Non-crossing Hamiltonian Paths and Cycles in Output-Polynomial Time

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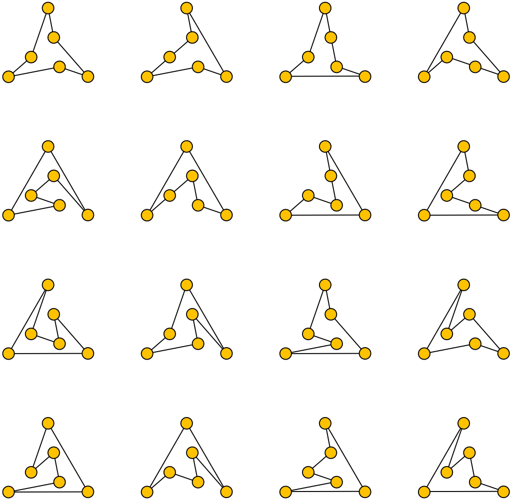
Polygonalizations

Given n points

Find a simple polygon with exactly those vertices

(allow 180° angles for collinear points)



Known

They always exist [Steinhaus 1964]

. . . even for non-general position but non-collinear

Easy construction:

Sort radially around any point in convex hull

[Deneen and Shute 1988]

There can be singly-exponentially many

and they can be listed in single-exponential time

[Sharir et al. 2013; García et al. 2000; Yamanaka et al. 2021]

Some optimization criteria are NP-hard (traveling salesman!)

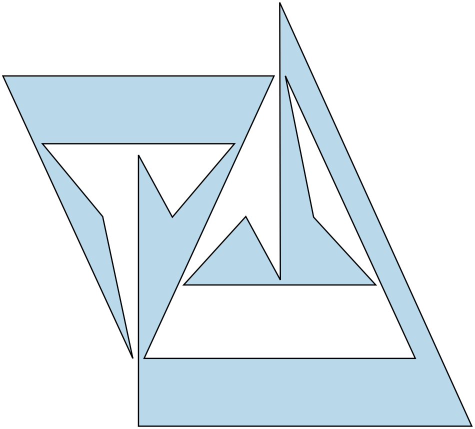
for more see: [Fekete 2000; Fekete and Keldenich 2018]

Unknown

Can we list them all in polynomial time/polygon?

We can for many other non-crossing structures by searching a *state space* connected by *local moves*

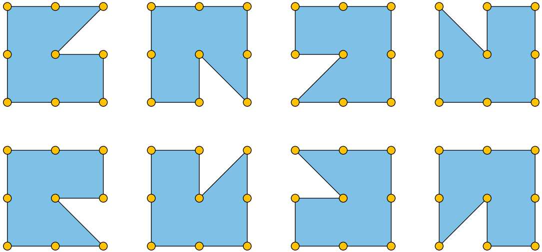
For polygonizations, natural moves do not work



Unflippable polygon
[Hernando et al. 2002]

Main result

We can list all polygonalizations of given points in time polynomial in the number of polygonalizations



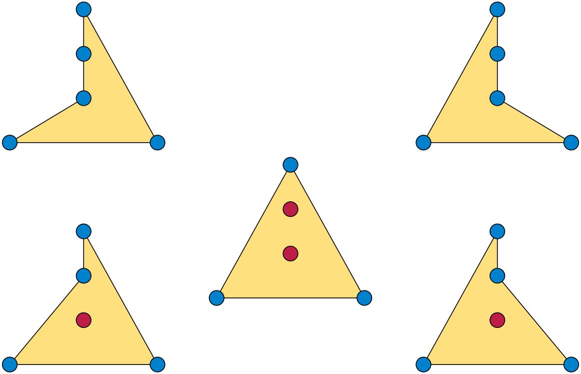
Singly-exponential in worst case (matching known algorithms)

Can be much faster when # polygonalizations is smaller

Also works (a little easier) for non-crossing Hamiltonian paths

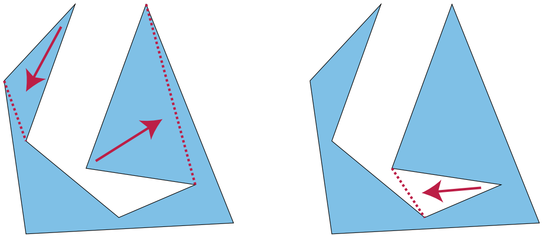
Surrounding polygons

Vertices = subset of points, enclosing the rest



Listing all surrounding polygons

Two ears theorem: A polygon that is not a triangle has ≥ 2 ears, triangles that can be cut off to form a simpler polygon [Meisters 1975; Guggenheimer 1977]



$\Rightarrow$ Every surrounding polygon, other than the convex hull, has a triangle that can be popped out to form a simpler polygon

$\Rightarrow$ Tree of surrounding polygons rooted at the convex hull

$\Rightarrow$ Explore this tree [Yamanaka et al. 2021]

Listing all polygonalizations

For each surrounding polygon:

If it is a polygonalization:

Output it

Already used to list polygonalizations in singly exponential time and polynomial space [Yamanaka et al. 2021]

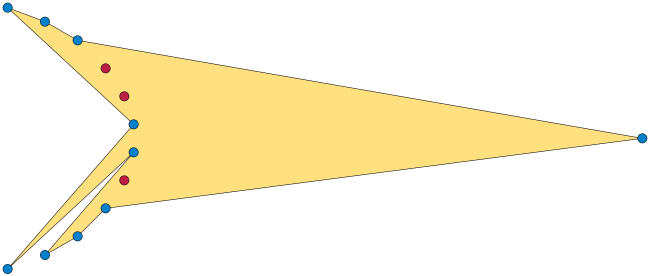
We prove this is output-polynomial!

