Graphs in Nature

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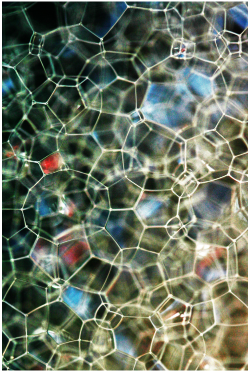
Inspiration: Steinitz's theorem

Purely combinatorial characterization of geometric objects:

Graphs of convex polyhedra are exactly
the 3-vertex-connected planar graphs

Image: Kluka [2006]

Overview



Cracked surfaces, bubble foams, and crumpled paper also form natural graph-like structures

What properties do these graphs have?

How can we recognize and synthesize them?

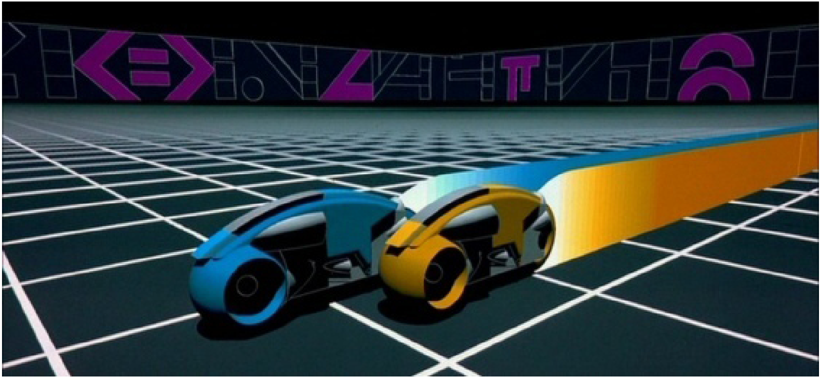
I. Cracks and Needles

Motorcycle graphs: Canonical quad mesh partitioning

Problem: partition irregular quad-mesh into regular submeshes

[Eppstein et al. 2008]

Inspiration: Light cycle game from TRON movies

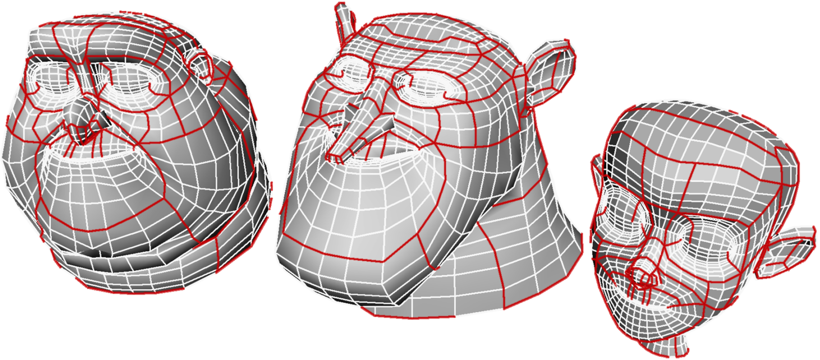


Mesh partitioning method

Grow cut paths outwards from each irregular (non-degree-4) vertex

Cut paths continue straight across regular (degree-4) vertices

They stop when they run into another path



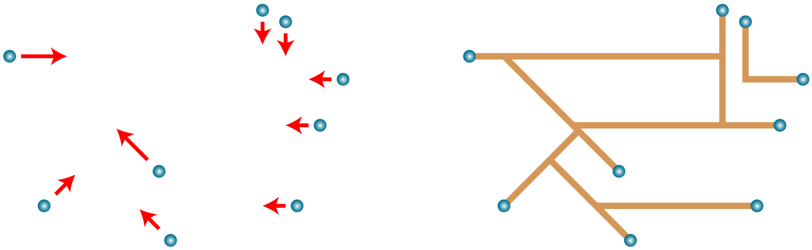
Result: approximation to optimal partition

(exact optimum is NP-complete)

Mesh-free motorcycle graphs

Earlier...

Motorcycles move from initial points with given velocities
When they hit trails of other motorcycles, they crash



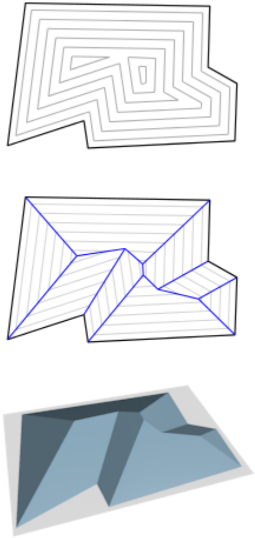
[Eppstein and Erickson 1999]

Application of mesh-free motorcycle graphs

Initially: A simplified model of the inward movement of reflex vertices in *straight skeletons*, a rectilinear variant of medial axes with applications including building roof construction, folding and cutting problems, surface interpolation, geographic analysis, and mesh construction

