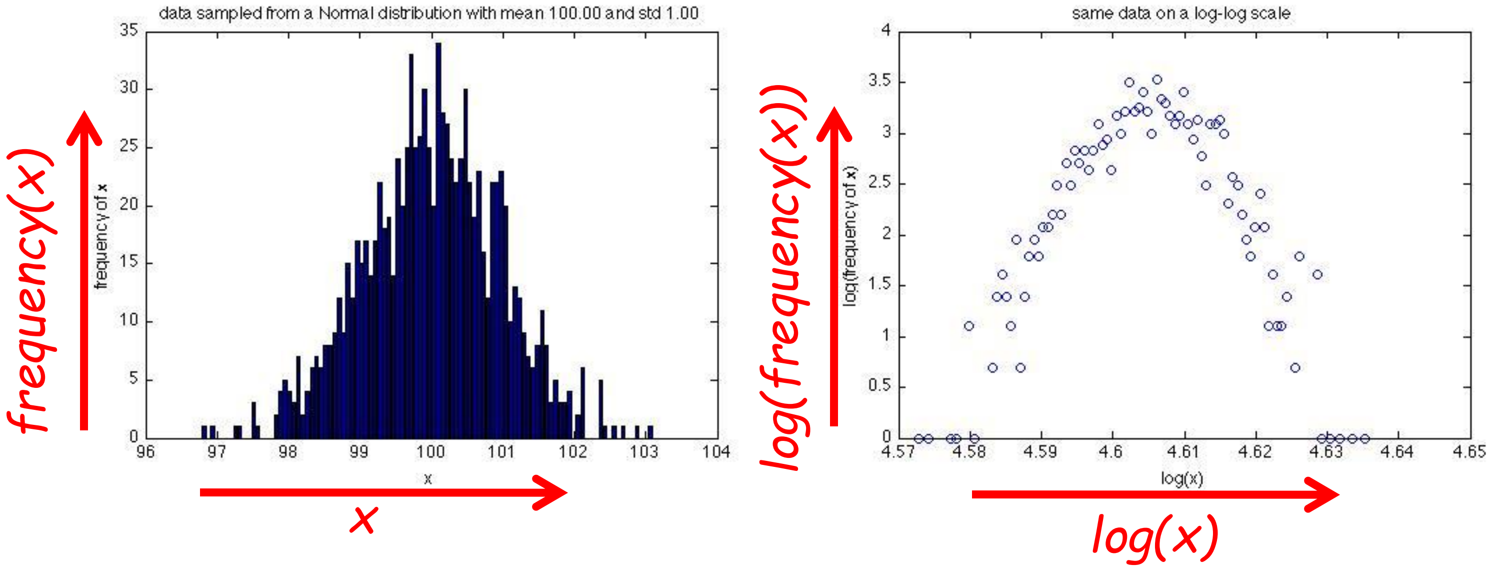
A long tail
(also known as a "heavy tail")

Degree Distribution Algorithm

1. Compute the degree, deg(v), of each vertex, v.
   - If G is represented as an adjacency list, count the number of elements in v's list.
2. Create a histogram count array, H, of size n, and initialize each $H[i] = 0$.
3. For each vertex, v, increment $H[\deg(v)]$.
4. Plot the values of H from 0 to n-1 on a regular and log-log scale
5. If the values on the log-log plot form a straight line, determine its slope to find the exponent of the power law degree distribution

