Minimum-Weight Steiner Triangulation of Convex Polygons Requires Interior Steiner Points

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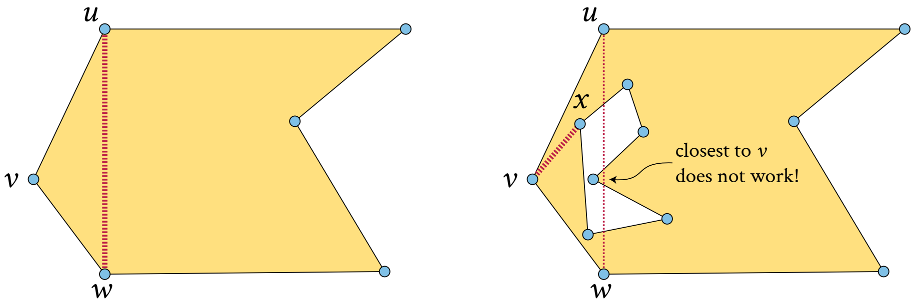
Every planar straight line graph has a triangulation

Can add one more edge at any interior angle $uvw < \pi$ that is not already part of a triangle

If we can add edge uw, do so

Else add edge from v to vertex in uvw farthest from line uw

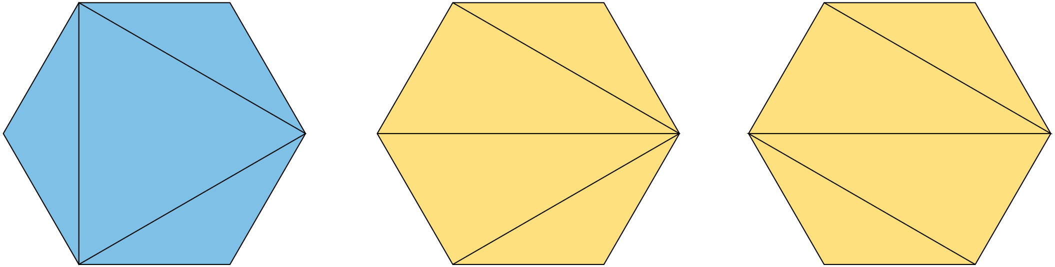


No extra Steiner points needed

Minimum-weight triangulation without Steiner points

Well-studied in the 1970s

[Düppe and Gottschalk 1970; Shamos and Hoey 1976; Lloyd 1977]



Polynomial via dynamic programming for simple polygons

[Gilbert 1979; Klincsek 1980]

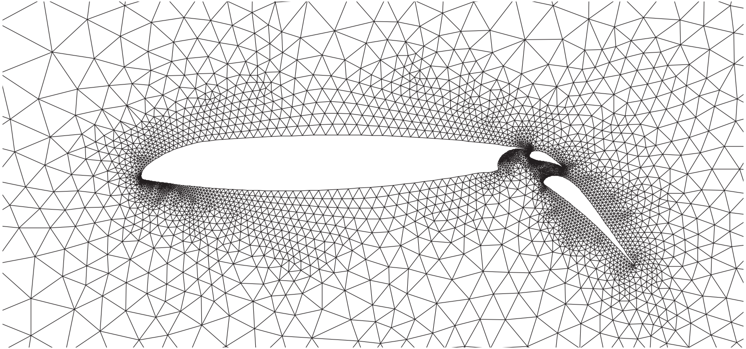
Famous as potentially NP-intermediate for point sets until finally shown to be NP-hard, with quasi-polynomial-time approximation scheme [Mulzer and Rote 2008; Remy and Steger 2009]

Steiner triangulation

Input: a polygon in the plane (or more general planar straight line graph)

Output: edge-to-edge subdivision into triangles

Allow added Steiner points as vertices, on boundary and interior



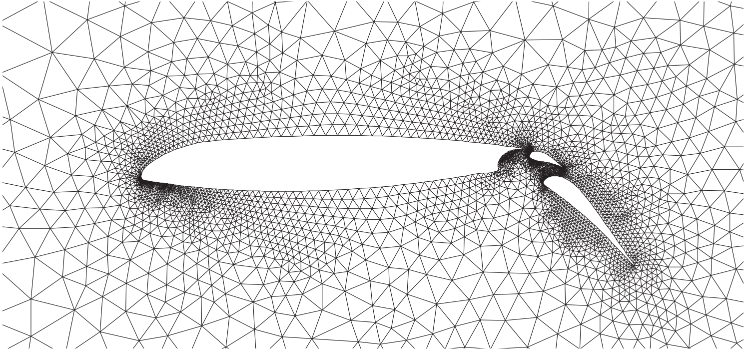
Often used for mesh generation in scientific computation

Why add Steiner points?