Later: Subroutine for constructing straight skeletons of simple polygons [Cheng and Vigneron 2007; Huber and Held 2012]

Image: Huber [2012]



Construction of mesh-free motorcycle graphs

Main ideas:

Define asymmetric distance:
Time when one motorcycle would crash into another's trail

Repeatedly find closest pair and eliminate crashed motorcycle

Image: Dancede [2011]

$O(n^{17/11+\epsilon})$ [Eppstein and Erickson 1999]

Improved to $O(n^{4/3+\epsilon})$ [Vigneron and Yan 2014]

Additional log speedup using mutual nearest neighbors instead of closest pairs [Mamano et al. 2019]

Gilbert tessellation

Even earlier...

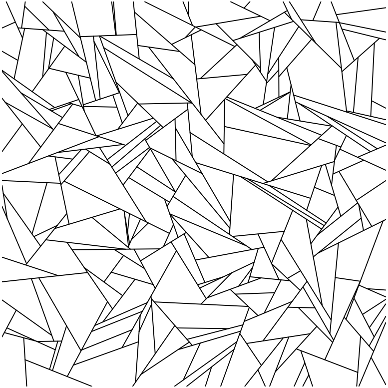


Image: Rocchini [2012b]

Gilbert [1967]:

Choose random points in the plane

Start *two* motorcycles in opposite (random) directions and equal speeds at each point

Form the motorcycle graph as before

Modeling the growth of needle-like crystals

(Gilbert's original motivation)

Image: Lavinsky [2010]

Cracks in dried mud

"Most mudcrack patterns in nature topologically resemble" Gilbert tesselations [Gray et al. 1976]

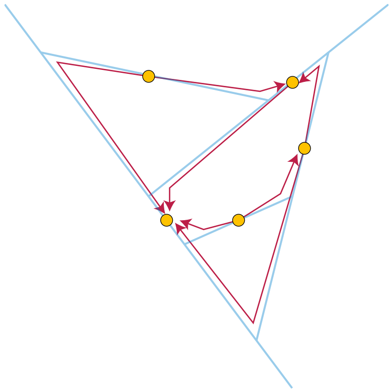
Image: Grobe [2007]

Combinatorial structure of a Gilbert tessellation

Represent as a graph:

Vertex for each segment

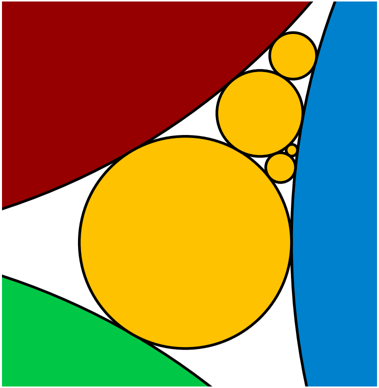
Edge for each crash



Contact graphs

Vertices = non-overlapping geometric objects of some type

Edges = pairs that touch but do not overlap



E.g. Koebe–Andreev–Thurston circle packing theorem:
Planar graphs are exactly the contact graphs of disks

Contact graphs of line segments

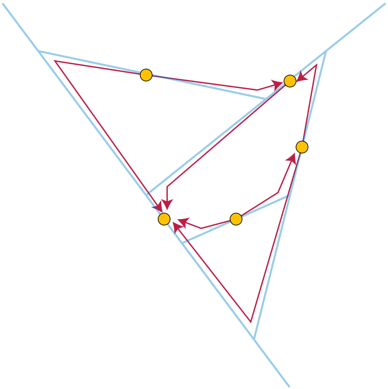
These graphs are:

Planar

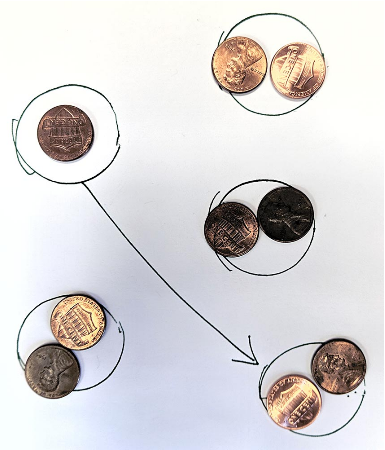
(2, 3)-sparse

(Each k-vertex subgraph has at most $2k - 3$ edges)

- $2k$ because each segment has 2 ends
- -3 because the convex hull has 3 vertices



Recognizing $(2,3)$-sparse graphs



Pebble game:

Start with all vertices, no edges, 2 pebbles/vertex

If a missing edge has > 3 pebbles, remove one pebble and draw edge directed away from removed pebble

If you need more pebbles, pull them backwards along directed paths, reversing the path edges

