Cubic Planar Graphs That Cannot Be Drawn On Few Lines

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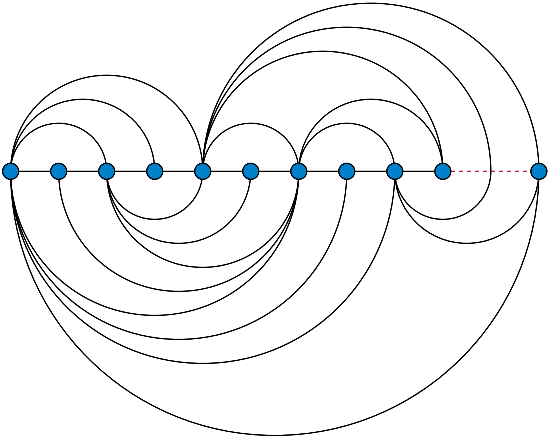
Symposium on Computational Geometry, June 2019

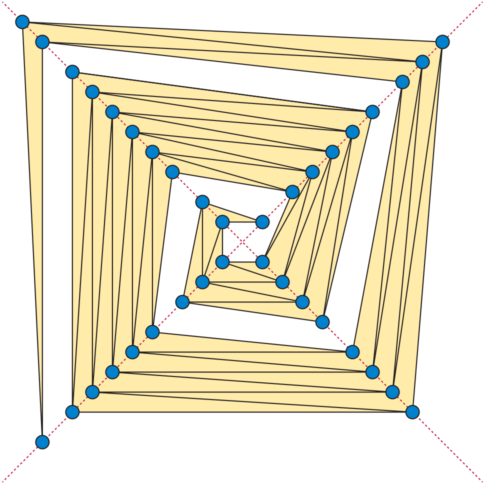
Planar graph drawing on one line

Always possible using ≤ 2 semicircles per edge



Planar graph drawing on two lines

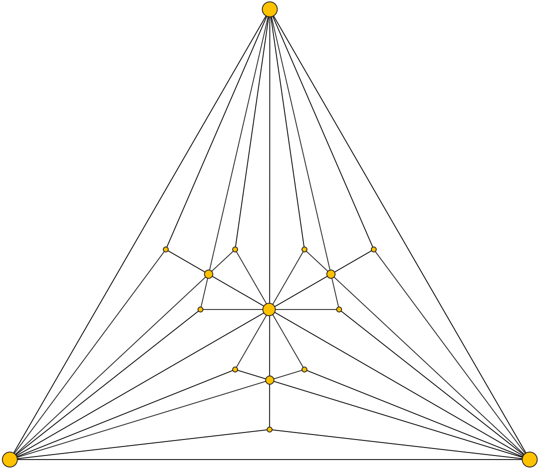
Can draw graphs of arbitrarily high treewidth
(here, a 6×6 grid)



Can also draw all trees and all outerplanar graphs

Known graphs requiring non-constant lines, I

Recursively subdivide triangles into 3 triangles

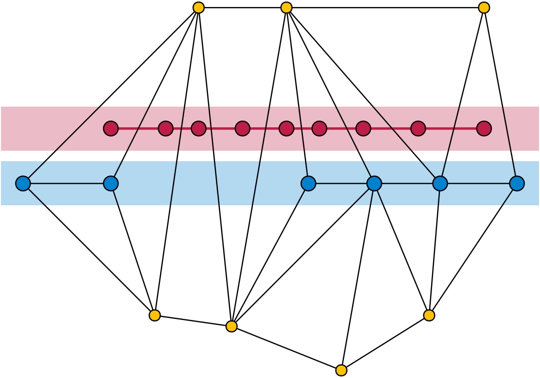


Line through subdivision point misses a smaller triangle

⇒ needs $\Omega(\log n)$ lines even though treewidth ≤ 3 [Eppstein 2018]

From many-point lines to long dual paths

In a planar graph whose faces are all triangles

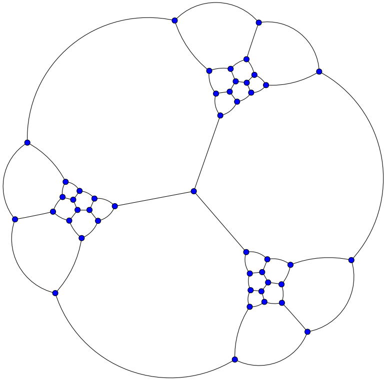


Line through k points $\Rightarrow$ path of $\geq k-1$ edge-to-edge triangles

(Doesn't work for non-triangles — they can cross line > 1 time)

Known graphs requiring non-constant lines, II

Find non-Hamiltonian 3-regular planar graph [Tutte 1946]



[Eppstein 2014]

Recursively replace vertices by copies of same graph

$\Rightarrow$ longest path $O(n^{0.99})$ [Grünbaum and Walther 1973]

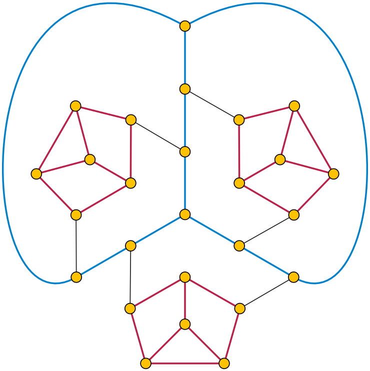
Dualize $\Rightarrow$ triangulated graph requiring $\Omega(n^{0.01})$ lines

