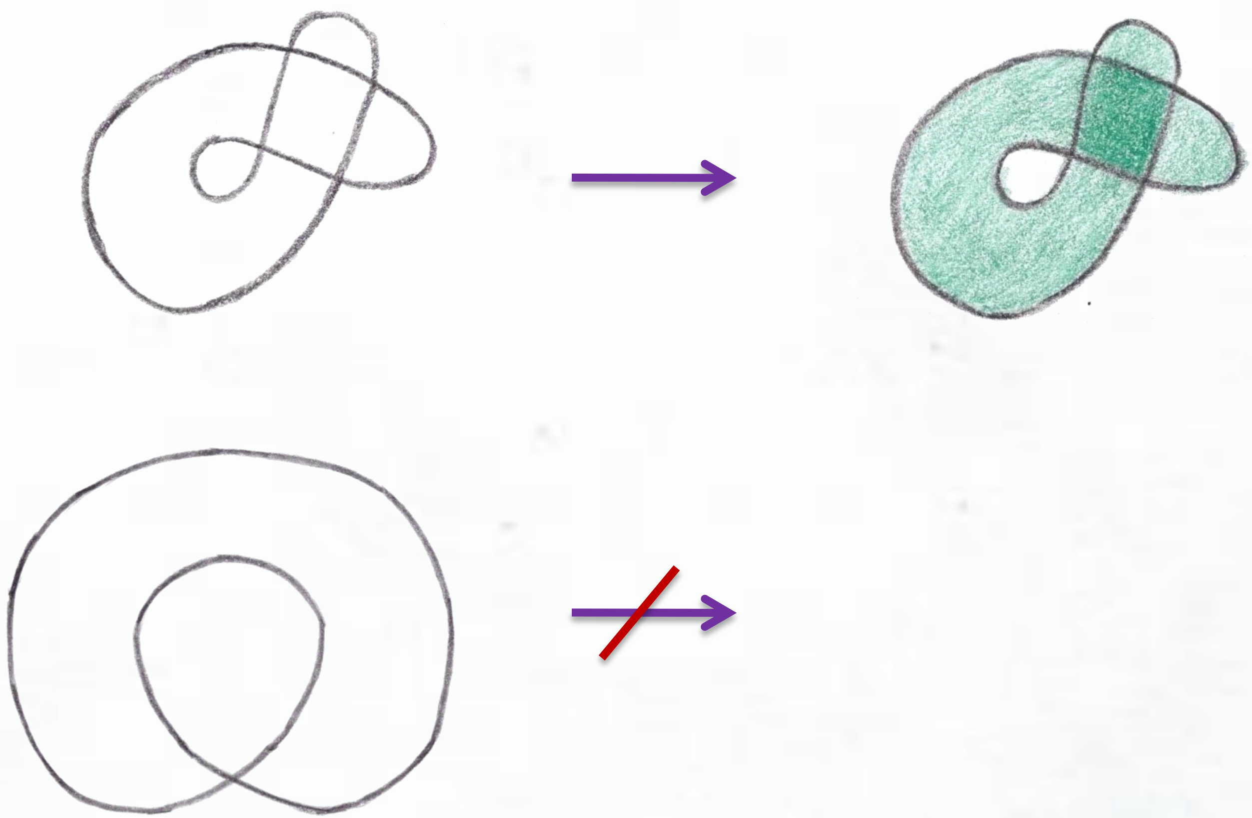
Self-overlapping Curves Revisited

David Eppstein
Elena Mumford

Curves as Surface Boundaries



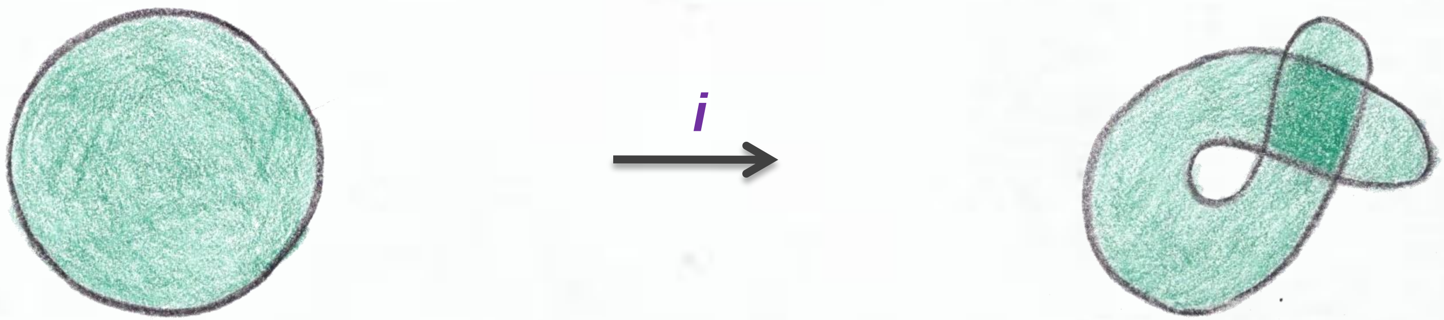
Immersion

An immersion of a disk D in the plane is a continuous mapping

$$i: D \to R^2$$

disk
in the plane

immersed disk
in the plane



$i_{n(p)}: n(p) \to i(n(p))$ is a *homeomorphism*.

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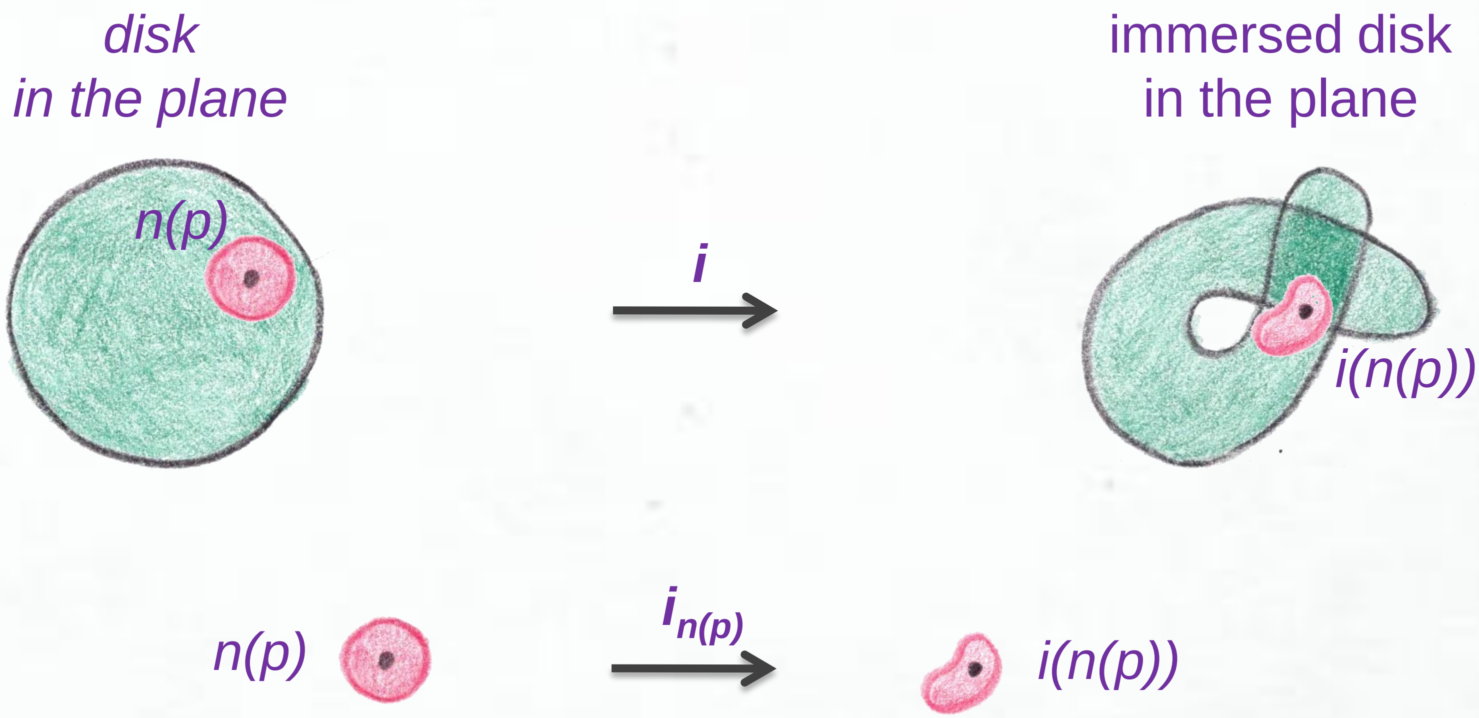


$i_{n(p)}: n(p) \to i(n(p))$ is a *homeomorphism*.

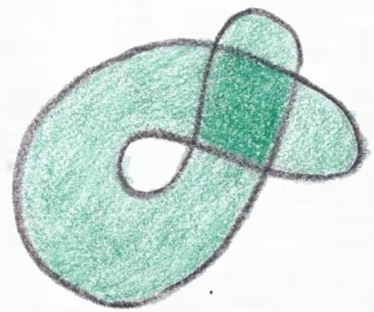
Immersion

The image of the boundary of the disk is a (self-intersecting) closed curve.

disk in the plane

i →

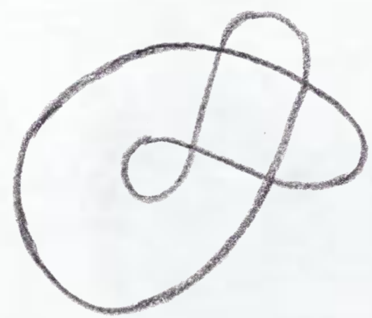
immersed disk in the plane



disk boundary in the plane

i →

self-intersecting curve in the plane



Embedding

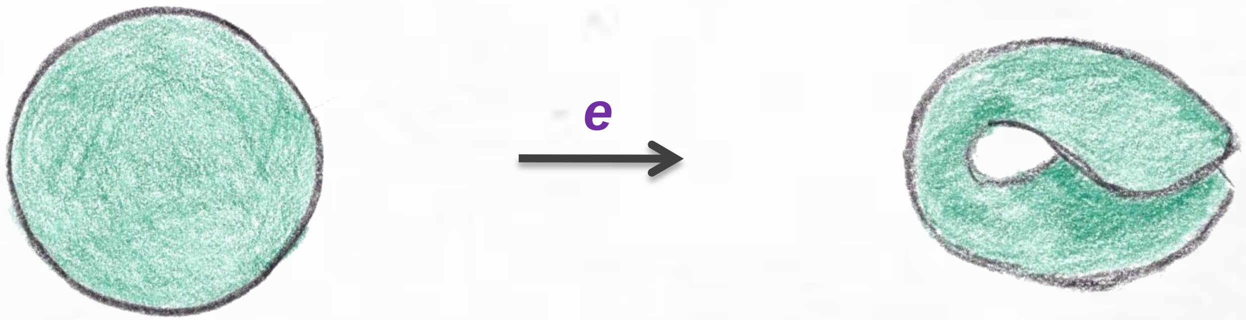
An embedding of a disk

$e\colon\ D \to R^3$

$e\colon D \to e(D)$ is a homeomorphism.

disk
in the plane

disk embedded in space
as a generalized terrain



We consider a special type of embeddings:
one side of e(D) consistently points up.