If $(2,3)$-sparse, draws all edges
If not: will get stuck

[Lee and Streinu 2008]

From pebbles to line segments

Theorem: **Contact graphs of line segments are exactly the planar (2,3)-sparse graphs**

Proof outline:

Edge directions from pebbling indicate which motorcycle crashed into which trail

Embed the graph using Tutte spring embedding
Straighten segments using infinitesimal weights

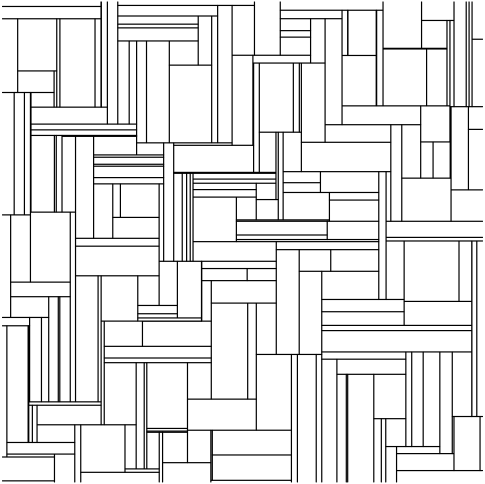
$(2,3)$-sparsity $\Rightarrow$ cannot degenerate to a line

[Thomassen 1993; de Fraysseix and Ossona de Mendez 2004]

(With planar separators, can pebble and recognize in time $O(n^{3/2})$)

Gilbert tessellations with restricted angles

E.g., random points with axis-aligned pairs of motorcycles:

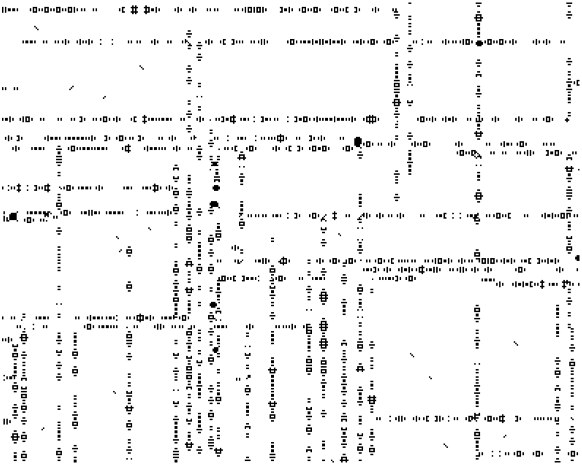


Mackisack and Miles [1996]; Burridge et al. [2013]

Image: Rocchini [2012a]

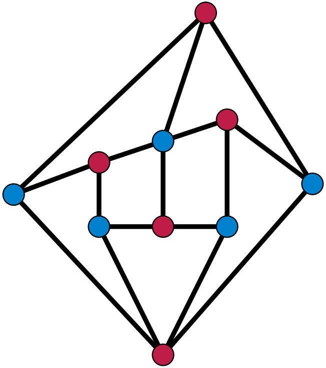
Replicator chaos

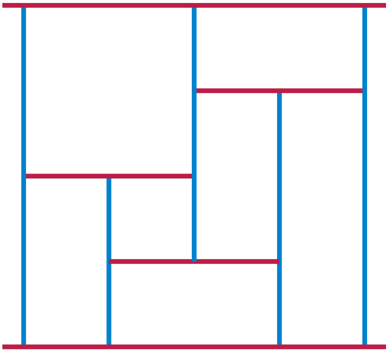
In 2d cellular automata that support 1d puffers or replicators (here B017/S1, possibly also Conway's Game of Life), sparse initial state ⇒ space fills with trails [Eppstein 2010]



Recognizing axis-parallel contact graphs

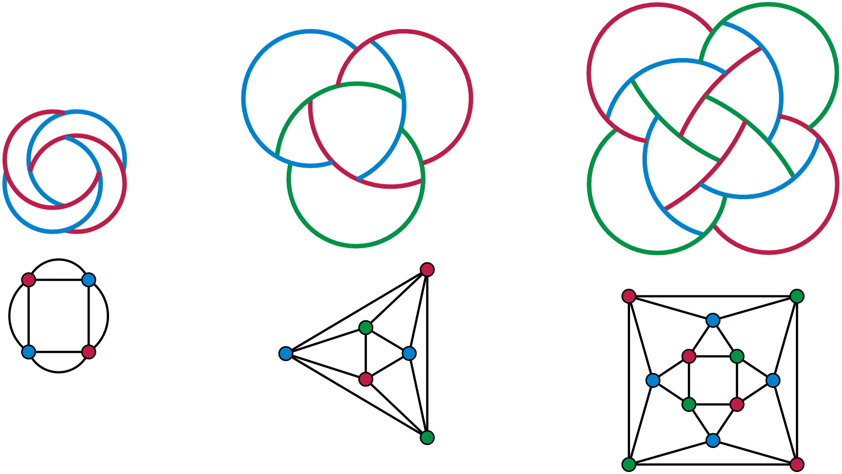
Contact graphs of axis-parallel segments = planar bipartite graphs