To improve quality of triangulation, according to various optimality measures

⇒ numerical robustness and improved convergence of calculations using the triangulation

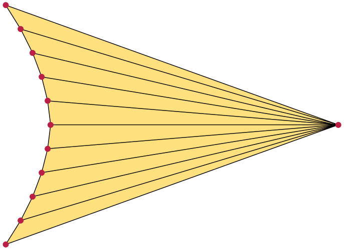
Much research on avoiding obtuse angles, avoiding sharp angles, etc.

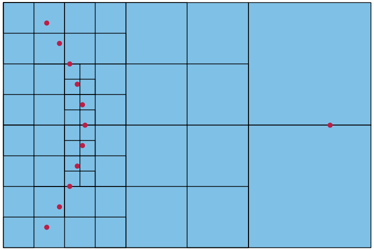


Instead, today: Minimize boundary length (sum of perimeters)

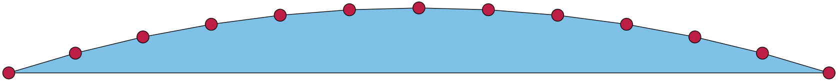
Steiner triangulations can be much shorter

Improvement can be linear for point sets and simple polygons





… and logarithmic even for convex polygons

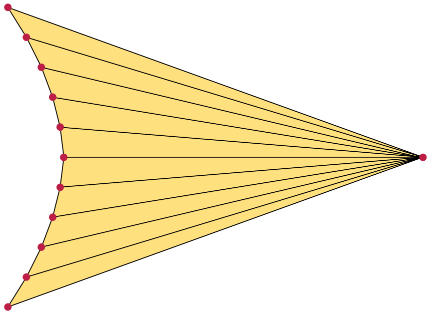


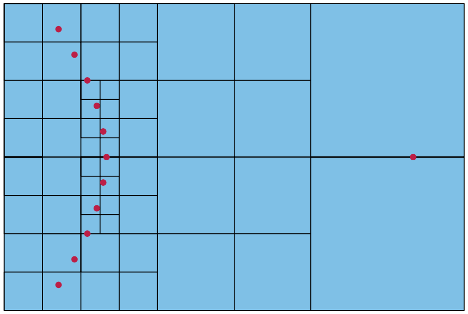
Complexity of finding short Steiner triangulations

Not known to be hard

Logarithmic approximation ratio [Clarkson 1991]

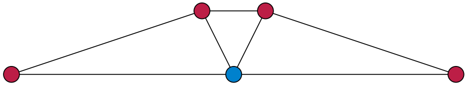
(Large) constant approximation using quadtrees [Eppstein 1994; Cheng and Lee 2002]





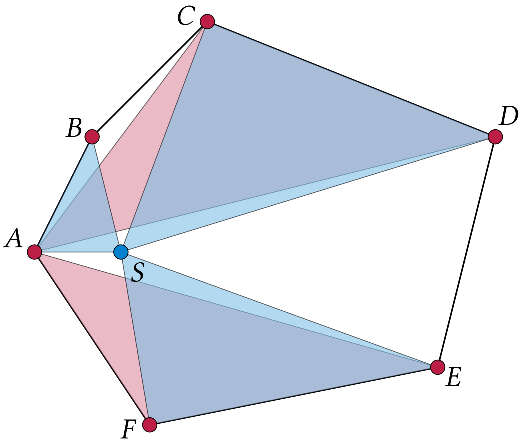
Can we solve the convex case exactly?