Embedding

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disk
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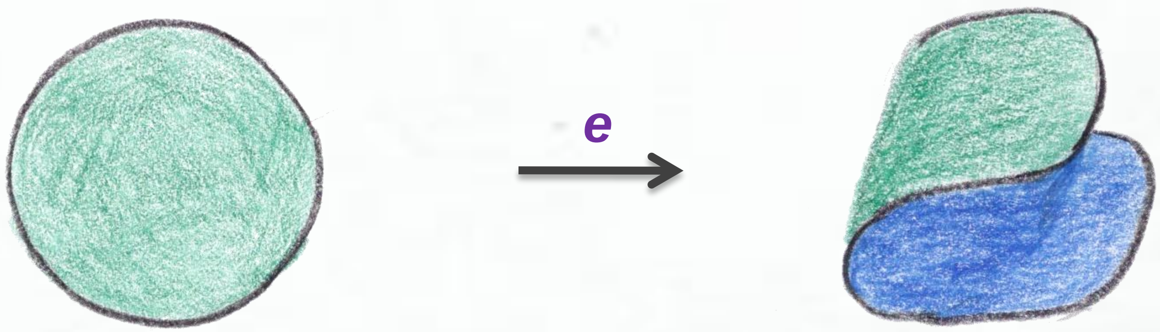
disk embedded in space
that is NOT a generalized terrain



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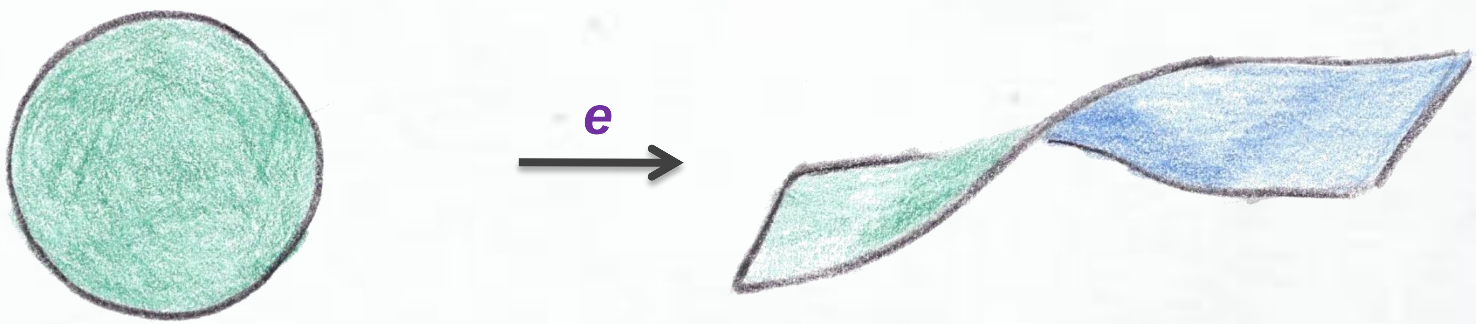


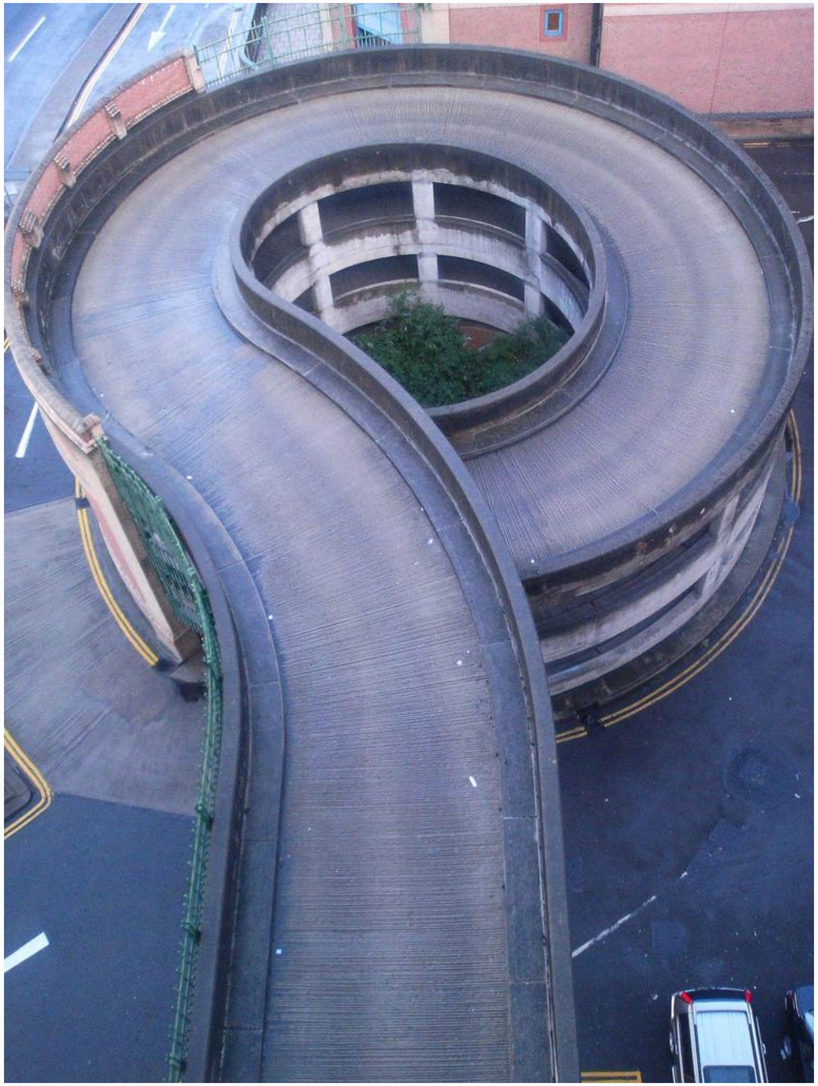
We consider a special type of embeddings:
one side of e(D) consistently points up.

EXAMPLES

By Flickr user **Mark Wheeler**

from *http://www.flickr.com/photos/markwheeler/246569058/*

Examples



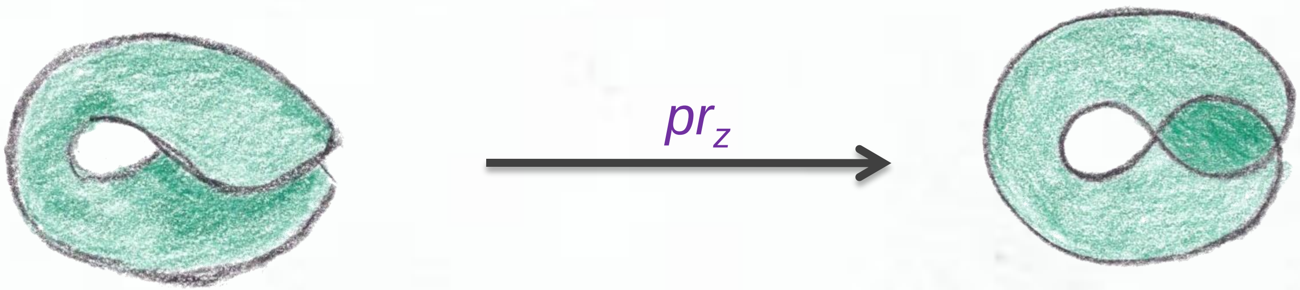
By Flickr user **Mark McLaughlin**

from *http://www.flickr.com/photos/clocky/257469851/*

Embedding

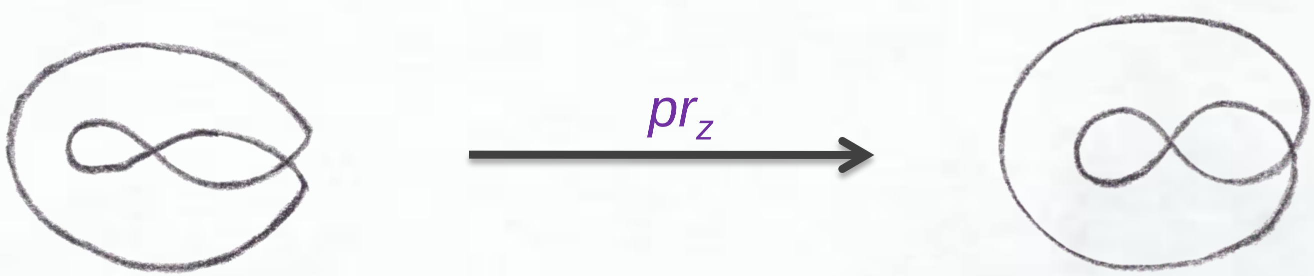
disk embedded in space
as a generalized terrain

immersed disk
in the plane



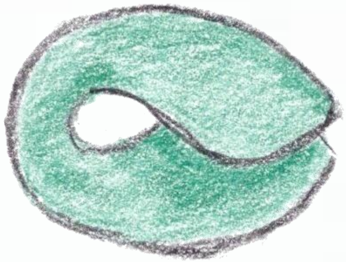
the boundary of a disk
embedded in space

self-intersecting curve
in the plane

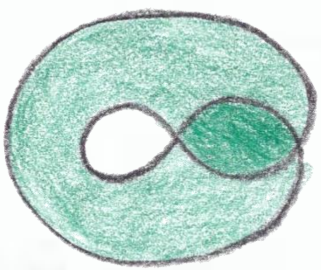


Problem Statement

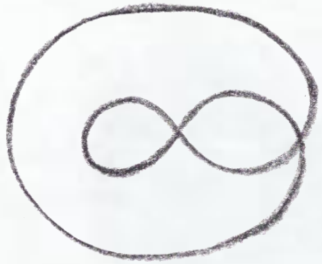
disk embedded in
space as a
generalized terrain