[Hartman et al. 1991]

Not fully characterized: Circular arcs



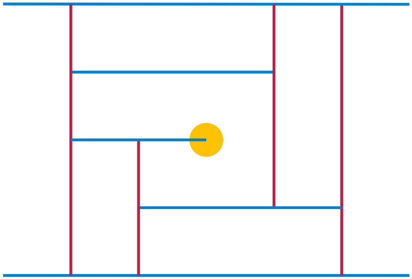
[Alam et al. 2015]

Back to Gilbert tessellations

Segment contact graphs: Fully characterized

Gilbert tessellation graphs are almost the same, but. . .

When there are fewer than $2n - 3$ edges,
when can segment endpoints be forced to lie on convex hull?



When all cracks grow at equal speed, does this impose additional combinatorial constraints?

II. Bubbles and Foams

Soap bubbles and soap bubble foams

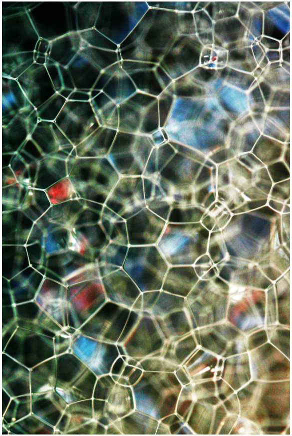


Image: woodleywonderworks [2007]

Soap molecules form double layers separating thin films of water from pockets of air

A familiar physical system that produces complicated arrangements of curved surfaces, edges, and vertices

What can we say about the mathematics of these structures?

Plateau's laws

In every soap bubble cluster:

- Each surface has constant mean curvature
- Triples of surfaces meet along curves at 120° angles
- These curves meet in groups of four at equal angles

Observed in 19th c. by Joseph Plateau

Proved by Taylor [1976]

Image: Unknown [1843]

Young–Laplace equation

Thomas Young

Image: Adlard [1830]

For each surface in a soap bubble cluster:

mean curvature
$= 1/$pressure difference

(with surface tension as constant of proportionality)

Formulated in 19th c., by Thomas Young and Pierre-Simon Laplace

Pierre-Simon Laplace

Image: Feytaud [1842]

Planar soap bubbles

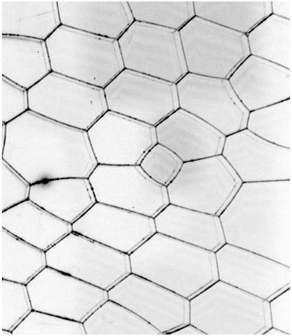


Image: Keller [2002]

3d is too complicated, let's restrict to two dimensions

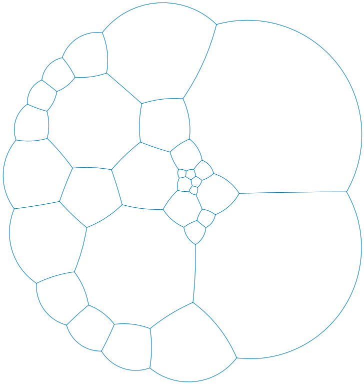
Equivalently, form 3d bubbles between parallel glass plates

Bubble surfaces are at right angles to the plates, so all 2d cross sections look the same as each other

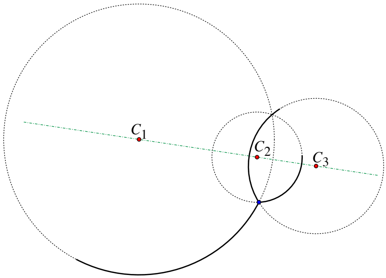
Plateau and Young–Laplace for planar bubbles

In every planar soap bubble cluster:

- Each curve is an arc of a circle or a line segment
- Each vertex is the endpoint of three curves at 120° angles
- It is possible to assign pressures to the bubbles so that curvature is inversely proportional to pressure difference



Geometric reformulation of the pressure condition



For arcs meeting at 120° angles, the following three conditions are equivalent:

- We can find pressures matching all curvatures
- Triples of circles have collinear centers
- Triples of circles form a "double bubble" with two triple crossing points

Möbius transformations

Fractional linear transformations

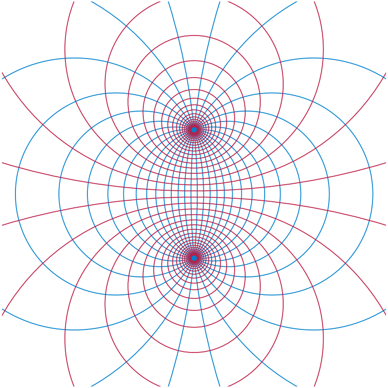
$$z \mapsto \frac{az+b}{cz+d}$$

in the plane of complex numbers

Take circles to circles and do not change angles between curves

Plateau's laws and the double bubble reformulation of Young–Laplace only involve circles and angles

so the Möbius transform of a bubble cluster is another valid bubble cluster



Bubble clusters don’t have bridges

(Bridge: same face on both sides of an edge.)