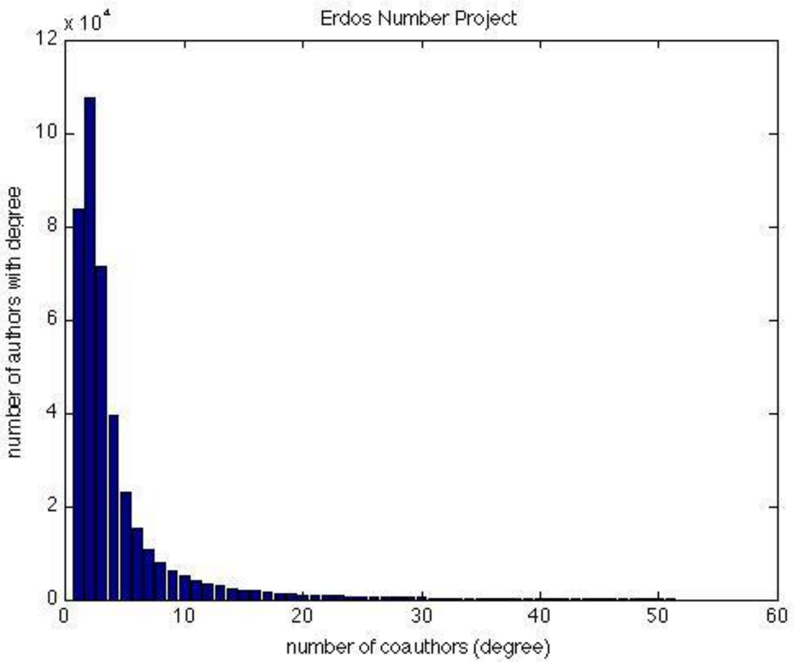
Example 1

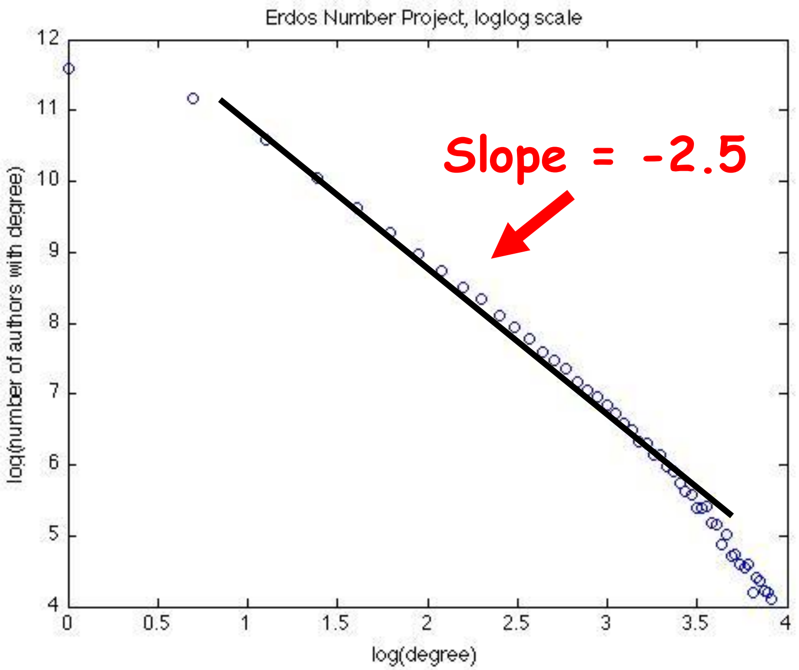
- Degree distribution without a long/heavy tail .
- Does not exhibit a power law.



Example 2

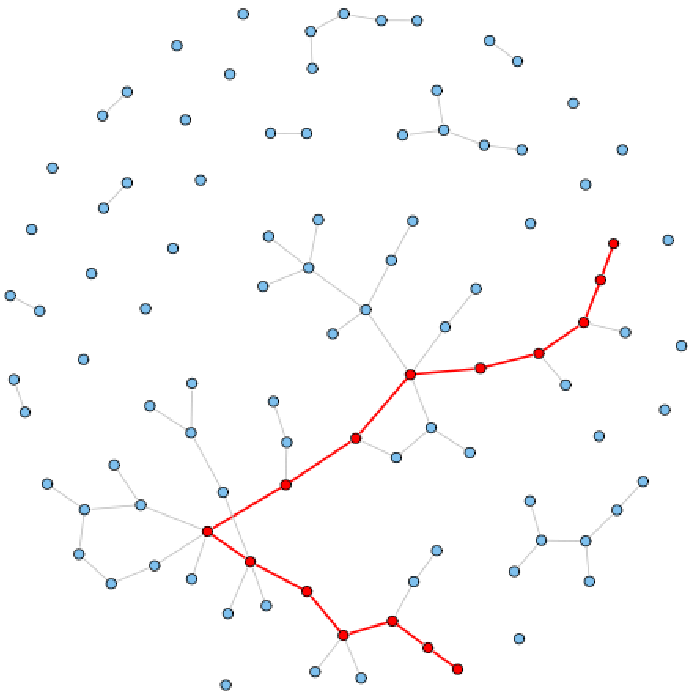
- Degree distribution with a long/heavy tail .
- Does exhibit a power law, with exponent -2.5.





