Games on Game Graphs

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The graphs of games

Games and graphs

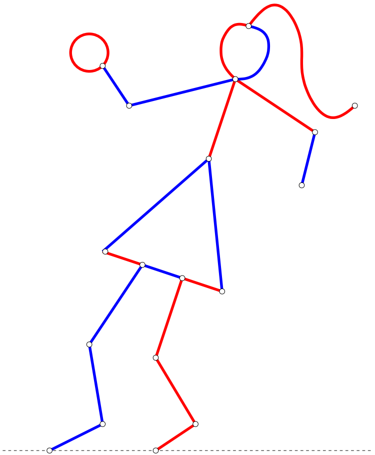
Many games are played on graphs!

Others have interesting graphs as their state spaces

These graphs make interesting test cases for other problems in graph theory

The "other problems" we will look at:

graph parameters based on pursuit–evasion games



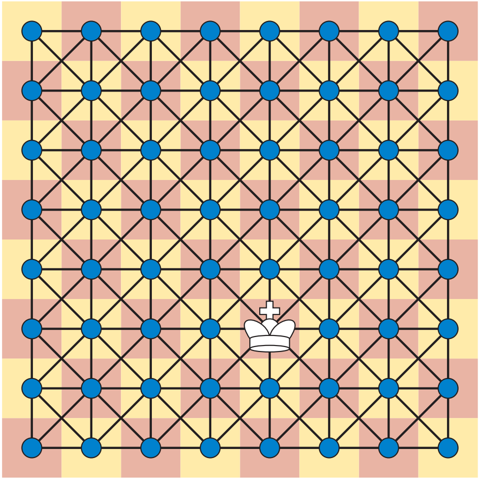
Chessboard graphs

Vertices: The squares of a chessboard (possibly larger than $n \times n$)

Edges: Moves of some chess piece

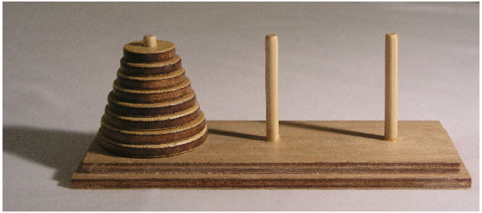
Knight's tour =
Hamiltonian cycle in knight's graph

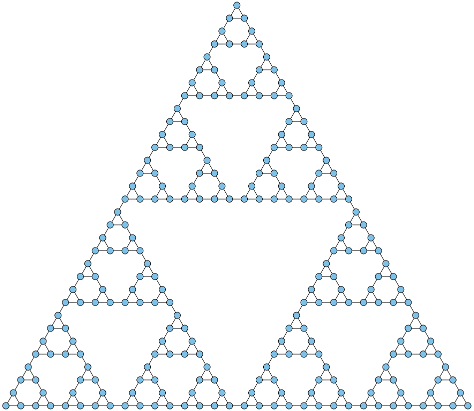
8 queens problem =
Independent set in queen's graph



Hanoi graphs

State space of "Tower of Hanoi" puzzle





Also produces interesting graphs for more than three towers

Not the same as higher-dimensional Sierpinski tetrahedra/simplices!

Twisty puzzles

State space is generally the Cayley graph of a group generated by puzzle moves

(Exception: 15-puzzle and relatives)

