Finding Maximal Sets of Laminar 3-Separators in Planar Graphs in Linear Time

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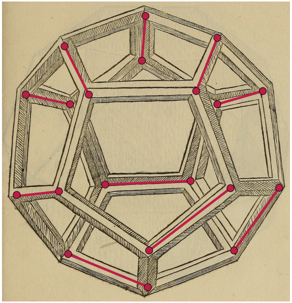
Principle: Connectivity $\Rightarrow$ more structure

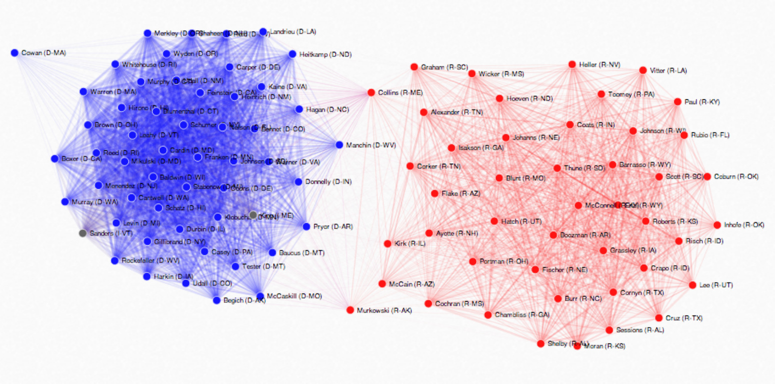
Examples:

- 2-edge-connected and 3-regular $\Rightarrow$ perfect matching [Petersen 1891]
- 3-vertex-connected and planar $\Rightarrow$ realization as convex polyhedron [Steinitz 1922]
- 4-vertex-connected and planar $\Rightarrow$ K_5-minor-free [Wagner 1937]
- 4-vertex-connected and planar $\Rightarrow$ Hamiltonian [Tutte 1977]



Algorithmic version of connectivity principle

Solve problems by dividing into more-connected pieces,
using structure, and gluing solutions together



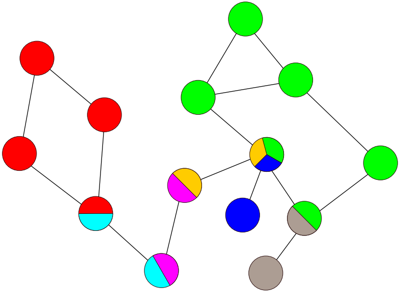
[Swallow 2013]

Canonical partition by 1-vertex cuts

Block (biconnected component): equivalence class of edges under relation of belonging to a simple cycle

Articulation point: vertex in ≥ 2 components

Block-cut tree: bipartite incidence graph of blocks and articulation points



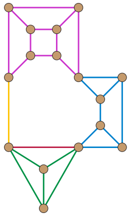
[Zyqqh 2010]

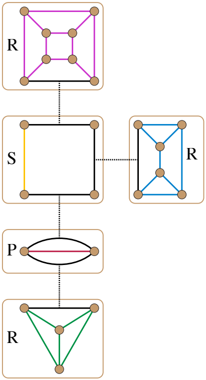
Canonical partition by 2-vertex cuts

SPQR tree: Tree with vertices labeled by
cycles (S), dipoles (P), and 3-vertex-connected graphs (R)

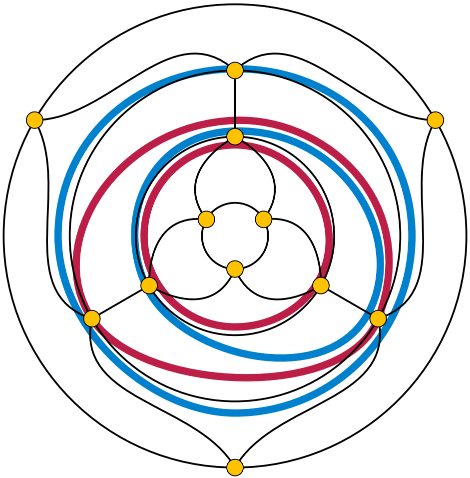
Tree edges ⇒ glue graphs on shared edge and delete the edge

[Mac Lane 1937; Hopcroft and Tarjan 1973; Bienstock and Monma 1988; Di Battista and Tamassia 1990]





But partition by 3-vertex cuts is not canonical!



Main theorem: Given a 3-vertex-connected planar graph we can find a maximal, laminar set of 3-cuts in linear time

Why?