Distance

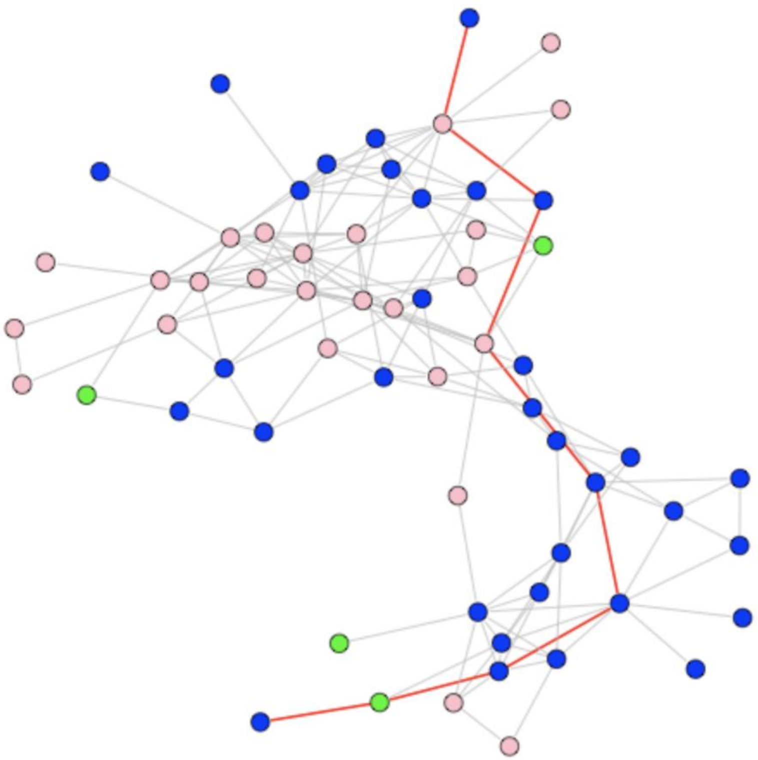
- The **distance** between two vertices is the length of the shortest path connecting them.
  - This assumes the network has only a single connected component
  - If two vertices are in different components, their distance is infinite



[Image from https://www.sci.unich.it/~francesc/teaching/network/geodesic.html](https://www.sci.unich.it/~francesc/teaching/network/geodesic.html)

Diameter

- The **diameter** of a network is the maximum distance between a pair of vertices in the network
  - It measures how near or far typical individuals are from each other



The dolphin network with the diameter (the longest shortest path) highlighted in red. The diameter is 8 edges long.

[From https://users.dimi.uniud.it/~massimo.franceschet/bottlenose/bottlenose.html](https://users.dimi.uniud.it/~massimo.franceschet/bottlenose/bottlenose.html)