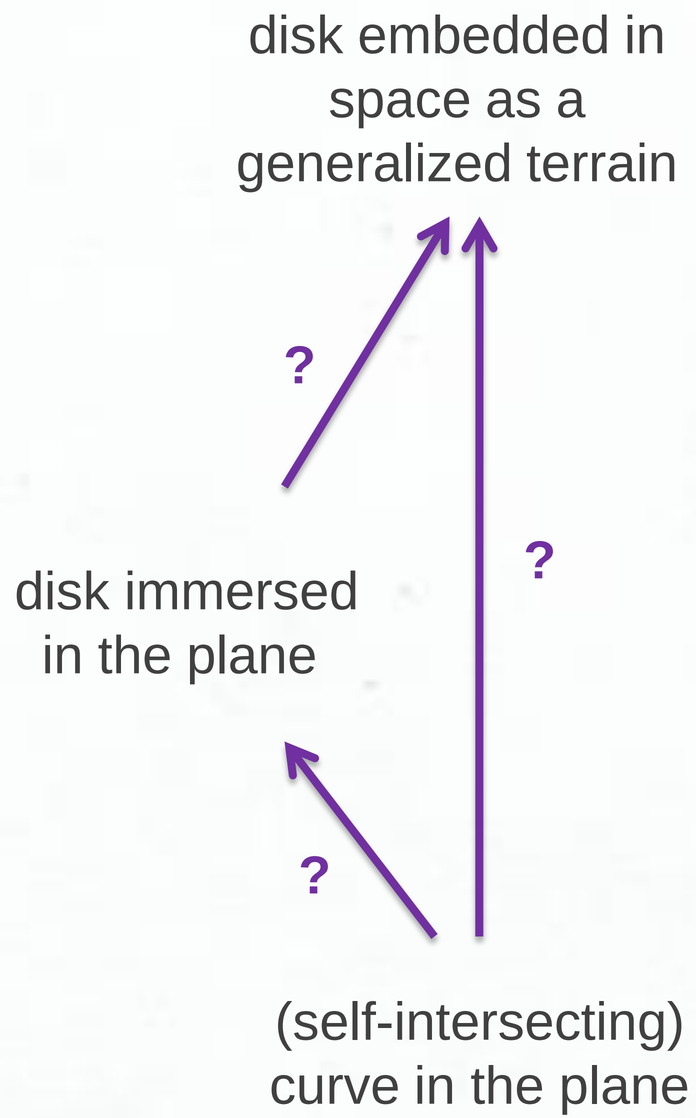
disk immersed
in the plane



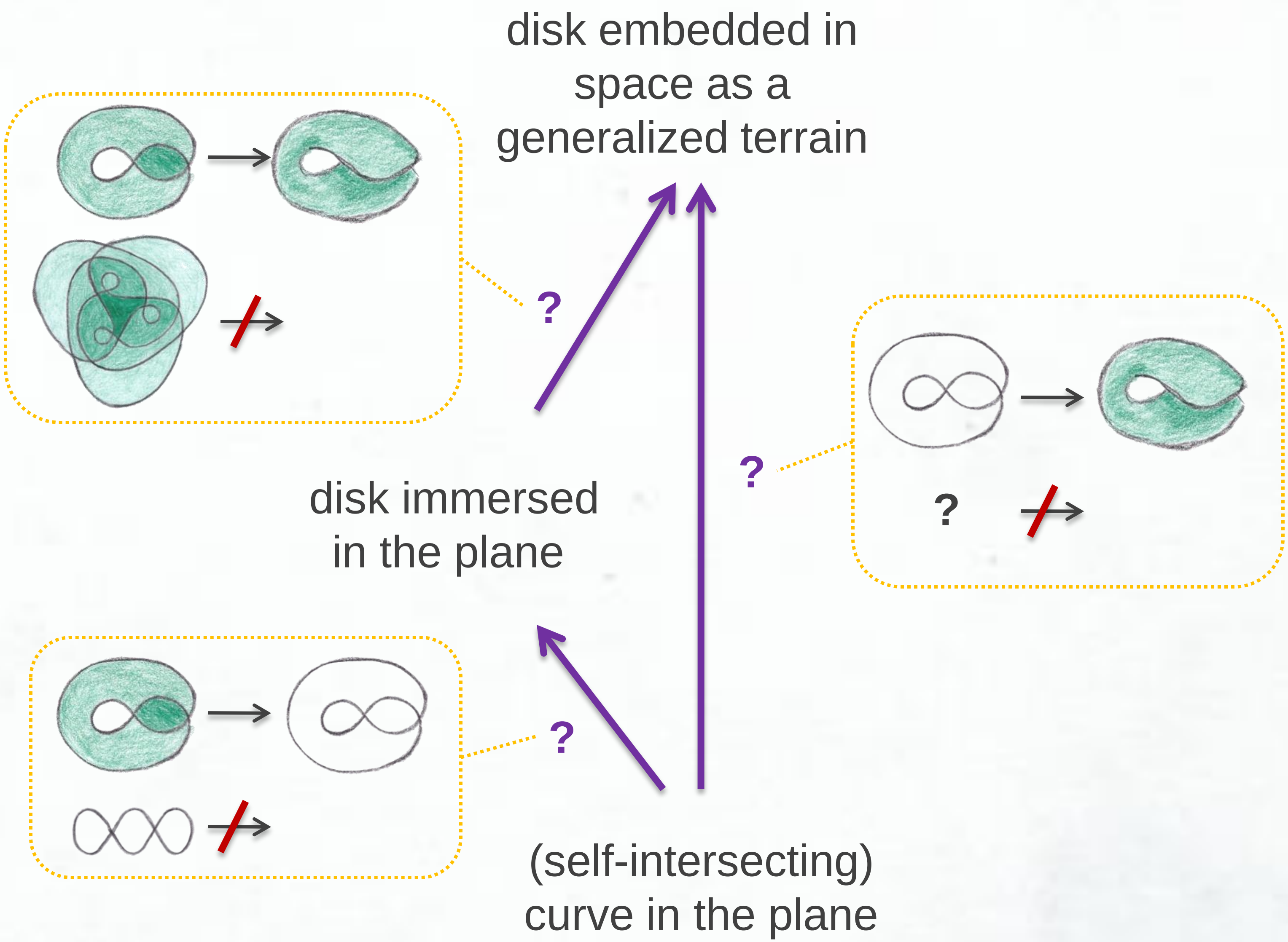
(self-intersecting)
curve in the plane



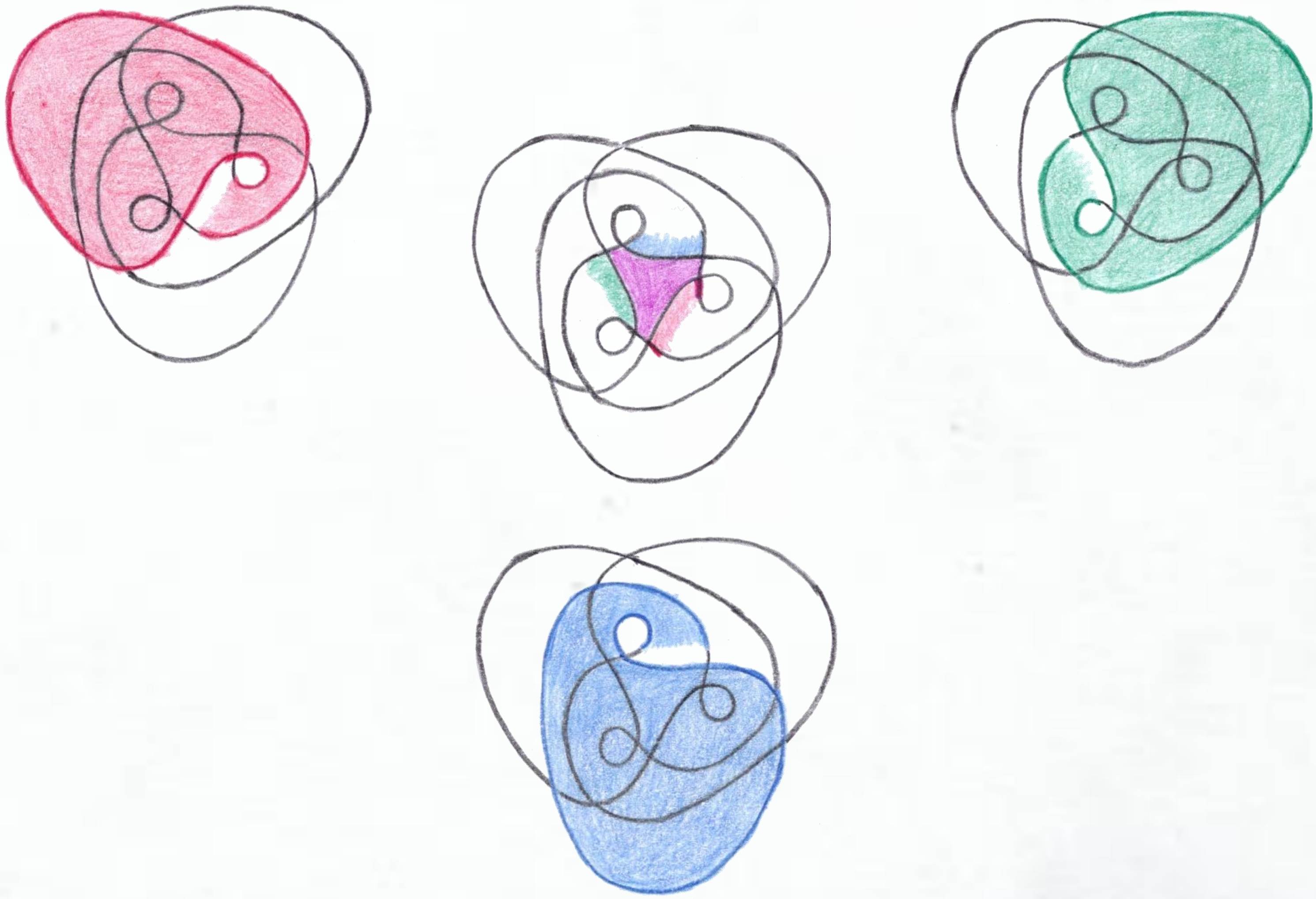
Problem Statement



Problem Statement

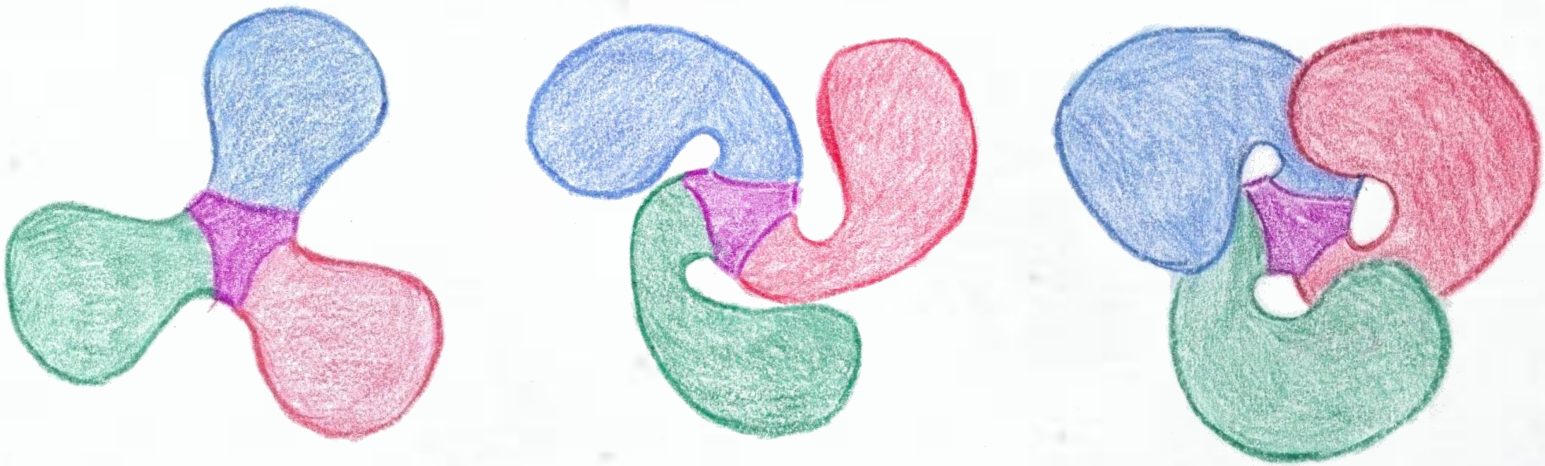


Bennequin Disk I



An immersed disk that is not a projection of a disk embedded in space

Bennequin Disk I



An immersed disk that is not a projection of a disk embedded in space

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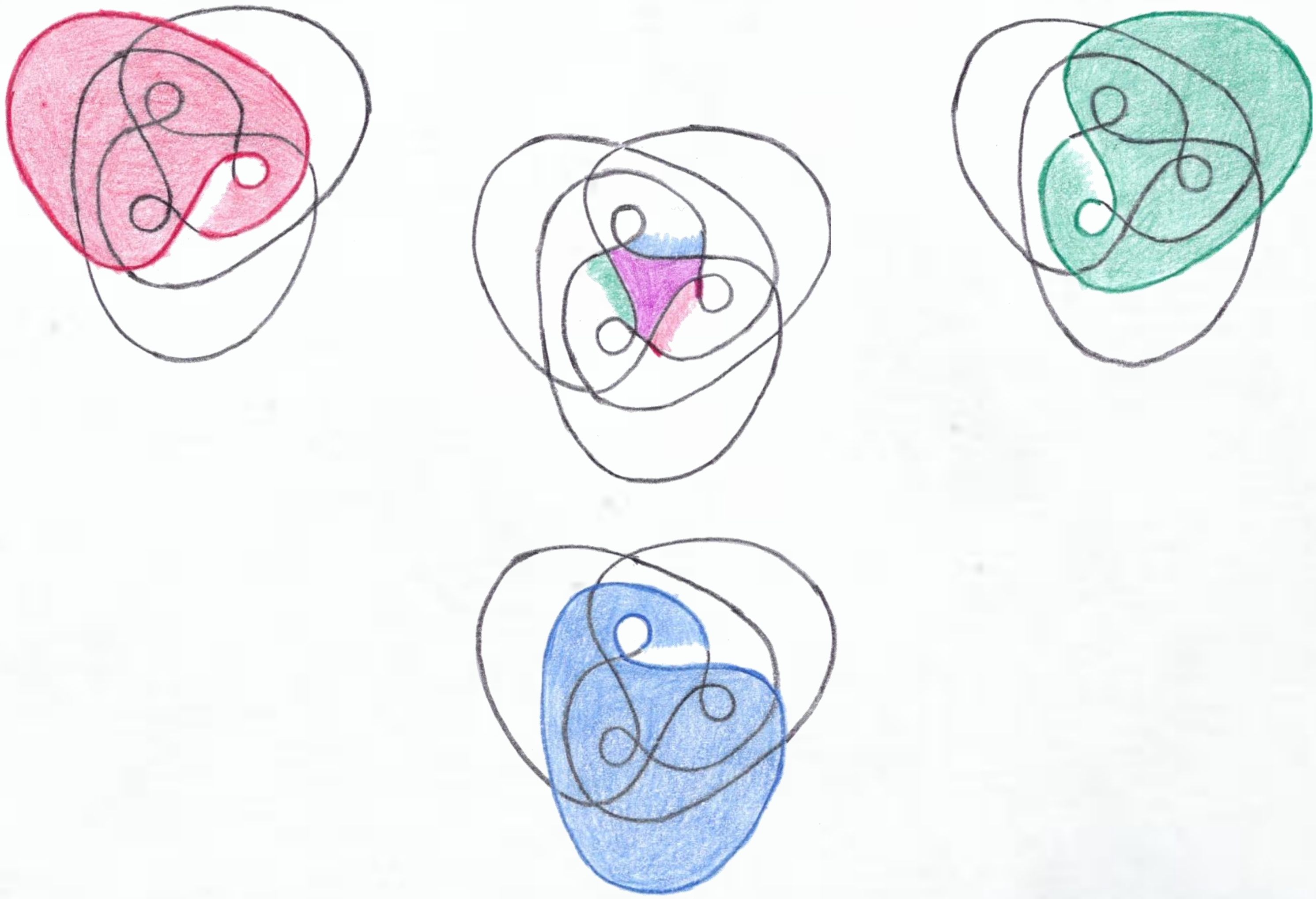