Image: Unknown [1940]

Main ideas of proof:

- A bridge that is not straight violates the pressure condition
- A straight bridge can be transformed to a curved one that again violates the pressure condition

Bridges are the only obstacle

For planar graphs with three edges per vertex and no bridges, we can always find a valid bubble cluster realizing that graph

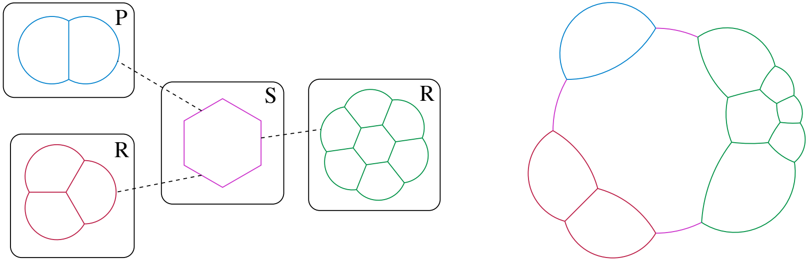
[Eppstein 2014]

Main ideas of proof:

1. Partition into 3-connected components and handle each component independently
2. Use Koebe–Andreev–Thurston circle packing to find a system of circles whose tangencies represent the dual graph
3. Construct a novel type of Möbius-invariant *power diagram* of these circles, defined using 3d hyperbolic geometry
4. Use symmetry and Möbius invariance to show that cell boundaries are circular arcs satisfying the angle and pressure conditions that define soap bubbles

Step 1: Partition into 3-connected components

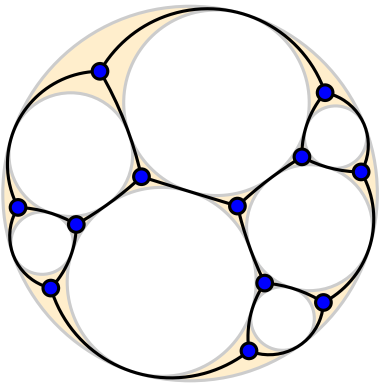
For graphs that are not 3-regular or 3-connected, decompose into smaller subgraphs, draw them separately, and glue them together



The decomposition uses *SPQR trees*, standard in graph drawing

Use Möbius transformations in the gluing step to change relative sizes of arcs so that the subgraphs fit together without overlaps

Step 2: Circle packing



After the previous step we have a 3-connected 3-regular graph

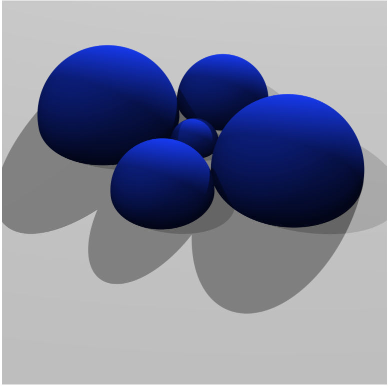
Koebe–Andreev–Thurston circle packing theorem guarantees the existence of a circle for each face, so circles of adjacent faces are tangent, other circles are disjoint

Can be constructed by efficient numerical algorithms

[Collins and Stephenson 2003]

Step 3a: Hyperbolic Voronoi diagram

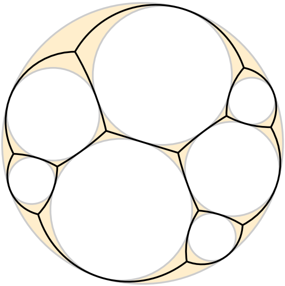
Embed the plane in 3d, with a hemisphere above each face circle



Use the space above the plane as a model of *hyperbolic geometry*, and partition it into subsets nearer to one hemisphere than another

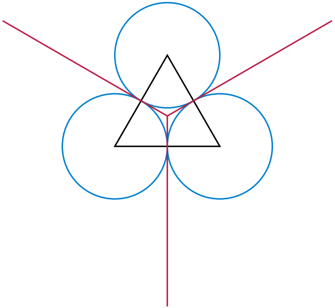
Step 3b: Möbius-invariant power diagram

Restrict the 3d Voronoi diagram to the plane containing the circles (the plane at infinity of the hyperbolic space).