Small puzzles can produce enormous graphs

E.g. $3 \times 3 \times 3$ Rubik's cube $\Rightarrow$ 4.3×10^{19} vertices

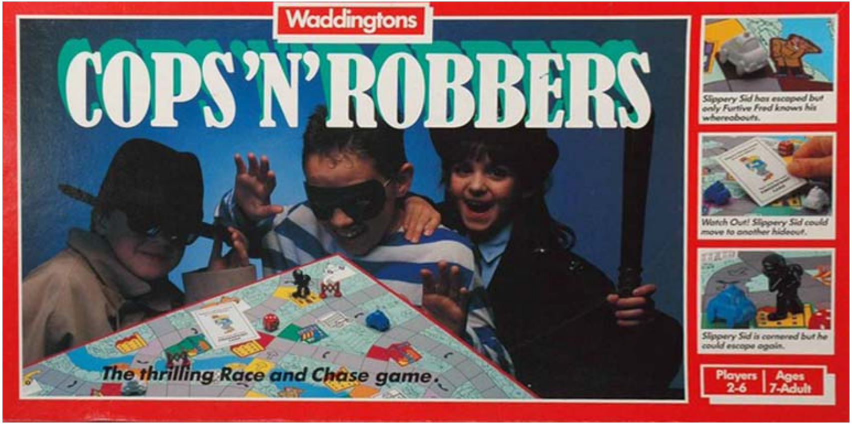
Cop-win graphs and cop-number

Pursuit–evasion games

One player controls one or more cops, who win by catching the robber

The other player controls the robber, who wins by remaining uncaught

Many variations depending on how each player can move

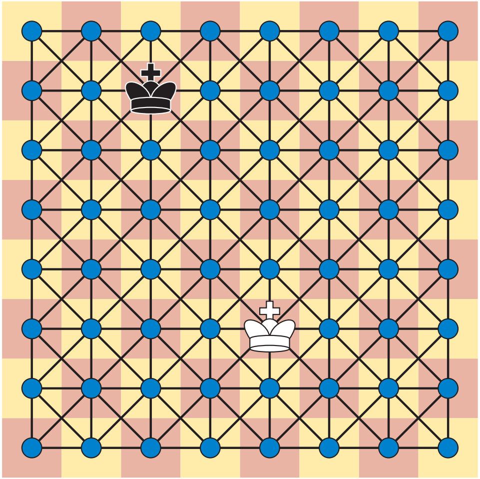


Cop-win graphs

Both players move one edge at a time (or pass), on the same graph [Quilliot 1978; Nowakowski and Winkler 1983]

E.g. when a black king (cop) chases a white king (robber) on a chessboard, the cop wins:

- First move to the same row as the white king
- Then, while remaining on same row as the white king, move to the same column



General winning strategies

Find a dominated vertex v
(closed neighborhood $N[v] \subset N[w]$ for some dominating vertex w)

Remove v and recurse, pretending other player is on w when really on v

Cop wins $\iff$ recursion bottoms out with a one-vertex graph (trivial win)

Robber wins $\iff$ removals get stuck, can move safely within remaining graph

None of the other chessboard graphs, Hanoi graph, Cayley graphs have dominated vertices, so the robber wins on all of them

Send in more cops!

Cop number of a graph: how many cops are needed to win

Two cops win on cycle graphs of length > 3

EXPTIME-hard and W[2]-hard for unbounded cop number

Easy $O(n^{k+2})$ algorithm for testing cop number k

Conjecture [Meyniel 1985]: Cop number is $O(\sqrt{n})$

Cop number on chessboard graphs

Easiest case: rook's graph

Two cops go to the row and column of the robber

The next move, one of them can catch the robber

The same strategy of blocking all straight-line robber paths, and then shadowing robber moves to stay in blocking position, should work for $O(1)$ cops on other chess graphs, but details are messy

Cop number on Cayley graphs

There is a large literature, mostly focused on Meyniel's conjecture
E.g. see [Frankl 1987]

Some Cayley graphs have cop number $\Theta(\sqrt{n})$ [Bradshaw et al. 2021]

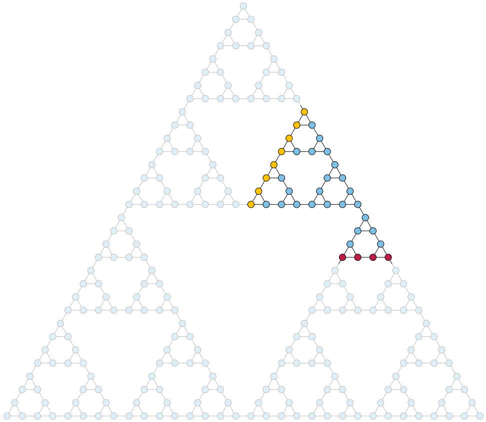
Unclear what is known for twisty puzzles

Cop number on Hanoi graphs

Planar $\Rightarrow$ three cops suffice [Aigner and Fromme 1984] but we can do better

Useful principle: A single cop can guard any shortest path, by moving along the path to stay within reach of the robber