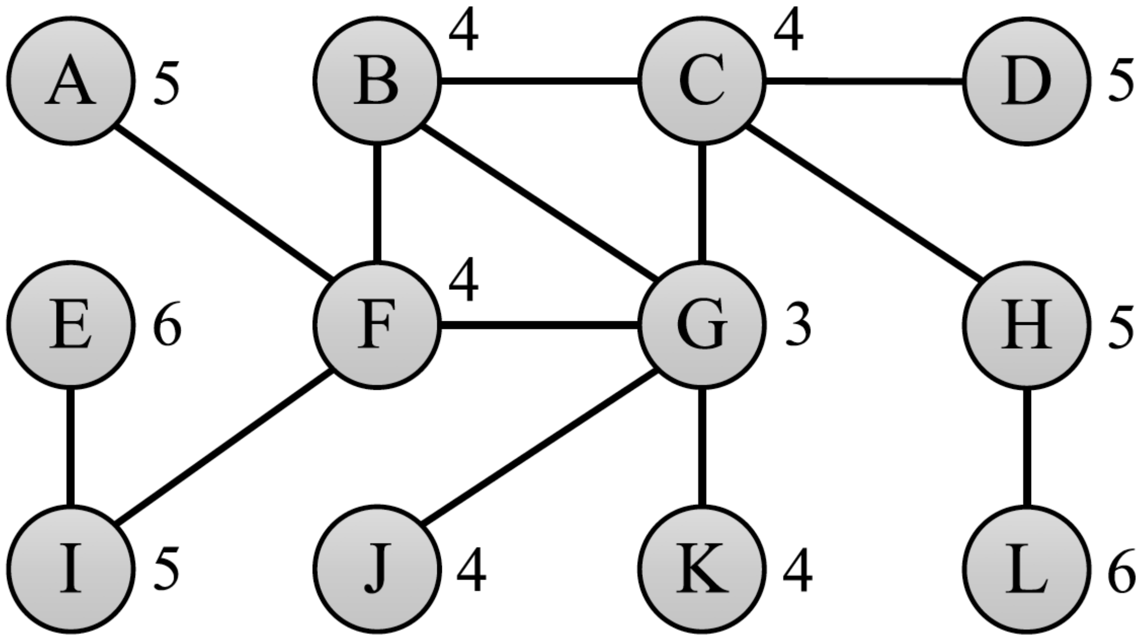
Definitions

- Consider a connected undirected graph $G = (V, E)$ with $n = |V|$ nodes and $m = |E|$ edges
- **Distance** $d(v, w)$: length of shortest path between nodes $v, w \in V$
- **Diameter** $D(G)$: maximal distance (longest shortest path length) over all node pairs: $\max_{v,w \in V} d(v, w)$
- **Eccentricity** $e(v)$: length of a longest shortest path from v: $e(v) = \max_{w \in V} d(v, w)$
- **Diameter** $D(G)$ (alternative definition): maximal eccentricity over all nodes: $\max_{v \in V} e(v)$
- Eccentricity distribution: (relative) frequency $f(x)$ of each eccentricity value x

$$f(x) = \frac{|\{u \in V \mid e(u) = x\}|}{n}$$

Example

- A graph with diameter 6
- Numbers next to nodes denote eccentricity values



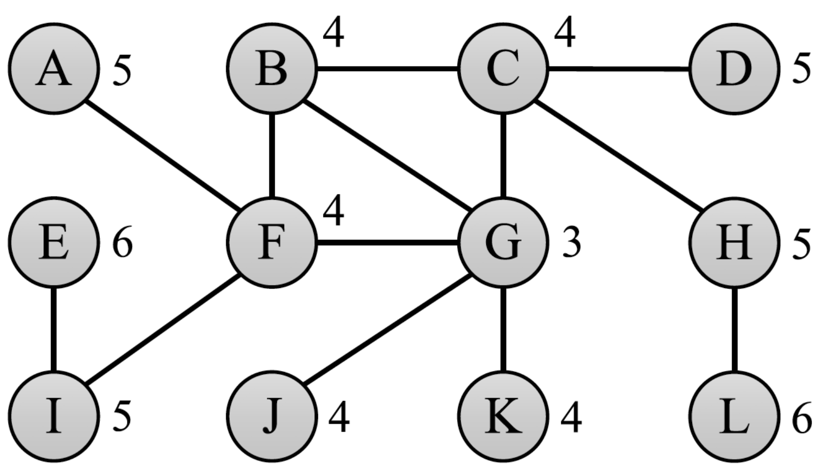
Naïve Algorithm

- **Diameter** is equal to the largest value returned by an All Pairs Shortest Path (APSP) algorithm
- Brute-force: for each vertex v, execute a Breadth First Search (BFS) from v in O(m) time to find v's **eccentricity**. Return the largest value found.
- Time complexity O(nm)
- Problematic if n = 8 million and m = 1 billion.
  - If one BFS takes 6 seconds on a 3.4GHz machine, this brute-force algorithm takes 1.5 years to compute the diameter . . .

Heuristic Idea 1

- If we can find one of the nodes in a diameter pair, we can compute the diameter with one more BFS.