Symmetries of hyperbolic space restrict to Möbius transformations of the plane ⇒ diagram is invariant under Möbius transformations

Step 4: By symmetry, these are soap bubbles



Each three mutually tangent circles can be transformed to have equal radii, centered at the vertices of an equilateral triangle.

By symmetry, the power diagram boundaries are straight rays (limiting case of circular arcs with infinite radius), meeting at 120° angles (Plateau's laws)

Setting all pressures equal fulfils the Young–Laplace equation on pressure and curvature

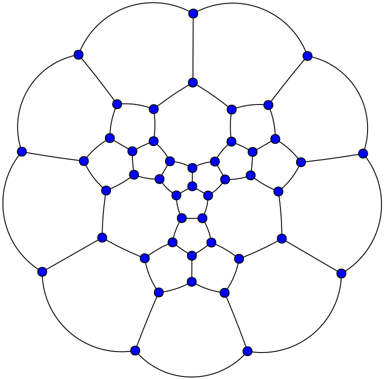
Bubble conclusions

Bubble graphs = planar 2-connected 3-regular graphs

Can be recognized and constructed in polynomial time

Also useful in network visualization (*Lombardi drawing*)

Depicted: a 46-vertex non-Hamiltonian graph from Grinberg [1968]



III. Crumples and Folds

Patterns in crumpled paper

Image: Pruitt [2011]

Studied experimentally [Andresen et al. 2007] (e.g. ridge lengths appear to obey power laws) but not well-understood theoretically

Similar patterns at nanoscale

Crumpled graphene has applications including power storage [Stoller et al. 2008] and artificial muscles [Zang et al. 2013]

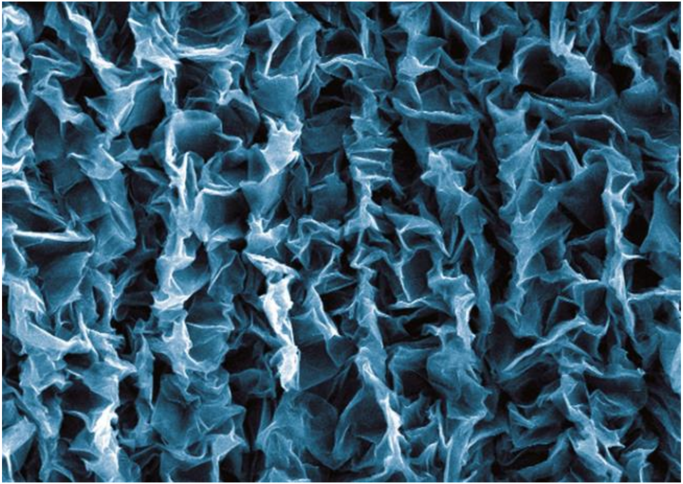
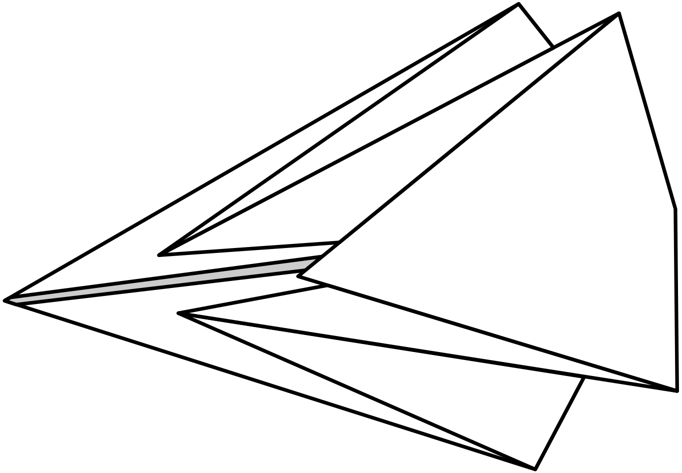


Image: Duke University [2013]

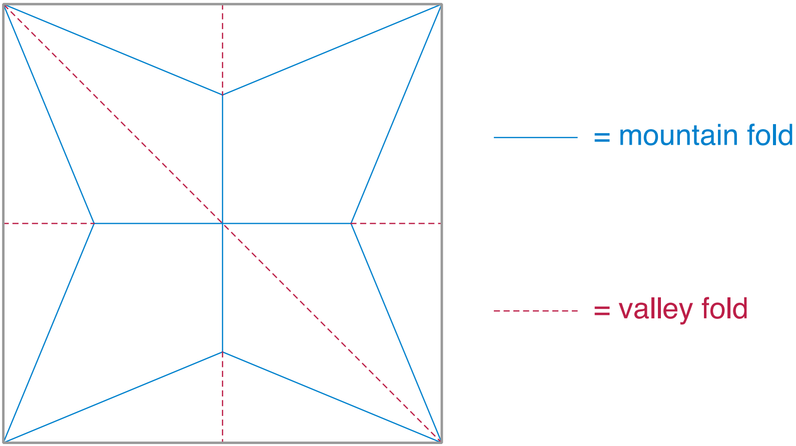
A discrete model of paper folding

Fold a piece of paper arbitrarily so that it lies flat again
(without crumpling)