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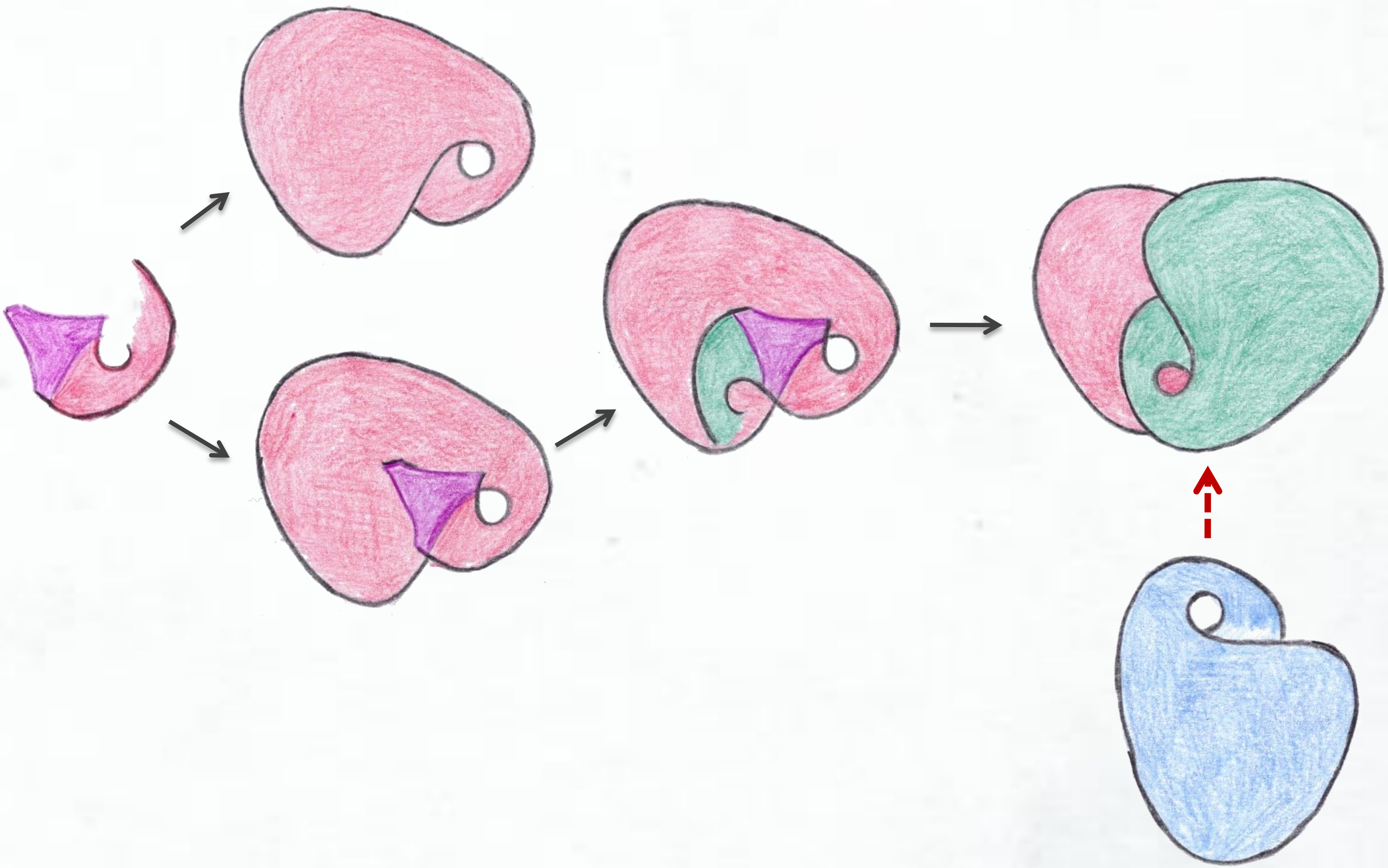
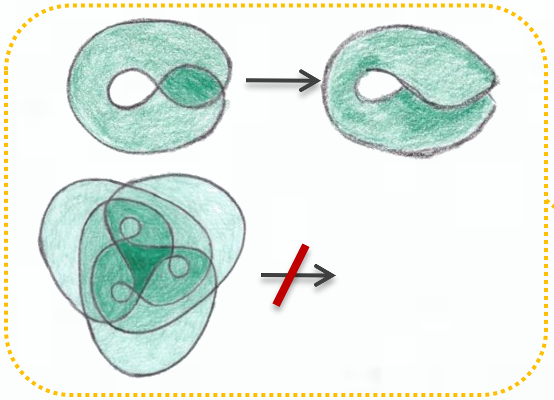
Bennequin Disk I



An immersed disk that is not a projection of a disk embedded in space

Problem Statement

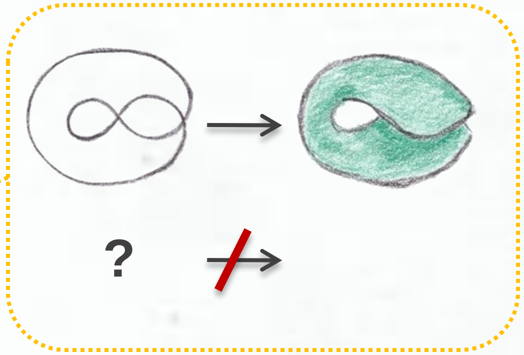
disk embedded in
space as a
generalized terrain



?

disk immersed
in the plane

?



Whitney(1937)
Shor and van Wyk(1992)

(self-intersecting)
curve in the plane

Problem Statement

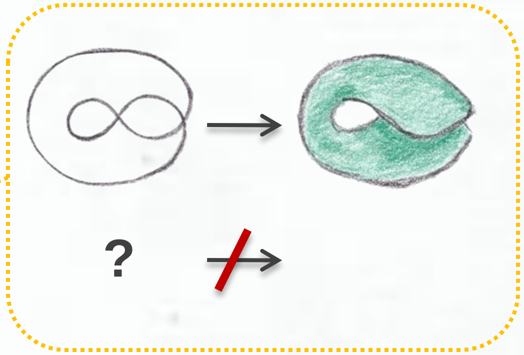
disk embedded in
space as a
generalized terrain

NP-complete

disk immersed
in the plane

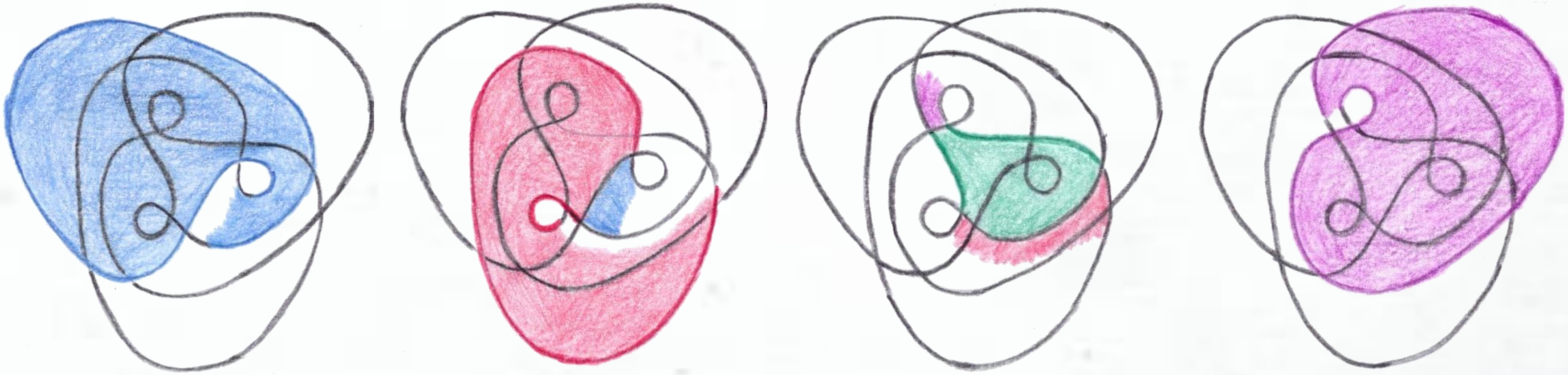
Whitney(1937)
Shor and van Wyk(1992)

?

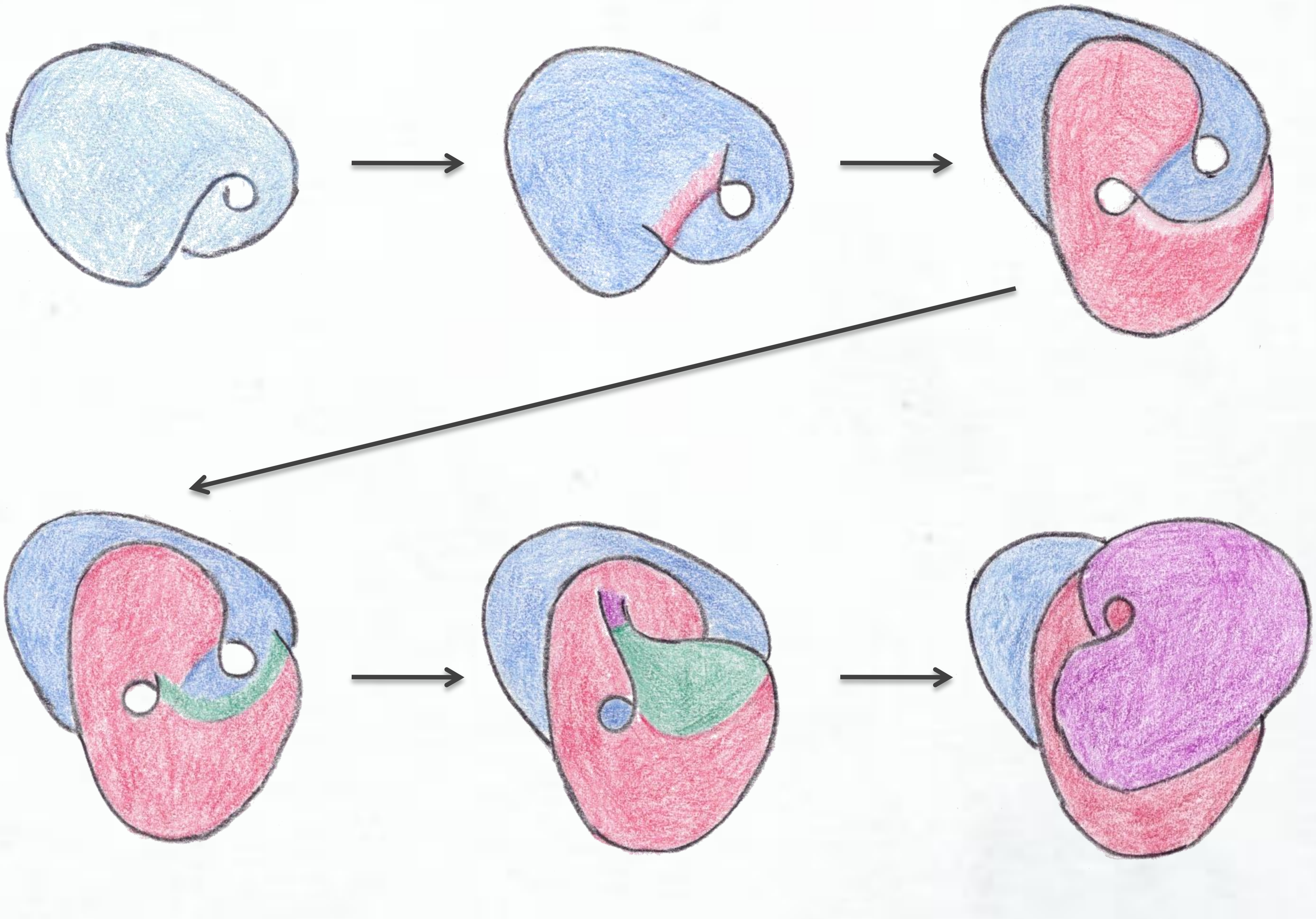


(self-intersecting)
curve in the plane

Bennequin Disk II



Bennequin Disk II



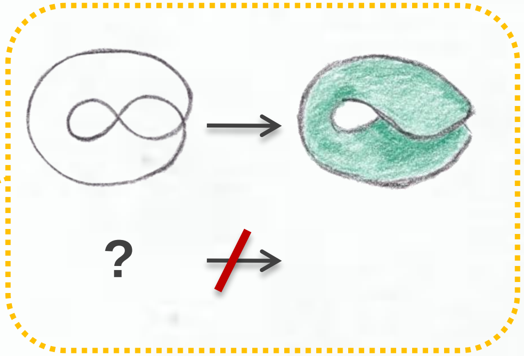
Problem Statement

disk embedded in
space as a
generalized terrain

NP-complete

disk immersed
in the plane

?