General strategy: two cops guard paths isolating a two-triangle subgraph; whenever the robber moves into one triangle, the cop in the other triangle can shift to a closer path



I.e.: If I have two Towers of Hanoi and you have one, I can synch one of mine to yours

Treewidth

A maze of equivalent definitions

Bounded treewidth $\Longleftrightarrow$

Hierarchical clustering of edges by vertex separators of size $O(1)$

No large grid minor

Subgraph of chordal graph with no large cliques

Tree decomposition with no large bags

$O(1)$ cops can win "cops with helicopters" pursuit-evasion game

No "bramble", touching subgraphs with high hitting number

No "haven" assigning "large component" to small vertex deletions

No "tangle" assigning "large side" to small vertex separators

Width from pursuit–evasion

Cops with helicopters occupy vertices, seeking a robber moving on graph paths

Each turn:

- The cops announce which cops are moving and where they will go
- The cops that are moving take off
- The robber moves through the graph, avoiding vertices occupied by non-moving cops
- The cops move as announced, winning if they land on the robber

treewidth = # cops needed to catch unlimited-speed robber (minus one)

[Seymour and Thomas 1993]

Cop-number vs treewidth

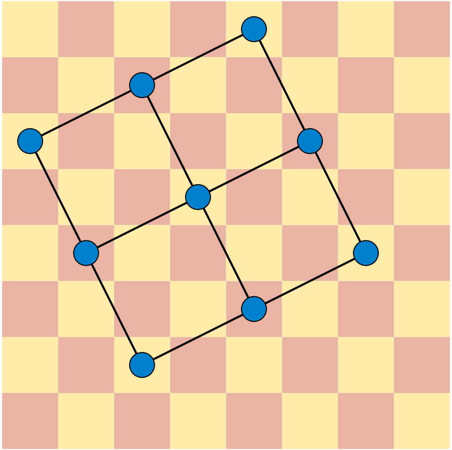
Cops on foot patrol are twice as effective as helicopters!

- Single turn of treewidth game $\Rightarrow$ many turns of cop-number game
- Each cop can guard path between two occupied locations
- One cop at a time walks to new position

So cop number $\leq 1 + \lfloor \text{treewidth}/2 \rfloor$ (but can be much less!) [Clarke 2002]

Chessboard graphs have high treewidth

All $n \times n$ chessboard graphs (not counting pawns) include large grids $\Rightarrow$ treewidth $\Omega(n)$



Tight to within constants for kings and knights, but what about other pieces?

Rook's graphs have quadratic treewidth

"Haven": function telling robber which component of unoccupied subgraph to be in

(must satisfy certain obvious consistency conditions)

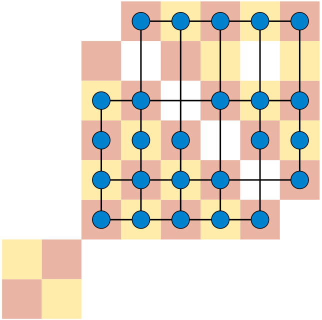
Observation: two "half-guarded" rows ($< n/2$ cops) share a column

So they all belong to the same component

Haven for $< n^2/2$ cops: component of half-guarded rows

$\Rightarrow$ treewidth $\geq \lceil n^2/2 - 1 \rceil$

Near-tight: $\lceil n(n+1)/2 \rceil$ cops can win



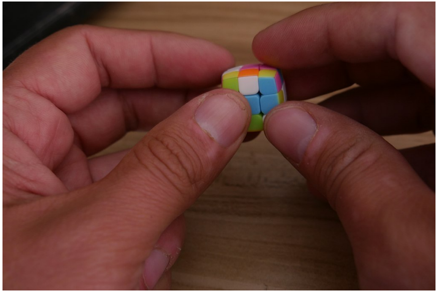
Same bound applies to queen's graph, (up to constant factor) bishop's graph

Twisty puzzles (typically) have high treewidth

For vertex-symmetric graphs, treewidth $= \Omega\left(\frac{\text{vertices}}{\text{diameter}}\right)$ [Babai and Szegedy 1992]