[Ravsky and Verbitsky 2011; Chaplick et al. 2016, 2017]

An incorrect conjecture

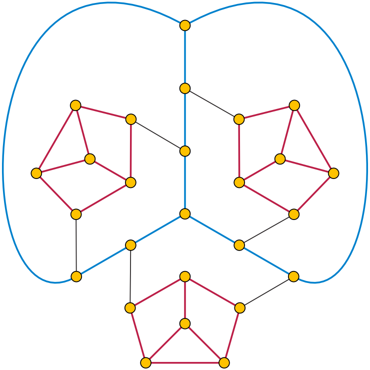
Conjecture [Firman et al. 2018]:
3-regular planar graphs can be drawn on 2 lines

Counterexample (D.E.):



(later verified computationally by Firman et al.)

Why is this a counterexample?

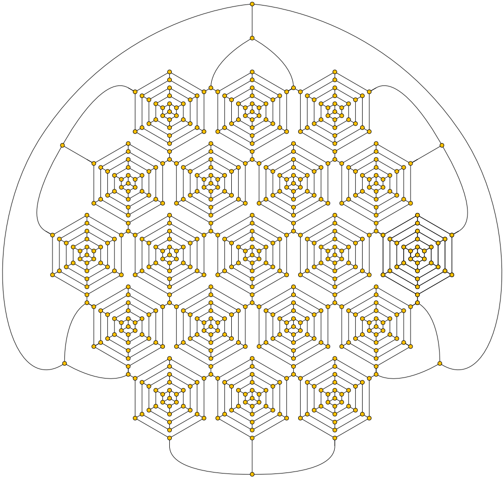


One of the three red pentagons avoids both
the crossing point of the two lines
and the outer face of the whole drawing

But it cannot be drawn only only two non-crossing line segments!

New results, I

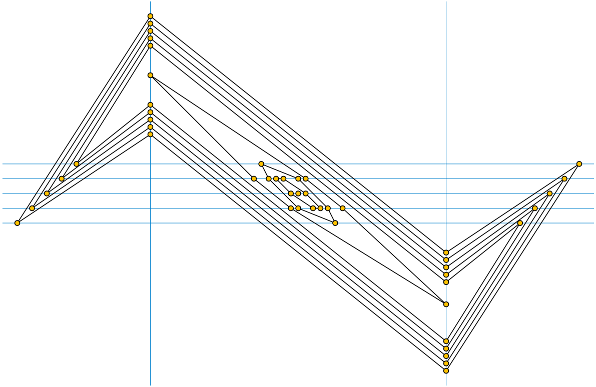
This graph requires $\Omega(n^{1/3})$ lines



(Rearrange hexagons $\Rightarrow$ bounded pathwidth)

Why?

For fewer lines, at least one hexagonal nest avoids all line crossings and the outer face

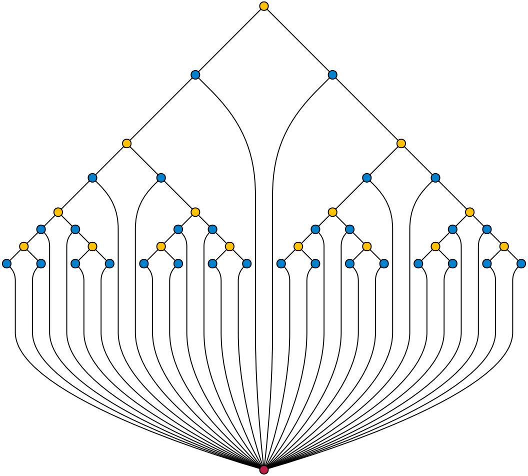


But each level of nesting uses up one of the segments formed by the lines that cross its outer hexagon

So it cannot be drawn on few lines without containing a crossing

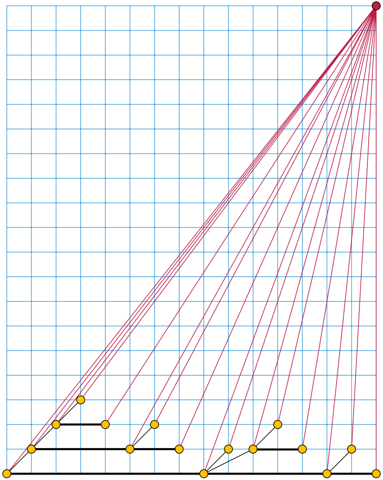
New results, II

This apex graph (tree + one vertex) requires $\Omega(\log n)$ lines



(by similar reasoning)

Drawing apex-trees on $O(\log n)$ lines



x-coordinate = preorder
y-coordinate = level in heavy path decomposition

Conclusions and new developments

From this paper:

- Planar graphs with n vertices may require $\Omega(n^{1/3})$ lines (even when 3-regular and bounded pathwidth)
- Series-parallel graphs and apex trees may require $\Omega(\log n)$ lines

The apex-tree bound is tight, but what about series-parallel?

More recently [Biedl et al. 2019]:

- NP-hard to find 2-line drawings
- 4-connected planar graphs can be drawn on $O(n^{1/2})$ lines

Can we close the gap between $\Omega(n^{1/3})$ and $O(n^{1/2})$?

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