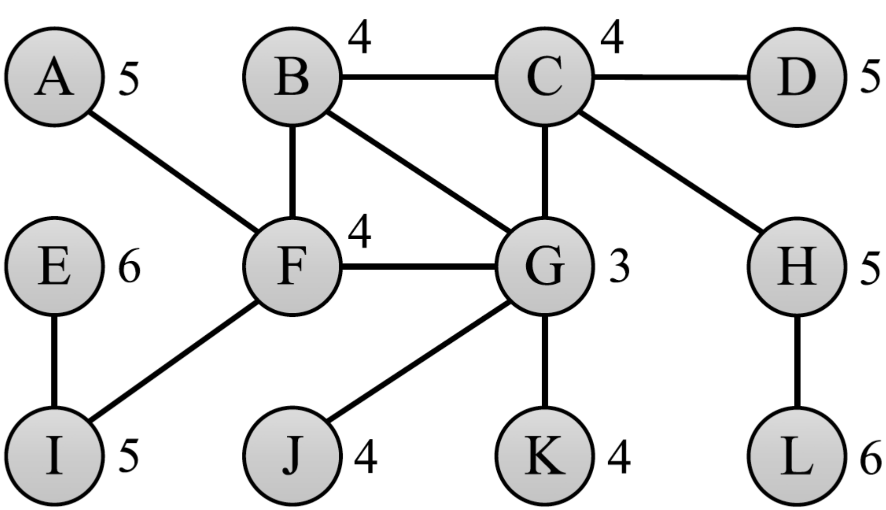
1. Perform a BFS from a random sample of nodes, recording nodes with maximum found distance, d.
2. Perform a BFS from all the far nodes (if small) or a random sample of this set (if large).



Heuristic Idea 2

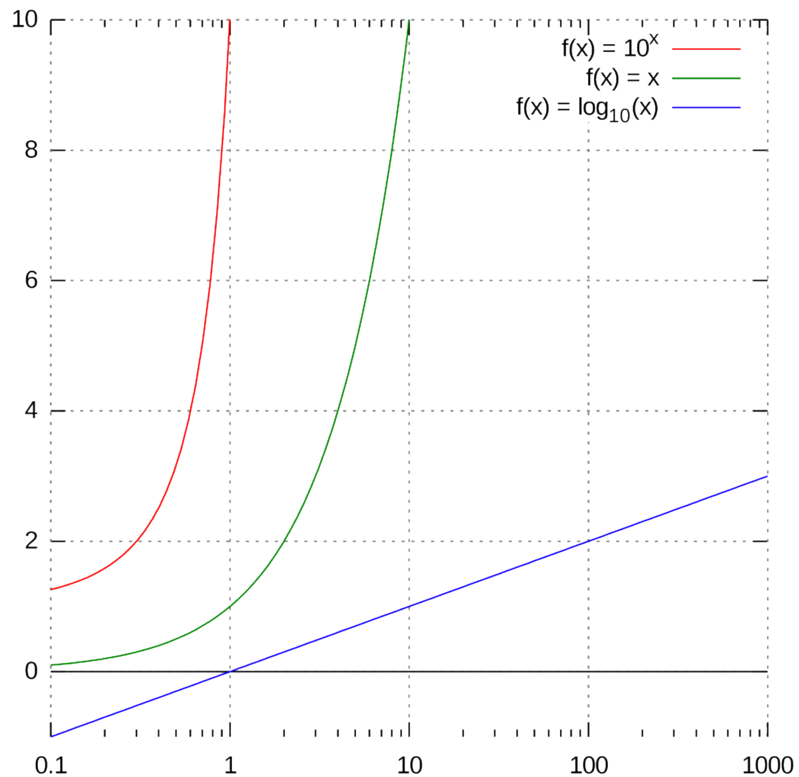
1. Let r be a random vertex and set $D_{max} = 0$.
2. Perform a BFS from r.
3. Select the farthest node, w, in this BFS.

- If the distance from r to w is larger than D_{max}, set D_{max} to this distance, let r = w, and repeat the above two steps.



Plot Results as a Function of n

- If the networks exhibit the **small world** phenomenon, then diameters are small.
- So plot diameters as a function of n on a lin-log scale:



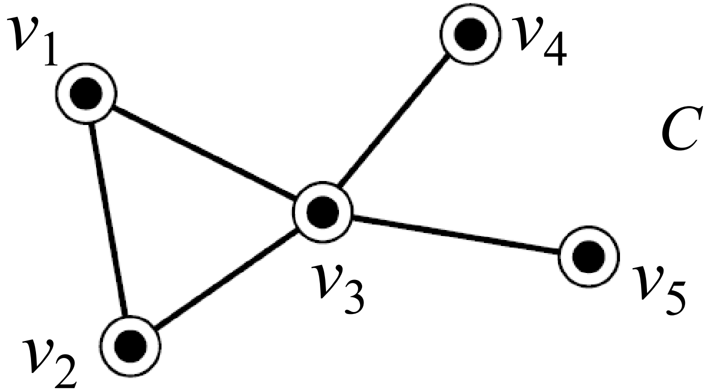
The log n function looks like a straight line