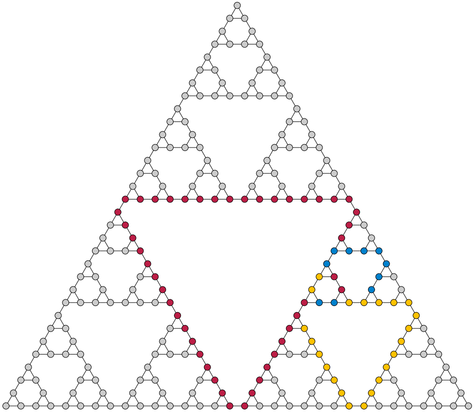
Cyclic groups (with one generator) have linear diameter (big), but
groups generated by n short cycles have diameter $O(n^2)$ (small) [Driscoll and Furst 1983]

Rubik generators $\neq$ cycles but $n \times n \times n$ diameter $= O(n^2/\log n)$ [Demaine et al. 2011]



$3 \times 3 \times 3$ Rubik has diameter 20 [Rokicki et al. 2014]

Hanoi graphs have treewidth four



Upper bound:

Recursive decomposition into triangles and trapezoids

Lower bound:

Subgraph that can be contracted into an octahedron

[Eppstein et al. 2020]

Hanoi graphs with more pegs

With n disks on $p > 3$ pegs, treewidth is $\dfrac{(p-2)^n}{n^{O(1)}}$ [Eppstein et al. 2020]

Upper bound: Recursive decomposition fixing the positions of the largest disks

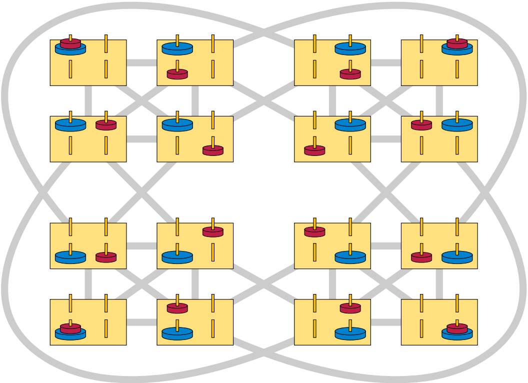
(# boundary states at top level, where biggest disk can move, is $p(p-1)(p-2)^{n-1}$)

Lower bound: Find a big low-diameter vertex-symmetric graph

Vertices = sets of states where

- $n/(p-2)$ disks are *frozen* to each of $p-3$ pegs
- remaining disks can move among remaining pegs

Edges = intersecting sets ($p-2$ frozen pegs, two empty)



Beyond treewidth

Slow robbers

Limit robber to paths of $\leq s$ steps per move

"Nowhere dense graphs" $\iff$ for all s, $O_s(1)$ cops win [Toruńczyk 2023]

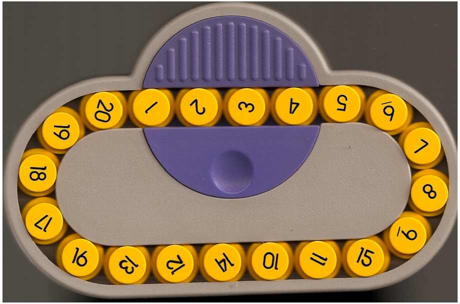
Uninteresting for near-regular graphs

- Bounded degree $\Rightarrow$ nowhere dense

  E.g. king's graph, knight's graph, Hanoi graphs, Top Spin, 15-puzzle

- Unbounded degree $\Rightarrow$ robber escapes in only one step ($s = 1$)

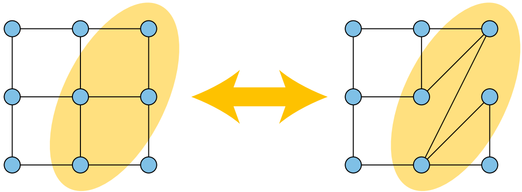
  E.g. other chess graphs Rubik's cube



More powerful cops

"Flip-width": another parameter defined via pursuit-evasion games [Toruńczyk 2023]

Instead of occupying one vertex, a cop can flip a subset of vertices, replacing edges by non-edges and vice versa in that subset



Each turn:

- Cops announce which subsets they will flip next
- Robber moves in the current flipped graph
- Cops undo their current flips and perform the announced flips

$O_s(1)$ cops catch speed-s robber at isolated vertex $\Rightarrow$ bounded flip-width

How powerful is flip-width?