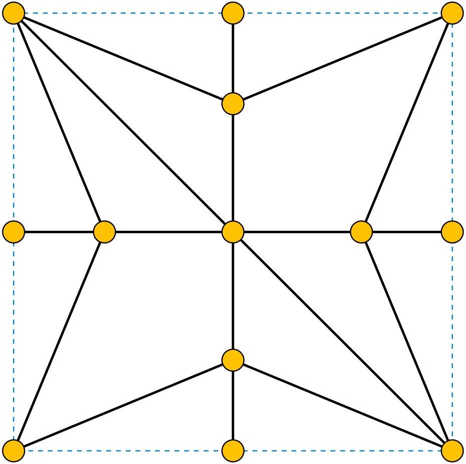
A discrete model of paper folding

Unfold it again and look at the creases from its folded state



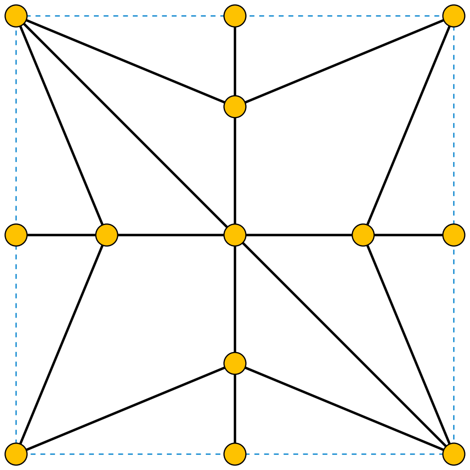
A discrete model of paper folding

It looks like a graph!



A discrete model of paper folding

It looks like a graph!

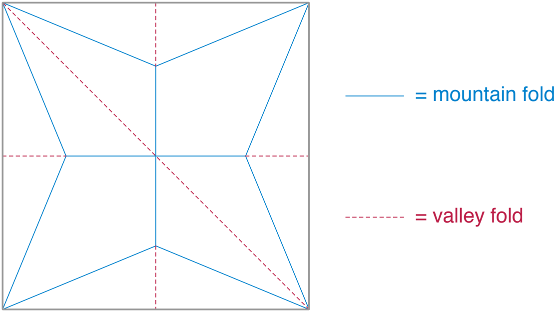


So, what graphs can you get in this way?

Local constraints at each vertex

Maekawa's theorem: at interior vertices,

$$|\#\text{ mountain folds} - \#\text{ valley folds}| = 2$$



So all vertex degrees must be even and ≥ 4

[Murata 1966; Justin 1986]

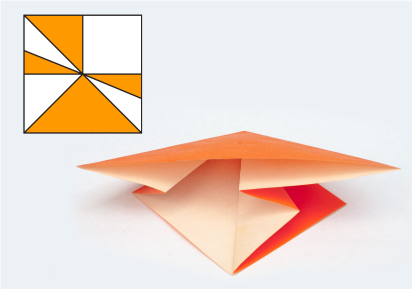
More local constraints at each vertex

Kawasaki's theorem: at interior vertices,

total angle facing up = total angle facing down

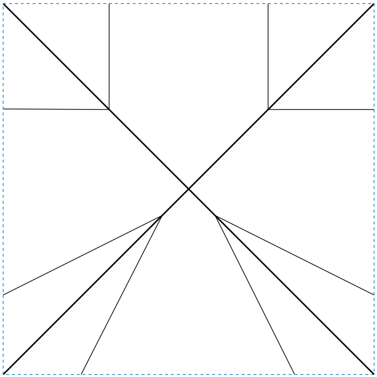
(alternating sum of angles must be zero)

[Robertson 1977; Justin 1986; Kawasaki 1989]



Unclear what effect this has on combinatorial structure

Local constraints are not enough



Even 4-regular trees meeting the angle conditions might not be foldable [Hull 1994]