LinLogScale.png: davidfg derivative work: Autopilot [CC BY-SA 3.0 (http://creativecommons.org/licenses/by-sa/3.0/)]

Clustering Coefficient

- "friend of a friend is a friend"
- If a connects to b, and b to c,
  then with high probability
  a connects to c.
- Clustering coefficient C:

C = 3*number of triangles / number of 2-edge paths



$C = 3*(1)/(1+1+6+0+0) = 3/8 = 0.375$

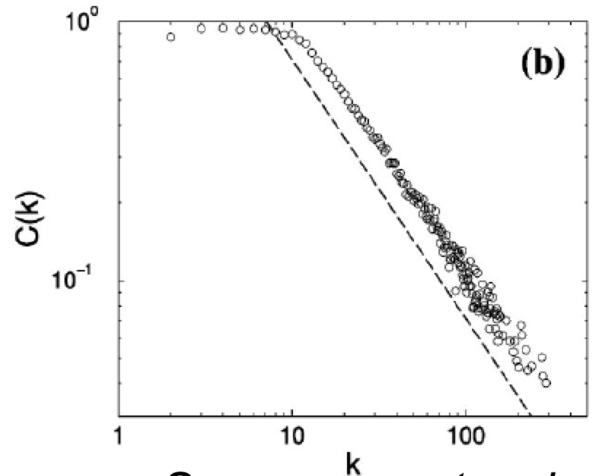
Clustering Coefficient (2)

- Clustering coefficient might have a power law:

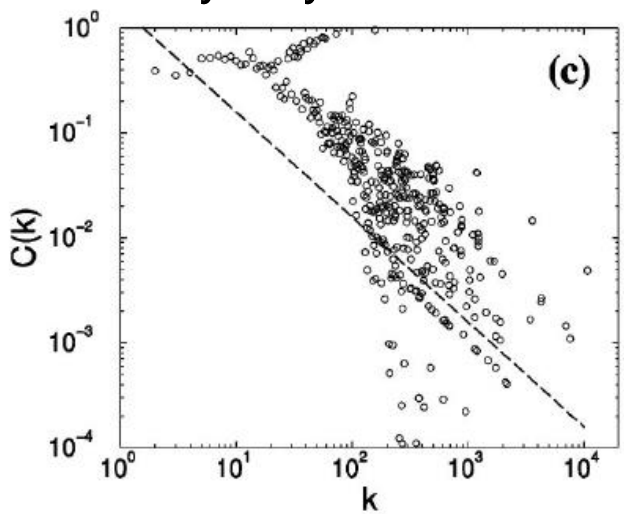
$$C(k) \sim k^{-1}$$

- It is speculated that in real networks:

$$C = O(1) \text{ as } n \to \infty$$



Synonyms network

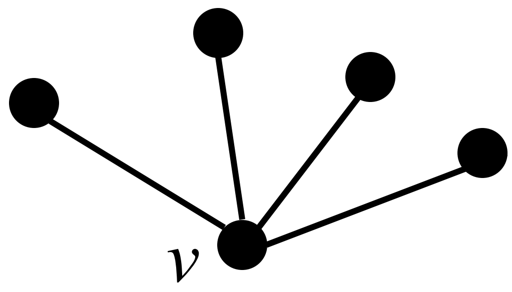


World Wide Web

Clustering Coefficient Algorithm

- Clustering coefficient C:

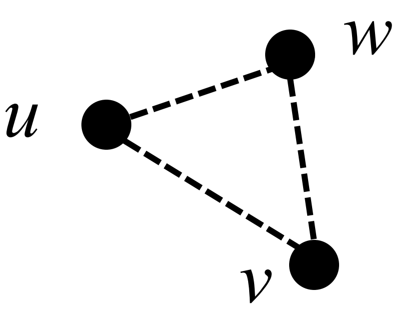
  C = 3*number of triangles / number of 2-edge paths
- Computing the **denominator** is **easy**:
  - For each vertex v, let deg(v) denote its degree.
  - The number of paths of length 2 with v in the middle is deg(v) choose 2 = deg(v)(deg(v)-1)/2.
  - So, to get the denominator for C, sum up deg(v)(deg(v)-1)/2 for all vertices, v, in G.



Number of 2-edge paths with v in the middle is 4(3)/2 = 6.