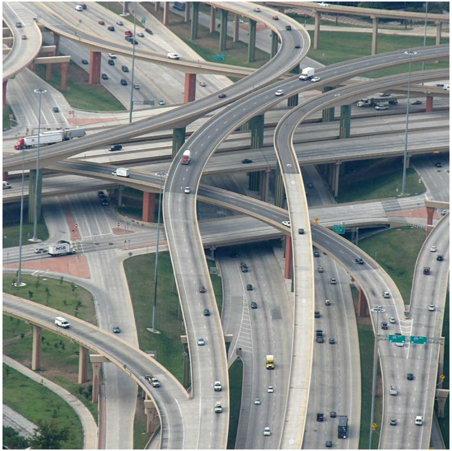
Encompasses most known graph width parameters including treewidth, twin-width, nowhere dense, etc.

Easy to find unnatural graph classes with bounded flip-width, other widths unbounded: union of bounded twin-width and nowhere-dense

Open: Is there a natural graph class with bounded flip-width, other widths unbounded?

Method for proving high width

Use a special subgraph called an interchange to find a winning strategy for the robber



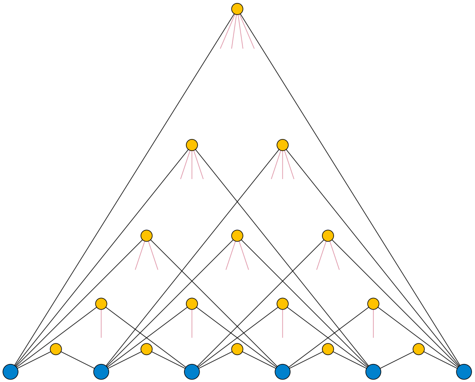
If some graphs in a graph family contain arbitrarily large interchanges then the family does not have bounded flip-width [Eppstein and McCarty 2023]

Definition of an interchange

Order the vertices of K_n and subdivide each edge into a two-edge path

Extra edges are allowed but not required:

- Between any two vertices of K_n
- Between any two subdivision points
- From subdivision point to vertices of K_n between the two it connects

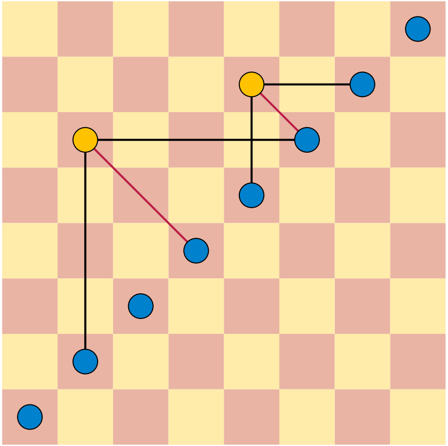


Two-step robber in a large-enough interchange (relative to # cops) escapes by moving to a vertex with two-edge paths to many other vertices

At least one of these paths will lead to another safe vertex in the next move

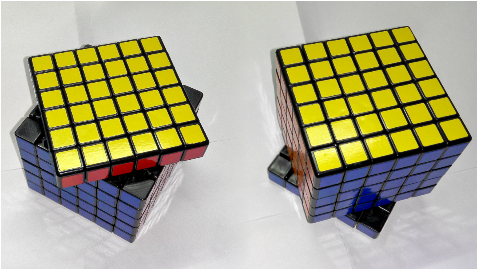
Interchanges in somewhere-dense game graphs

Rook's graph and queen's graph:



Rubik's cube:

- Use only xy-parallel twists
- Vertices of K_{n-1}: twist one plane
- Subdivision points: twist two planes



Therefore these graphs have unbounded flip-width

Summary

Three interesting parameters defined by pursuit-evasion games on game graphs: cop-number, treewidth, flip-width

Cop-number and treewidth are old and well-studied; flip-width is new and powerful

Game graphs provide interesting test cases for these parameters and their games

Some unanswered questions:

- Is cop-number bounded on Rubik's cube graphs?
- Can low diameter and high treewidth of Rubik's cube graphs be generalized to wider families of twisty puzzles?
- Do any somewhere-dense game graphs have bounded flip-width?

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