NC Algorithms for Computing a Perfect Matching, the Number of Perfect Matchings, and a Maximum Flow in One-Crossing-Minor-Free Graphs

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Decision vs search

Easy: Is there a zebra in this picture?

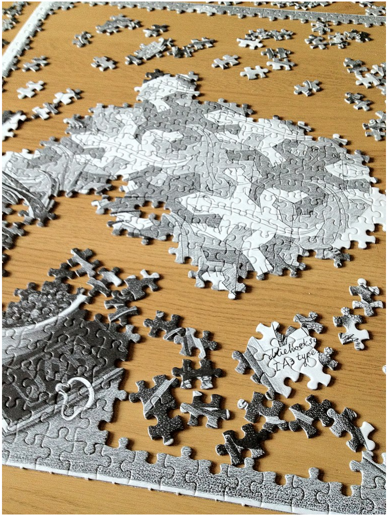
Harder: Find one zebra in this picture

[Hillewaert 2007]

Sequential search

Can build up a solution one piece at a time,
using decision algorithm to avoid mistakes

[Lilley 2012]



Randomized parallel search

Isolation lemma [Valiant and Vazirani 1986; Mulmuley et al. 1987]:
Random weights ⇒ unique solution

Synchronizes parallel solvers into all looking for the same solution

[Pereira 2017]

But is randomness necessary?

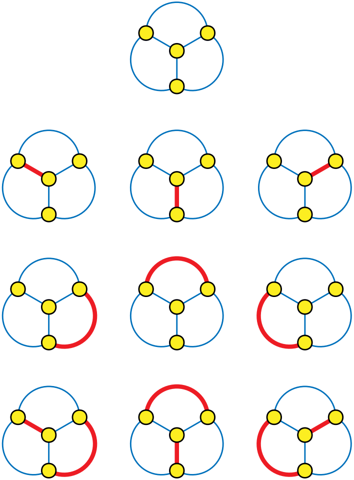
[Gaz∼enwiki and Wolfdog406 2004]

Parallel perfect matching in graphs

Important both in applications and as a test case

Known to be in RNC since Karp et al. [1986]

Still unknown whether in NC

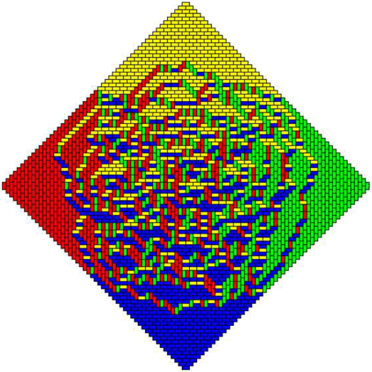


Stronger assistance for search: Counting

Counting perfect matchings is #P-complete [Valiant 1979]

But polynomial for planar graphs by transformation to a determinant [Kasteleyn 1967]

Used in NC algorithms for finding planar matchings [Anari and Vazirani 2018; Sankowski 2018]



[MiaFr 2012]

The limitations of counting

The determinant method works for graphs with no $K_{3,3}$ minor

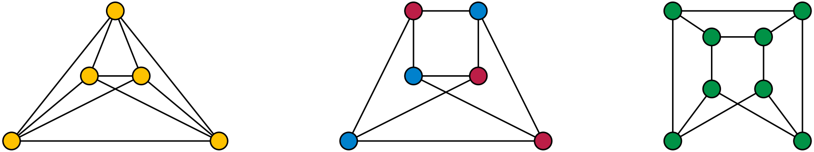
But it fails for $K_{3,3}$ and for any minor-closed family containing $K_{3,3}$

Vazirani [1989]: We can count perfect matchings in $K_{3,3}$-minor-free graphs in NC. But can we find one?

New results

We can find perfect matchings in $K_{3,3}$-minor-free graphs in NC

... or in any H-minor-free graph where H can be drawn in the plane with only one crossing



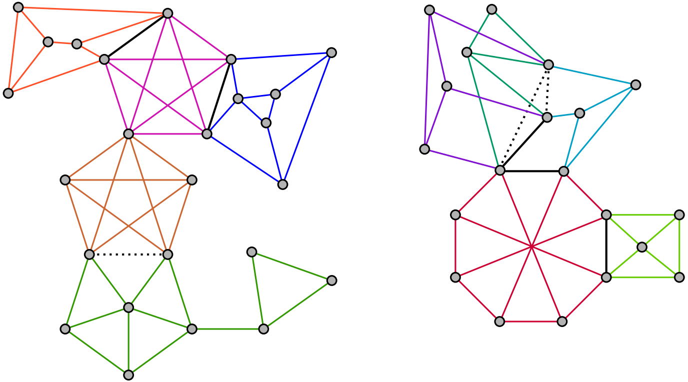
So the $K_{3,3}$ counting barrier is not actually a barrier

Same methods also provide NC counting algorithms for these graphs

The structure of one-crossing-minor-free graphs

These graphs all have a tree structure:

Planar graphs and graphs of bounded size (depending on the forbidden minor) glued together on cliques of size ≤ 3



[Wagner 1937; Robertson and Seymour 1993]

Parallel decision or function algorithms on trees

Typically:

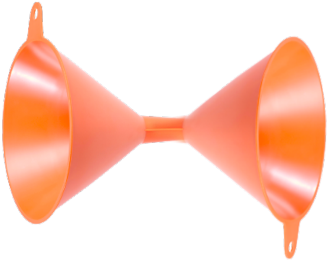
- *Rake* leaves and
- *compress* degree-two vertices
- preserving problem solution
- repeating until one vertex

Each repetition reduces size by a constant factor

[Sobolewski 2016]

Double-funnel search algorithm strategy

Given a tree-structured problem...



Rake and compress as before preserving existence of a solution

Find a solution on the constant-sized remaining problem

Then unrake and uncompress,
expanding solution back to original input

Replacing pieces of graphs by smaller pieces

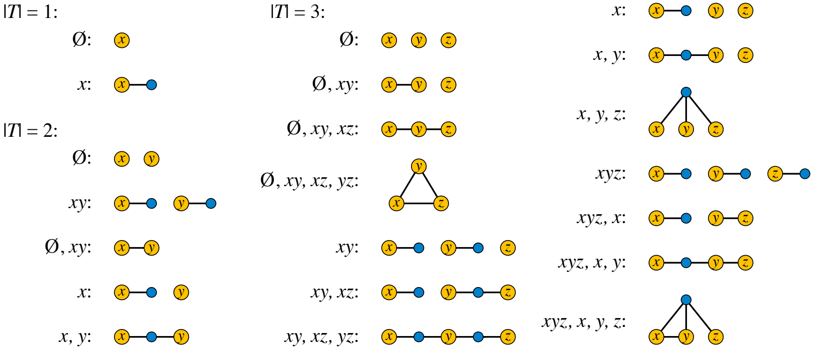
When we combine subgraphs in the decomposition tree:

Terminals: vertices connected to rest of the graph

Mimicking network: Same subsets of terminals are covered by matchings that cover all non-terminals

Double funnel: Replace and later un-replace by mimicking networks

Case analysis of three-terminal mimicking networks



Key property: gluing the replacement into face triangle of a planar graph preserves planarity

(allows us to use NC planar matching algorithms to construct and later un-replace mimicking networks)

Conclusions and open problems

Solved 30-year-old open problem: NC matching in $K_{3,3}$-free graphs

Extends more generally to one-crossing-minor-free graph families

Same method works for other problems including flow

Open: Extend to the more complicated tree structure of arbitrary minor-closed graph families

Open: Perfect matching in NC for arbitrary graphs

Open: How big do matching-mimicking networks need to be?

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