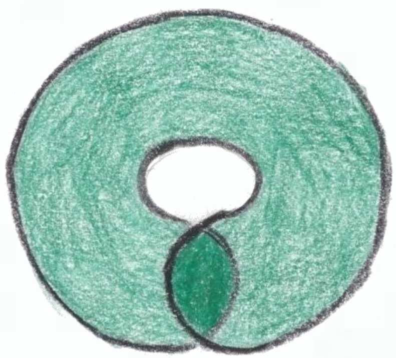
open

Whitney(1937)
Shor and van Wyk(1992)

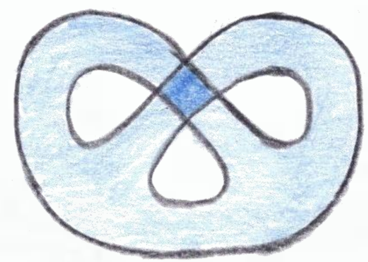
(self-intersecting)
curve in the plane

Generalize the Problem

disk with a boundary

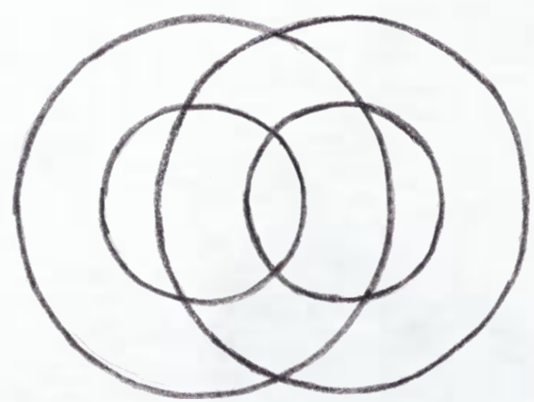


surface (two-dimensional manifold) with a boundary

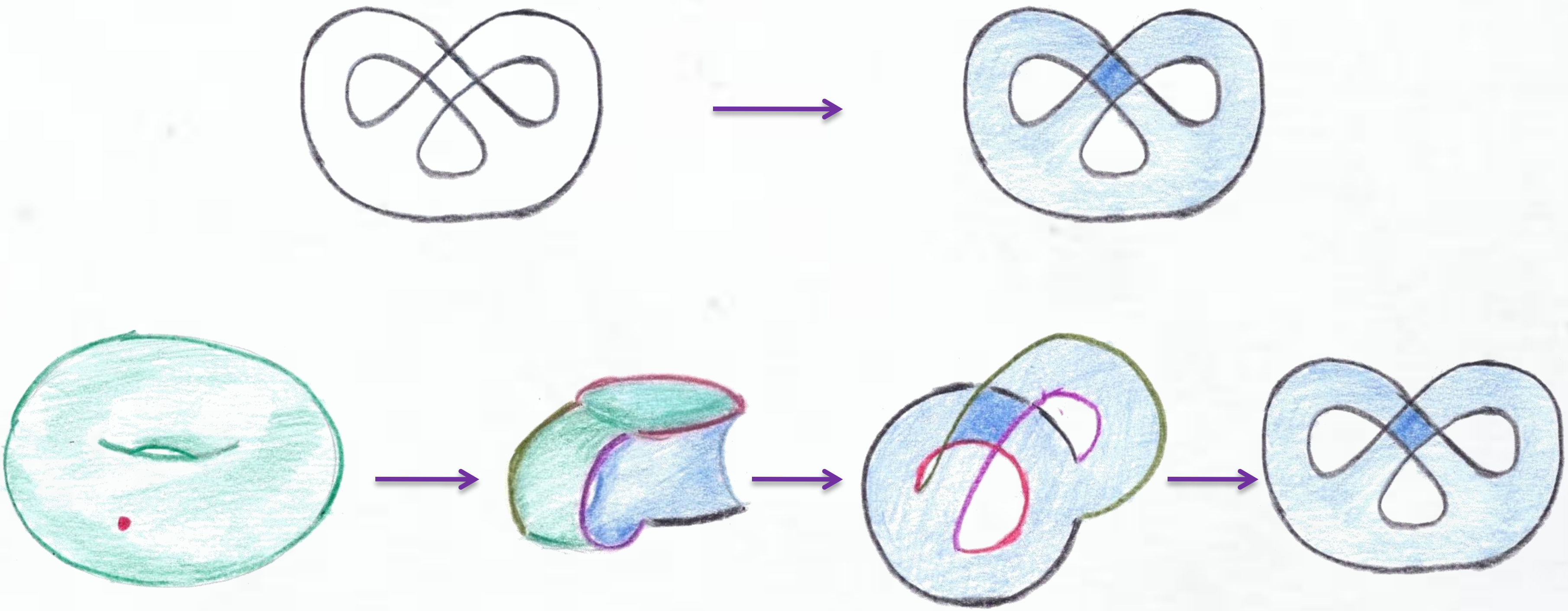


closed curve

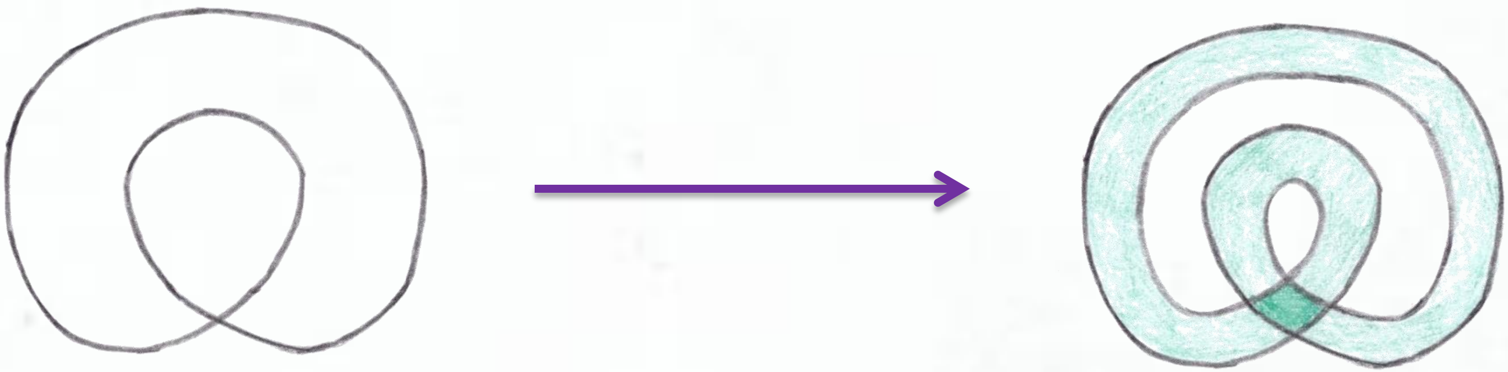
multiple closed curves

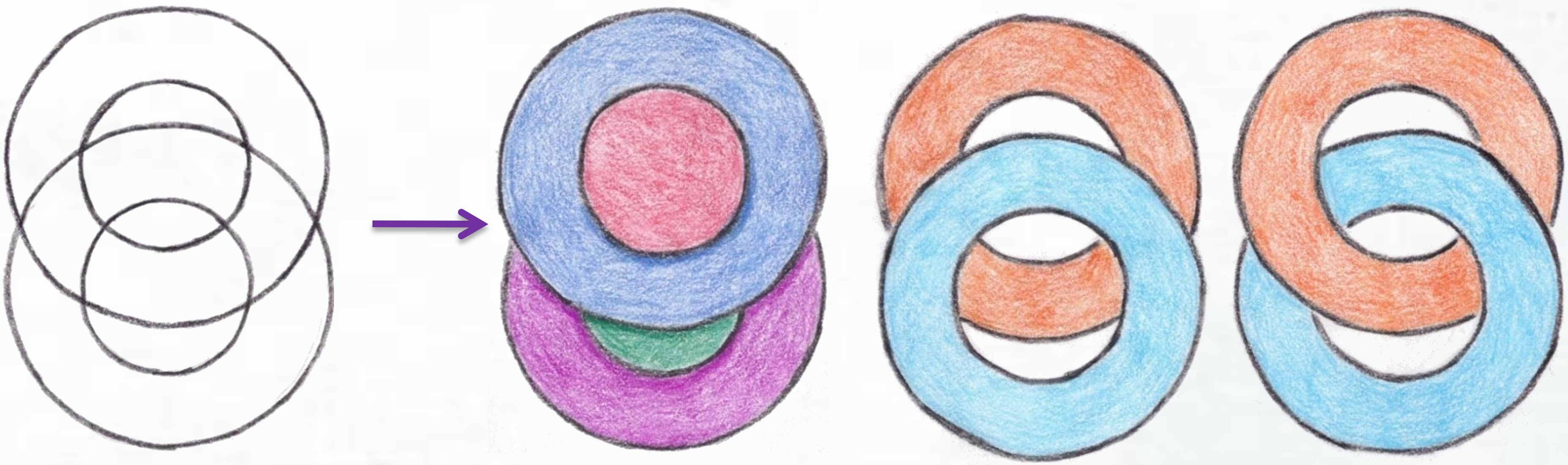


Disk → Manifold

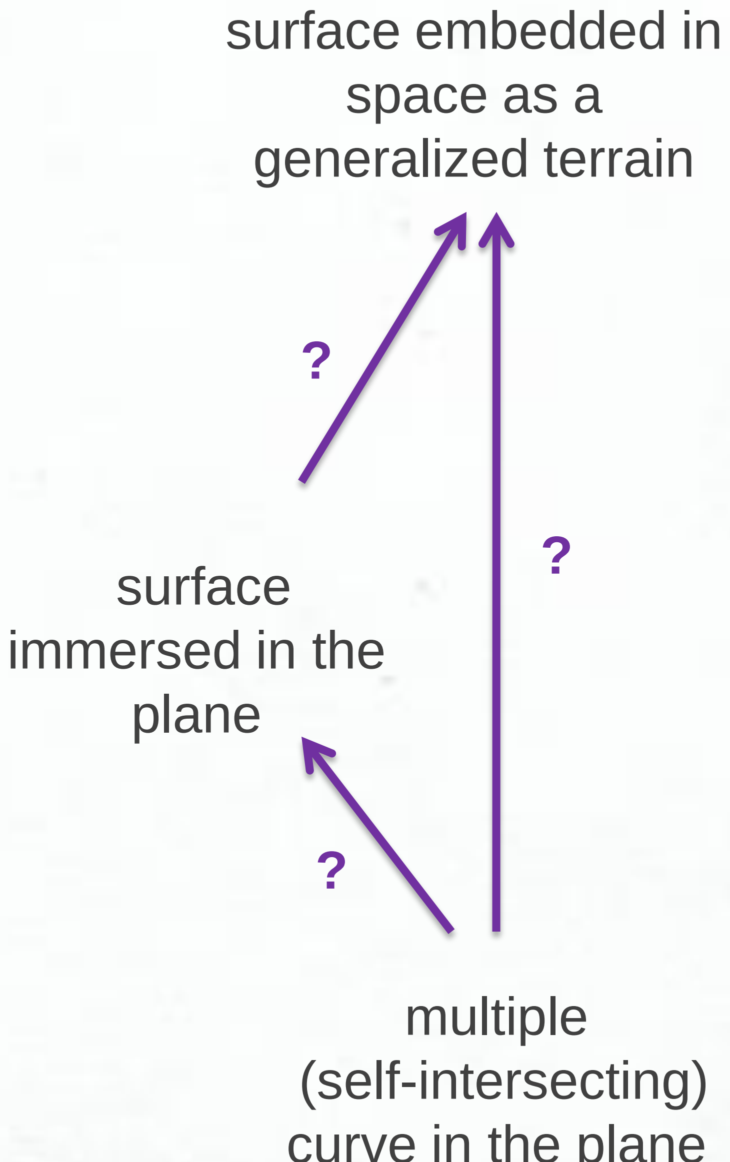


Multiple Curves

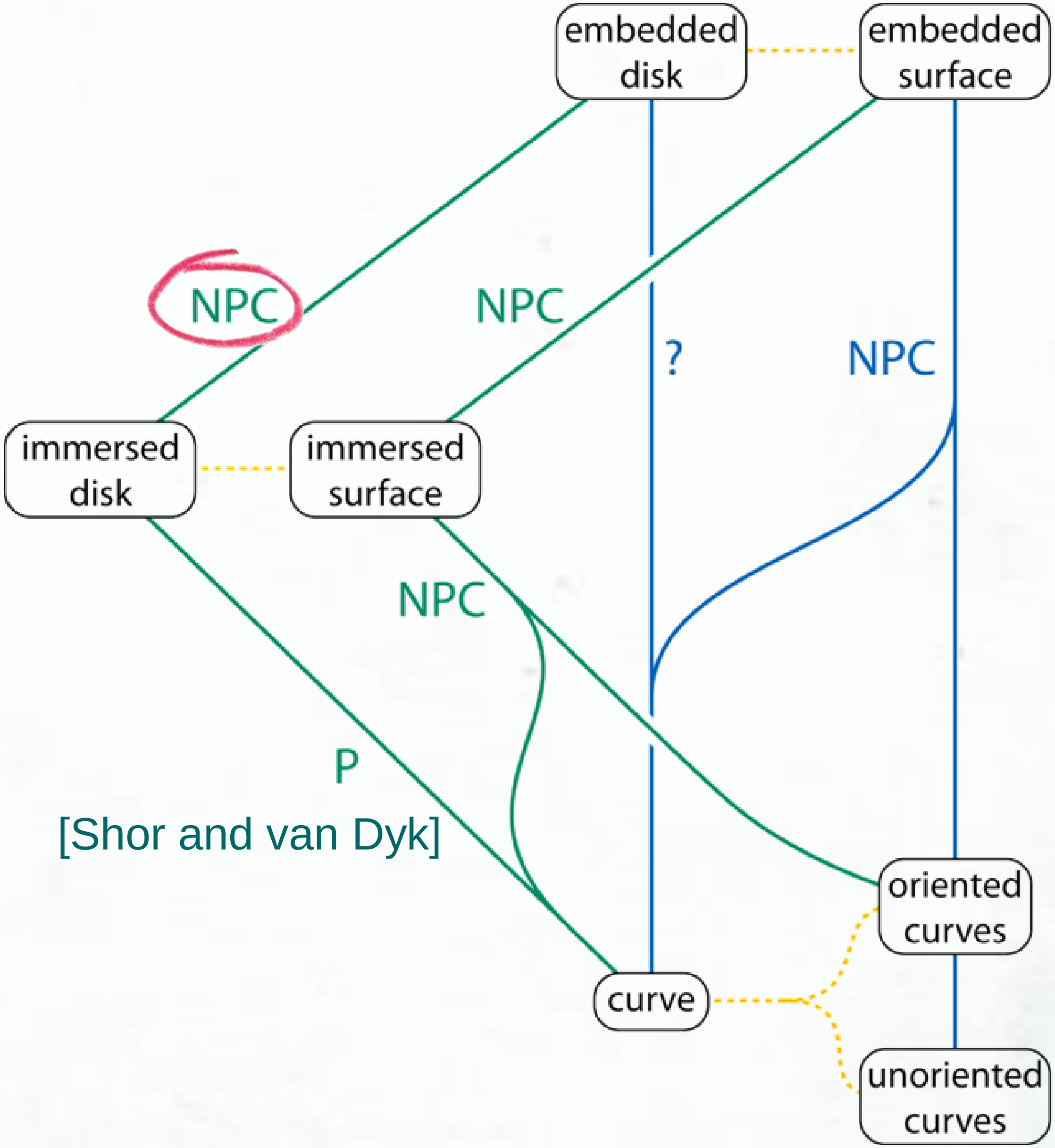




Generalize the Problem



RESULTS

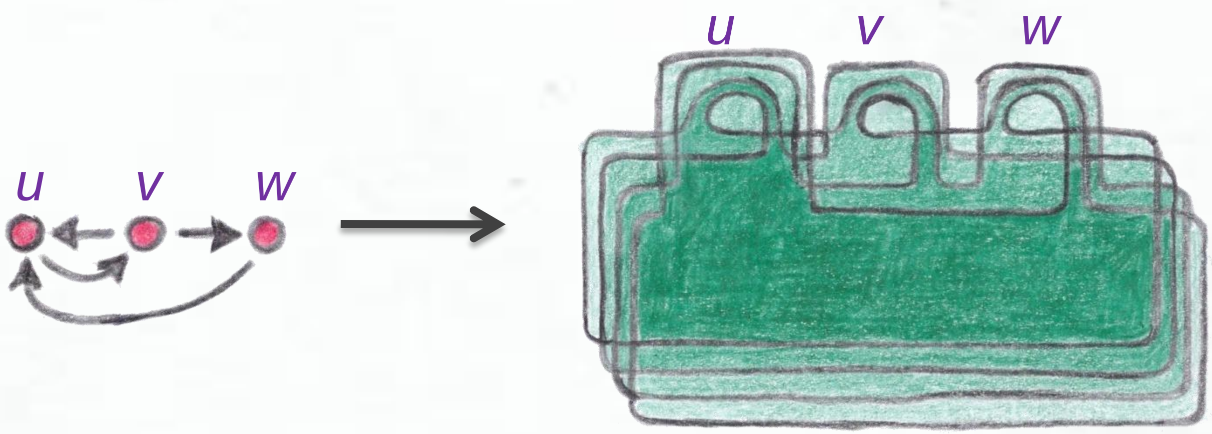


embedded disk
embedded surface
NPC
NPC
?
NPC
immersed disk
immersed surface
NPC
P
[Shor and van Dyk]