Counting Triangles

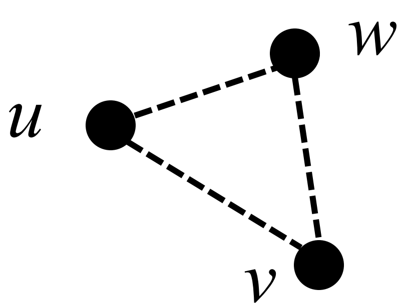
- To get the numerator for C, we need to count the number of triangles in the graph, G.
- Naïve algorithm:
  – For every triple, u, v, w in G, see if they form a triangle. If so, add 1 to a running count.
  – Running time is $O(n^4)$ if G is represented with an adjacency list.



This is bad.

Counting Triangles: Slight Improvement

- Put every edge, (v,w), into a hash table, T, so we can do a lookup to see if an edge exists in O(1) expected time, i.e., with a get((v,w)).
- Slightly better naïve algorithm:
  - For every triple, u, v, w in G, see if they form a triangle. If so, add 1 to a running count.
  - Running time is $O(n^3)$ expected if edges in G are stored in a hash table.

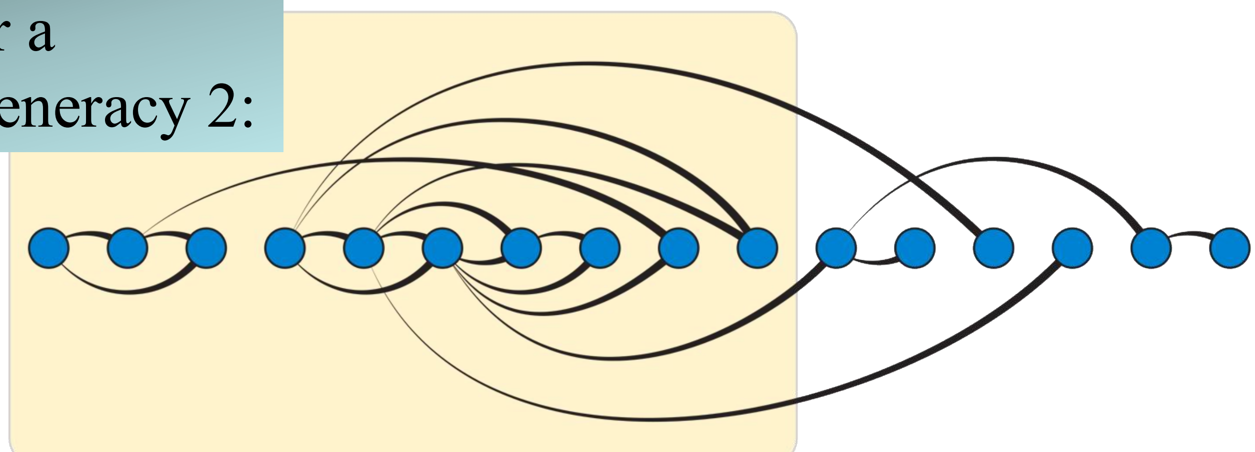


This is still bad.

Graph Degeneracy

- The **degeneracy** of a graph is the smallest value of d for which every subgraph has a vertex of degree at most d.
- If a graph has degeneracy d, then there exists an ordering of the vertices of G in which each vertex has at most d neighbors that are earlier in the ordering.

An ordering for a graph with degeneracy 2:



Public domain image by David Eppstein

Real-World Graphs

- Real-world graphs tend to have small degeneracy, d.