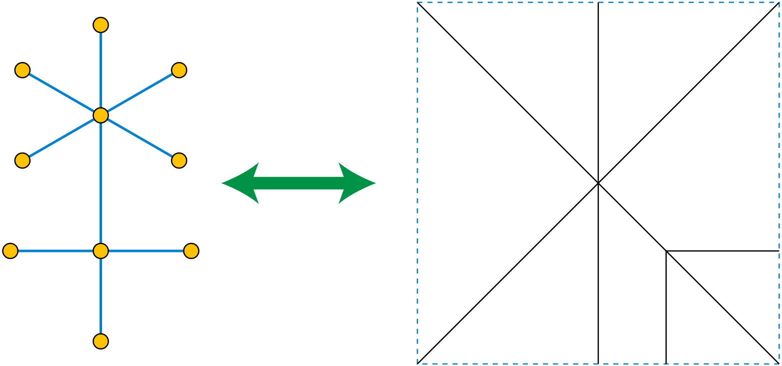
Central diagonal cross forces two opposite creases to nest tightly inside each other

Additional folds on the outer nested crease bump into the inner nested crease

...but all even-degree trees are realizable

Tree T is realizable with internal vertices interior to paper and leaves on boundary $\iff$ all internal degrees are even and ≥ 4

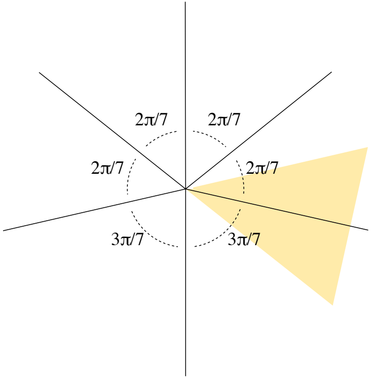
[Eppstein 2018]

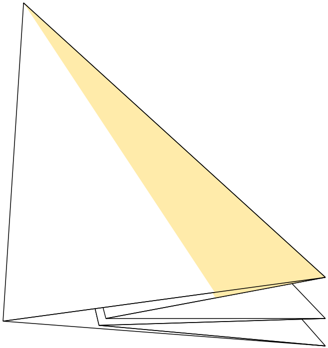


Main idea of tree realization

Construct tree top-down from root

Maintain buffer zones to prevent creases from nearing each other





Alternative graph model for infinite paper

Instead of interpreting infinite rays as leaves,
add a special vertex at infinity as their shared endpoint

Image: Hossain [2015]

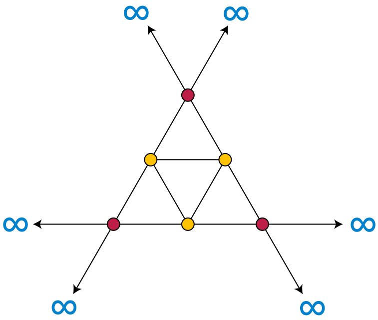
...so trees become series-parallel multigraphs

Some combinatorial constraints

The graphs of flat folding patterns with a vertex at infinity are:

- 2-vertex-connected
- 4-edge-connected
- not separable by removal of any 3 finite vertices

Proof ideas:
convexity of subdivision
rigidity of triangles

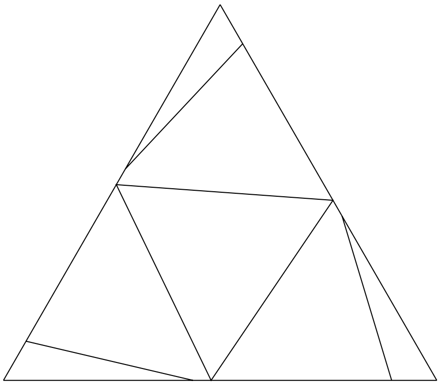


An unrealizable graph

[Eppstein 2018]

Return to finite paper sizes

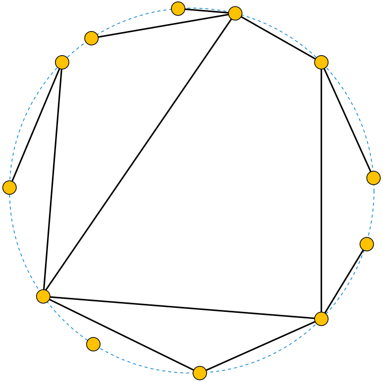
A different simplifying assumption:
All vertices are on the boundary of the paper



This triangle cannot be folded flat
(the three corners get in each others' way)

Characterization of boundary-vertex graphs

On circular or square paper, every folding pattern without interior vertices can be flat folded [Eppstein 2018]



Corollary: All outerplanar graphs are realizable on circular paper

Summary

Contact graphs of segments:
Well characterized; fast recognition and reconstruction

Inspiration for mesh partitioning, roof design

Combinatorial model missing some features of Gilbert tessellations

Planar soap bubble foams
Well characterized; fast recognition and reconstruction

Application to network visualization

What about 3d?

Flat-folded surfaces:
Partial characterization

Connections to mechanical design, nanostructures

References and image credits, I

Henry Adlard. Portrait of Thomas Young. Public-domain image, 1830. URL https://commons.wikimedia.org/wiki/File:Thomas_Young_(scientist).jpg.

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