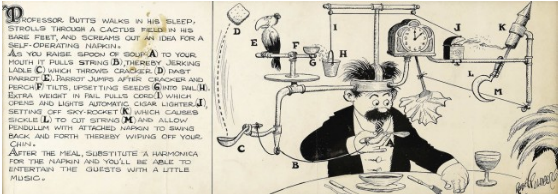
Faster separator construction for minor-closed graph families [Kawarabayashi, Li, and Reed, announced]

uses as subroutine

Finding pairs of vertex-disjoint paths between given terminals in arbitrary graphs [Kawarabayashi et al. 2015]

uses as subroutine

Finding maximal laminar family of 3-separators in planar graphs [this paper!]

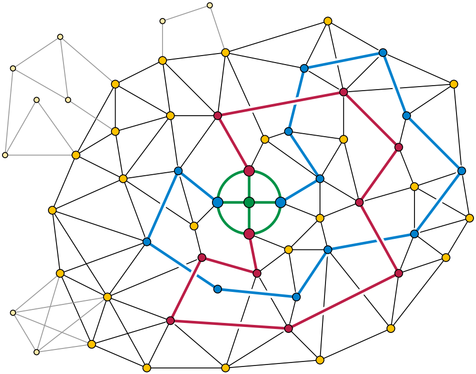


[Goldberg 1931]

Certifying the results for two disjoint paths

Add 4-wheel on path terminals to input graph. Then either:

- Find two paths $\Rightarrow \exists\ K_5$ minor
- Reduce graph on 3-vertex cuts to planar component containing wheel $\Rightarrow \nexists$ paths



Recursive algorithm for two paths (sketch)

1. Find a large set of contractable edges and contract them

2. Recurse!

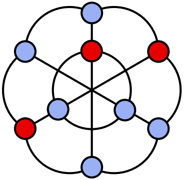
3(a). If found two paths, expand them back out

3(b). If found planar component, solve the problem using laminar 3-vertex cuts within the component to decompose it into subproblems

[danipaul 2018]

Naive algorithm for laminar cuts

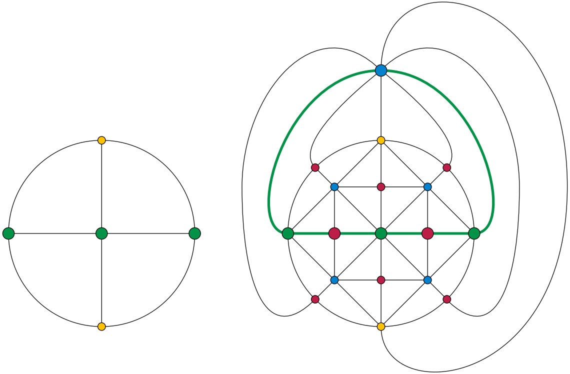
1. Find all cuts, and all non-laminar pairs of cuts
2. Build a graph, vertices = cuts, edges = non-laminar pairs
3. Find a maximal independent set (linear time in size of graph)



But: How to find everything? And how big is the graph?

Finding cuts and non-laminar pairs

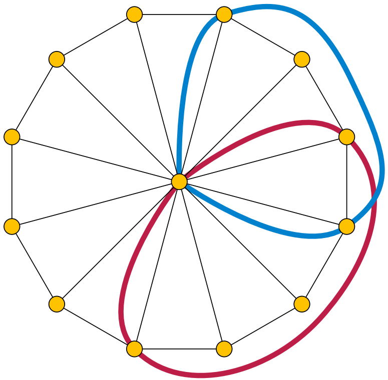
Replace input graph by its vertex-edge-face incidence graph



Turns 3-vertex cuts into certain 6-cycles,
non-laminar pairs into 12-edge subgraphs

Planar subgraph isomorphism can find them all in
$O(1)$ time per subgraph [Eppstein 1999]

. . . but the cut–crossing graph is too big!



Wheels have $\Theta(n^2)$ 3-vertex cuts,
and $\Theta(n^4)$ non-laminar pairs

Our solution (sketch)

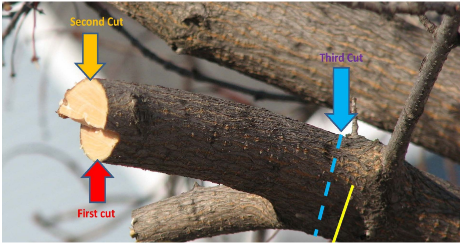
Wheels are the only bad case! So...

1. Find wheel-like subgraphs in vertex-edge-face incidence graph
2. Find cuts within each subgraph (easy)
3. Cut H into pieces along the edges of the subgraphs; each piece has only $O(n)$ cuts and crossings
4. Construct each piece's cut–crossing graph and find a maximal independent set in each piece

[Lombroso 2015]

Conclusions

Linear-time decomposition of planar graphs by 3-vertex cuts



[Pandian 2018]

Allows extra constraints on the cuts (needed in application)

Application to disjoint paths and separators; more applications?

Is there a nice linear-space description of all 3-vertex cuts,
like the SPQR tree for the 2-vertex cuts?

What about nonplanar graphs?

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