oriented curves
curve
unoriented curves

LIFT A DISK

Theorem Lifting an immersed disk is NP-complete.

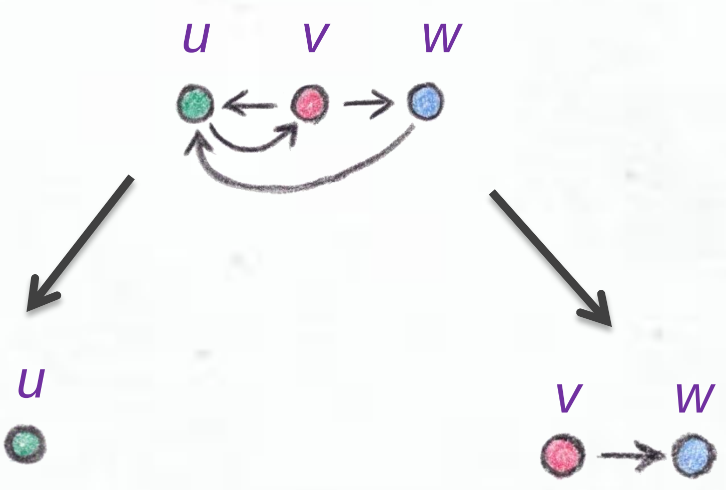
Proof: By reduction from ACYCLIC PARTITION.



ACYCLIC PARTITION

Given: a digraph $G = (V, E)$

Find: partition of V into sets V_1 and V_2 : $G(V_1)$ and $G(V_2)$ in G are acyclic.



Theorem ACYCLIC PARTITION is NP-complete.

Proof: By reduction from PLANAR 3-SAT.

Lift a Disk

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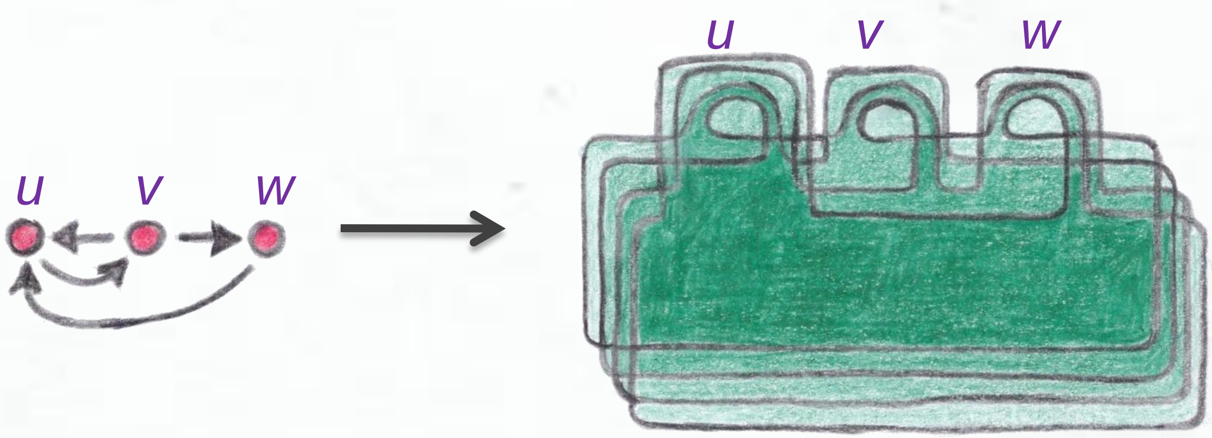
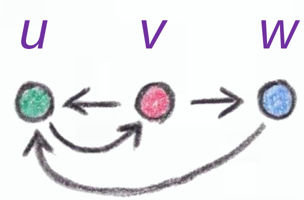
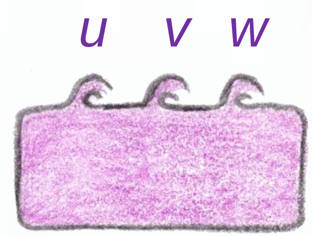
Proof: By reduction from ACYCLIC PARTITION.

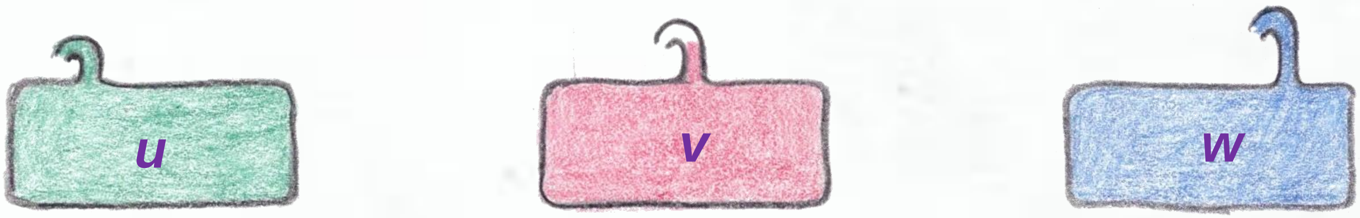


Lift a Disk

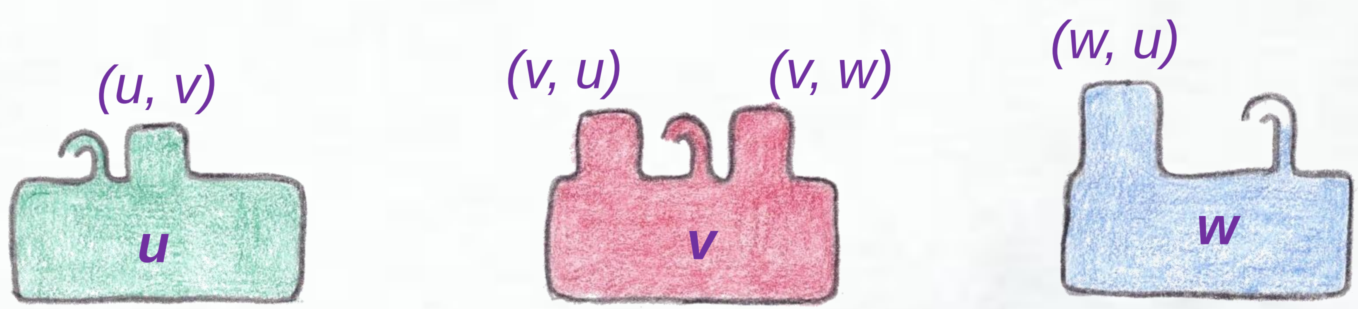




vertices:

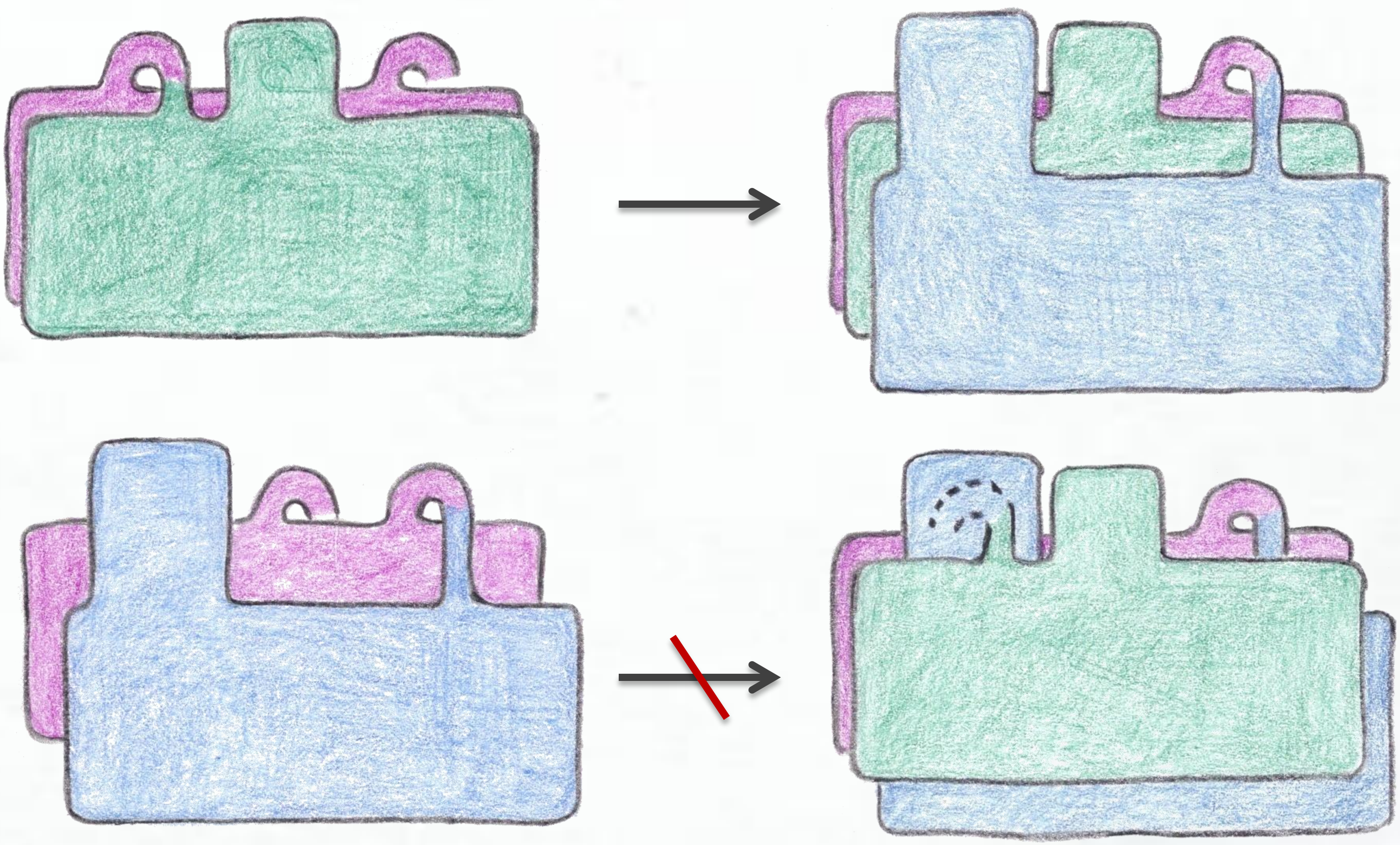


edges:



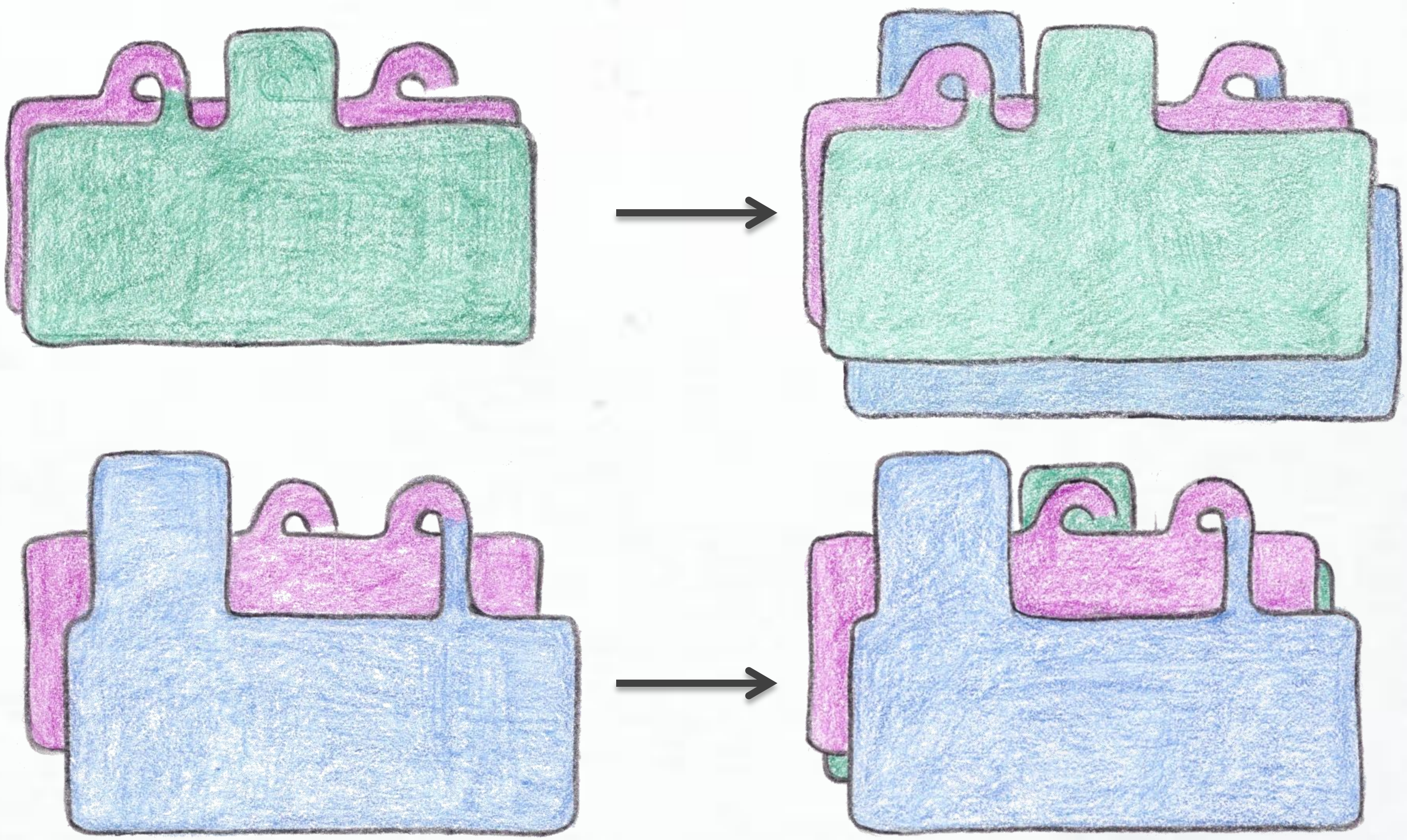
Lift a Disk

on the same side of the purple disk

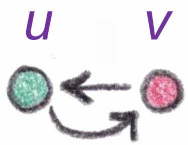


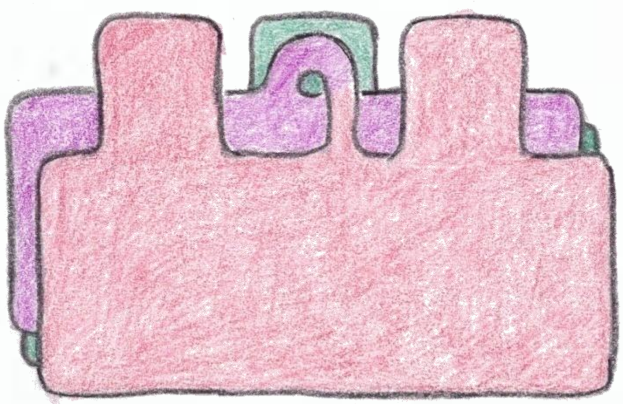
Lift a Disk

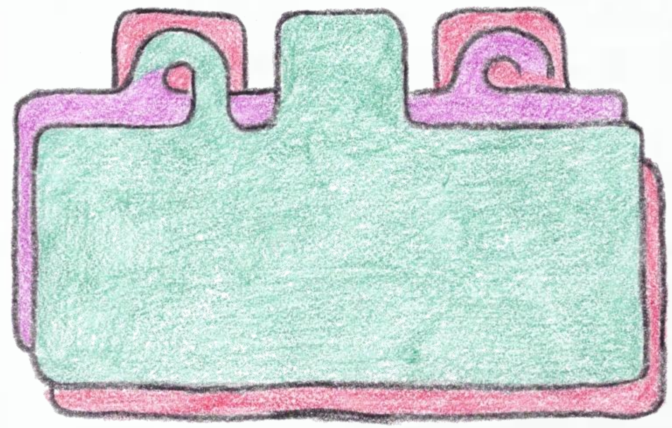
on different sides of the purple disk



Lift a Disk







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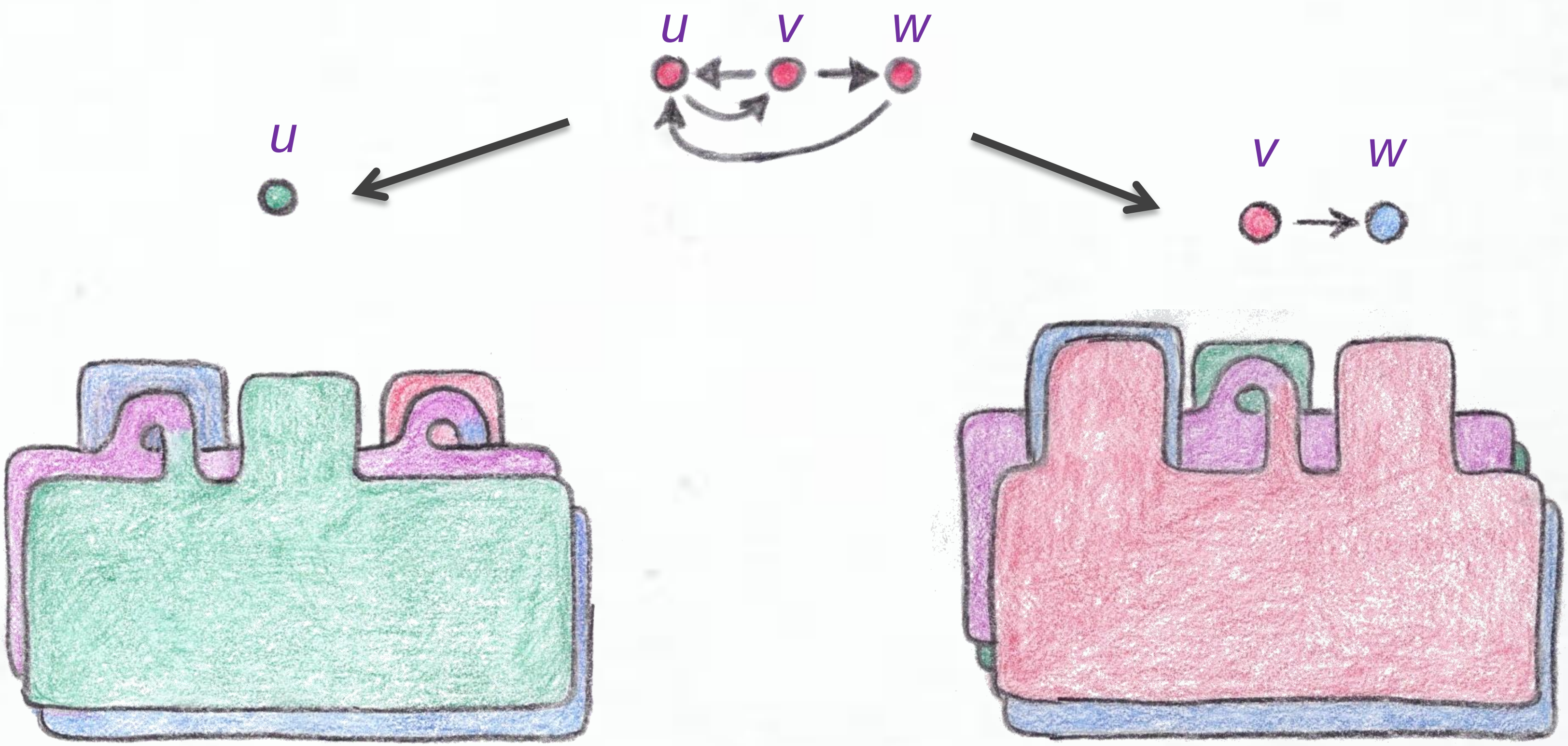
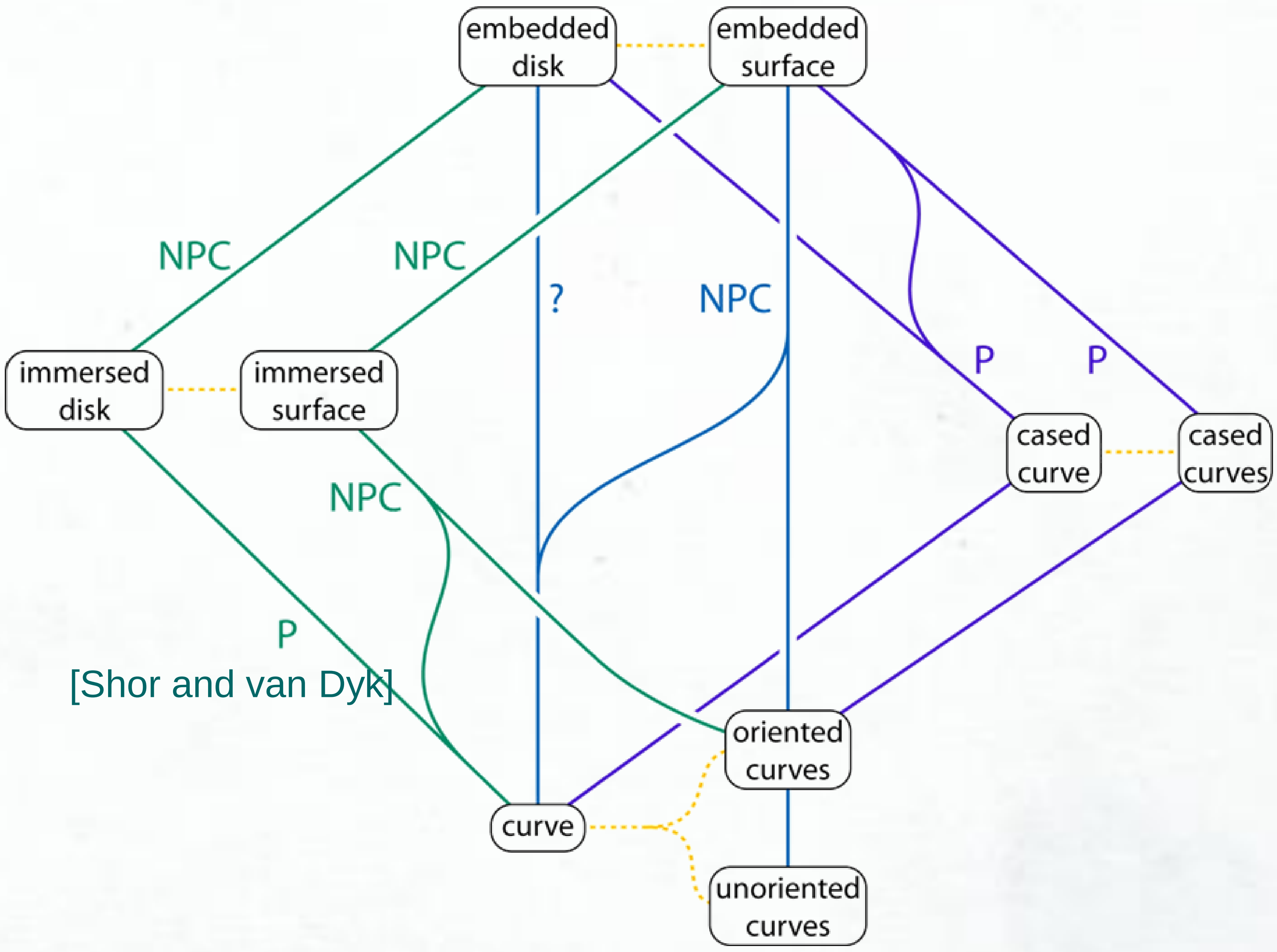
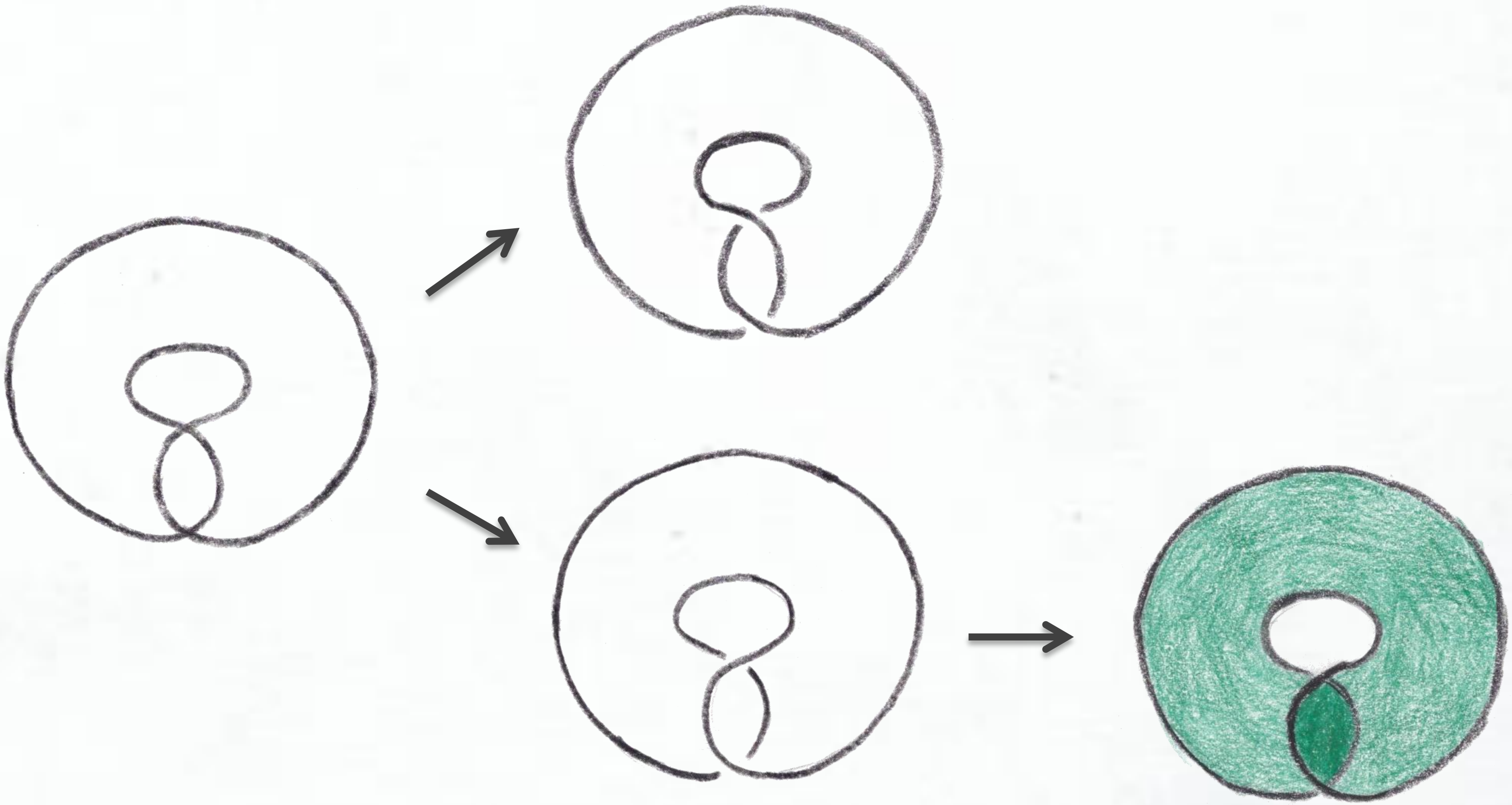


Summary

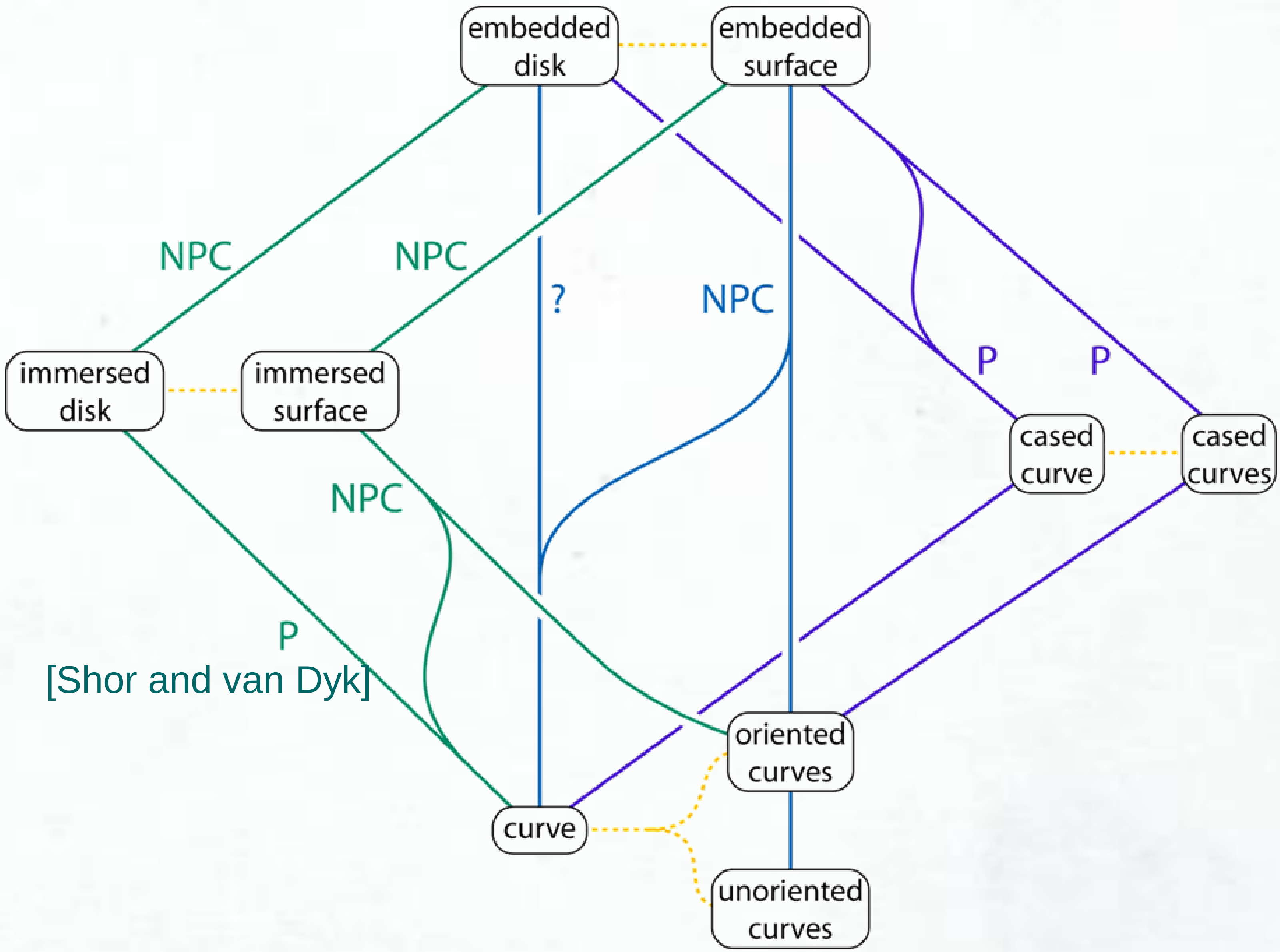

embedded disk
embedded surface
NPC
NPC
?
NPC
P
P
immersed disk
immersed surface
cased curve
cased curves
NPC
P
[Shor and van Dyk]
oriented curves
curve
unoriented curves

Cased Curves



A cased curve can be decided in $O(\min(nk, n+k^3))$.

Summary



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The End