[Mollerus 2007]

Equivalently: The tree of surrounding polygons cannot have many branches but few polygonalizations at its leaves

Example where numbers differ

Concave chain of $n-1$ vertices inside a triangle



$$\#\text{ polygonalizations} = (n-1)2^{n-4}$$

$$\#\text{ surrounding polygons} = \sum_{a+b+c=n-3} (a+1)\binom{a+b}{a}\binom{b+c}{b}$$

$$\approx (1+\text{golden ratio})^n \approx 2.618^n \approx (\#\text{ polygonalizations})^{1.388}$$

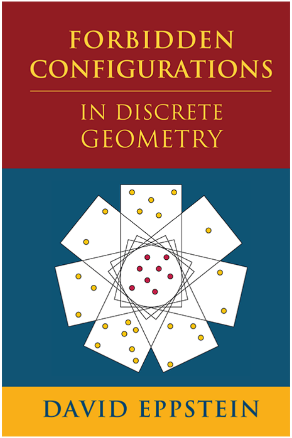
The main idea

Analyze point sets with few polygonalizations

Controlled by two hereditary parameters of order types of point sets

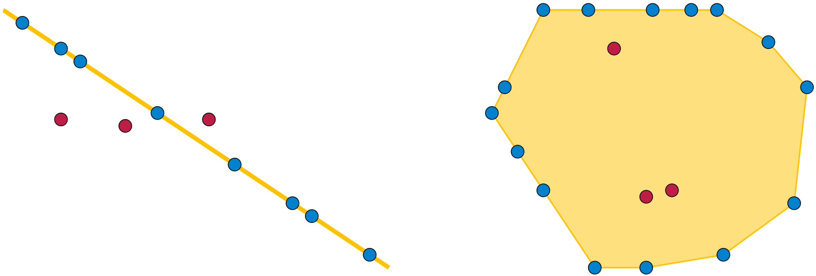
Approximate log # polygonalizations and log # surrounding polygons by a formula involving these two parameters

Both counts have the same approximation formula $\Rightarrow$ polynomial relation between them



Point sets with few polygons

The number of polygons is small when, for small k, either:



All but k points lie on a single line, or
all but k points lie on the convex hull

Determined by the k points, their neighbors, and their connections

$$\#\text{polygons} \leq \binom{n-k}{\leq 2k} 2^{O(k)} \quad \log \#\text{polygons} = O\left(k\left(\log \frac{n}{k} + 1\right)\right).$$

Hard part: Lower bound on #polygonalizations

Three separate lower bounds:

- All but k points on a line, but not on hull: $\geq \binom{n/2}{k/2}$
- All but k points on hull: $\geq \binom{(n+k)/4 - O(1)}{k/2}$
- Both hull and max line have $\leq n/7$ points: singly exponential

Combine to give $\log \#\text{polygonalizations} = \Omega\left(k\left(\log \frac{n}{k} + 1\right)\right)$

Many points on a line, not on hull

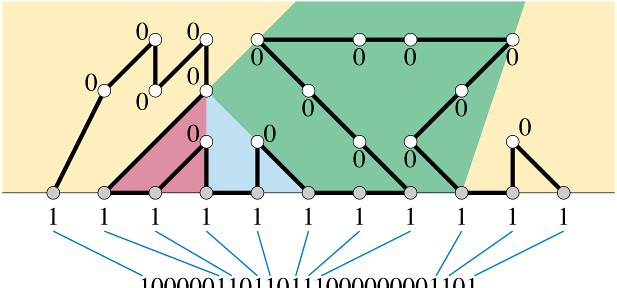
Find $\geq k/2$ points on one side of the line

Find many 010-avoiding binary sequences with $n - k$ one-bits, starting and ending with 1

1: point on line; 0: point not on line

After each block of 0's, rotate a ray from the next 1 to separate a block of that many off-line points from the rest

Each sequence corresponds to at least one polygonalization



Conclusions

We can:

List all polygonalizations in output-polynomial time

Approximate log #polygonalizations with constant approximation ratio in polynomial time

Solve traveling salesperson or any other optimization or counting problem on polygonalizations in XP time, $n^{O(k)}$

Work in progress but I think we can:

Count polygonalizations, solve TSP, and find min/max area polygonalization in time fixed-parameter tractable in same k

References and image credits, I

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