May need boundary Steiner points



For $n \leq 6$, one interior Steiner point is worse than contracting to closest vertex:

$$AC + AD + AE$$
$$\leq (AS + SC) + (AS + AD) + (AS + AE)$$
$$\leq (BS + SC) + (AS + AD) + (FS + AE)$$

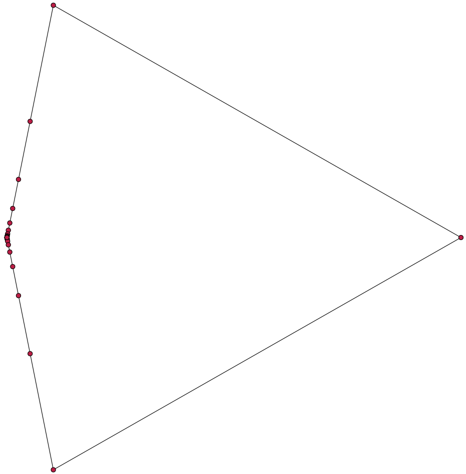


Conjecture [Eppstein 1994]: Interior points never help

Motivation: If so, tree-like dual structure $\Rightarrow$ dynamic program

Our result

The conjecture is false: ∃ convex 28-gon for which one interior Steiner point is better than any number of boundary Steiner points



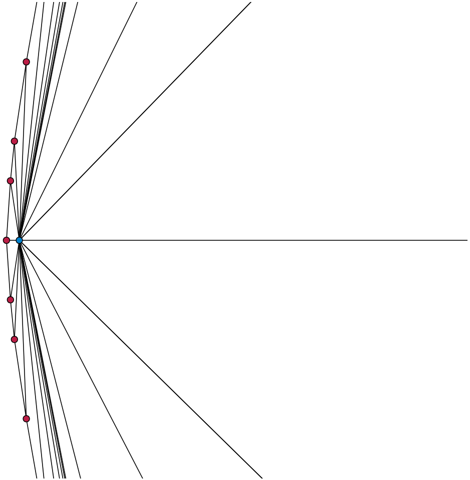
Overall shape is like a kite, slightly rounded to make it convex

Exponentially-spaced additional polygon vertices on the short sides of the kite

With the whole polygon visible, impossible to distinguish most vertices: they cluster at the corner of the kite where these sides meet

The Steiner point is very close to where these points cluster

Close-up, 40x



Optimal non-Steiner triangulation contracts Steiner point to closest vertex

(Verified by implementing the dynamic program for optimal polygon triangulation)

Most edges incident to the Steiner point extend to the right $\Rightarrow$ this contraction lengthens them more than it would save by eliminating short edges

(Verified numerically)

Therefore: interior Steiner triangulation is better than non-Steiner triangulation

The hard part: boundary Steiner points don't help

Main idea: Any counterexample could be changed locally into a better triangulation with fewer Steiner points $\Rightarrow$ counterexample with fewest Steiner points cannot exist

(Similar to how low-degree interior Steiner points can be removed without increasing total length)

Details: Long case analysis

- Steiner points on top or bottom short edges
  - With a neighbor on a long edge
  - Without a neighbor on a long edge
- Steiner point on long edge
  - with all neighbors on same side of symmetry axis
  - with neighbors on other side of symmetry axis
- More than one Steiner point on same polygon edge
- (several more increasingly complicated cases)

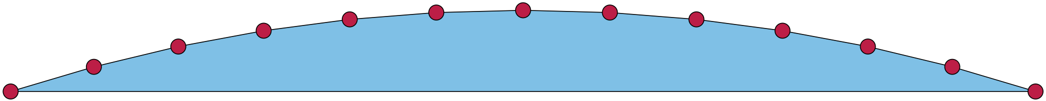
Conclusion and related problems

Main result: Optimal Steiner triangulation of convex polygons may need interior points

Previously known: Convex polygon optimal triangulation can be approximated within $O(1)$ (in polynomial time) using only boundary Steiner points; triangulation without Steiner points is worse by at least a logarithmic factor [Eppstein 1994]

Unknown: A reasonably small and explicit approximation ratio

Unknown [Arora]: Approximation scheme, ratio arbitrarily close to 1



Previously known: Minimum-weight non-Steiner triangulation of simple polygons is polynomial, but for point sets it is NP-hard

[Mulzer and Rote 2008]

Unknown: Any hardness results on minimum weight Steiner triangulation

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