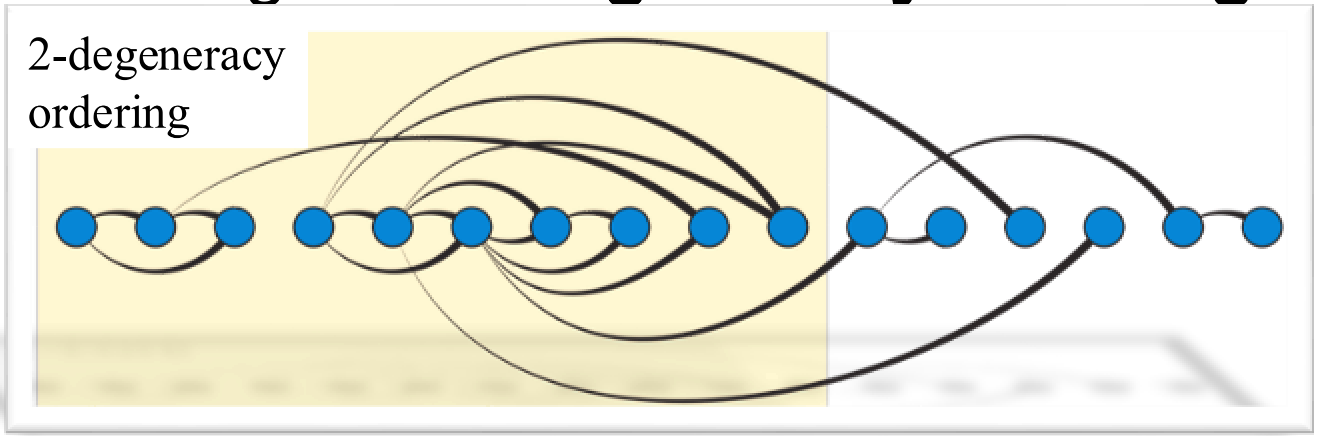
graph	n	m	d
zachary [48]	34	78	4
dolphins [35]	62	159	4
power [47]	4,941	6,594	5
polbooks [28]	105	441	6
adjnoun [29]	112	425	6
football [15]	115	613	8
lesmis [25]	77	254	9
celegensneural [47]	297	1,248	9
netscience [39]	1,589	2,742	19
internet [40]	22,963	48,421	25
condmat-2005 [38]	40,421	175,693	29
polblogs [4]	1,490	16,715	36
astro-ph [38]	16,706	121,251	56

graph	n	m	d
roadNet-CA [34]	1,965,206	2,766,607	3
roadNet-PA [34]	1,088,092	1,541,898	3
roadNet-TX [34]	1,379,917	1,921,660	3
amazon0601 [30]	403,394	2,443,408	10
email-EuAll [31]	265,214	364,481	37
email-Enron [24]	36,692	183,831	43
web-Google [2]	875,713	4,322,051	44
soc-wiki-Vote [33]	7,115	100,762	53
soc-slashdot0902 [34]	82,168	504,230	55
cit-Patents [18]	3,774,768	16,518,947	64
soc-Epinions1 [42]	75,888	405,740	67
soc-wiki-Talk [33]	2,394,385	4,659,565	131
web-berkstan [34]	685,231	6,649,470	201

Data from "Listing All Maximal Cliques in Large Sparse Real-World Graphs," by David Eppstein and Darren Strash

Degeneracy Ordering Algorithm

- Degeneracy can be computed by a simple **greedy algorithm**:
  - Repeatedly find and remove the vertex of smallest degree, adding it to the end of the list.
  - The degeneracy is then the highest degree, d, of any vertex at the moment it is removed.
  - The ordering is a **d-degeneracy ordering**.



Public domain image by David Eppstein

Linear-time Implementation

1. Initialize an output list, L, to be empty.
2. Compute a number, d_v, for each vertex v in G, which is the number of neighbors of v that are not already in L. Initially, d_v *is* just the degrees of v.
3. Initialize an array D such that $D[i]$ contains a list of the vertices v that are not already in L for which $d_v = i$.
4. Let N_v be a list of the neighbors of v that come before v in L. Initially, N_v is empty for every vertex v.
5. Initialize k to 0.
6. Repeat n times:
   - Let i be the smallest index such that $D[i]$ is nonempty.
   - Set k to max(k, i).
   - Select a vertex v from $D[i]$. Add v to the beginning of L and remove it from $D[i]$. Mark v as being in L (e.g., using a hash table, H_L).
   - For each neighbor w of v not already in L (you can check this using H_L):
     - Subtract one from d_w
     - Move w to the cell of D corresponding to the new value of d_w, i.e., $D[d_w]$
     - Add w to N_v

Triangle Counting Algorithm

- Compute a d-degeneracy ordering of the vertices, e.g., using the algorithm of the previous slide.
- Process the vertices according to this ordering, L:

  For each vertex, v:

    For each pair of vertices, u and w, adjacent to v and earlier in the ordering, i.e., u and v are in the list N_v from the degeneracy algorithm:

      If (u,w) is an edge in the graph, then add one to the triangle count.

- Running time is $O(d^2n) = O(dm)$ expected, assuming edges are stored in a hash table.