

Connectivity, Graph Minors, and Subgraph Multiplicity

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Abstract

It is well known that any planar graph contains at most $O(n)$ complete subgraphs. We extend this to an exact characterization: G occurs $O(n)$ times as a subgraph of any planar graph, if and only if G is three-connected. We generalize these results to similarly characterize certain other minor-closed families of graphs; in particular, G occurs $O(n)$ times as a subgraph of the $K_{b,c}$ -free graphs, $b \geq c$ and $c \leq 4$, iff G is c -connected. Our results use a simple Ramsey-theoretic lemma that may be of independent interest.

1 Introduction

It follows from the sparsity of planar graphs that each such graph contains at most $O(n)$ complete subgraphs K_3 and K_4 [6]. All cliques in a planar graph can be listed by an algorithm with $O(n)$ worst-case time complexity [3, 4, 6]. Enumeration of subgraphs has a number of uses, including a recent application in testing inscribability [5].

These results naturally raise the question of determining which other planar graphs occur $O(n)$ times as subgraphs of planar graphs. A necessary condition is that the subgraph G be 3-connected: otherwise, G can be split into two parts A and B by a separating pair of vertices, and by connecting many copies of A and B at the same two vertices, we find a family of planar graphs in which G occurs $\Omega(n^2)$ times as a subgraph.

Our main result is that this is also a sufficient condition. Every three-connected planar graph can only occur $O(n)$ times as a subgraph of other planar graphs.

We extend this to minor-closed families of graphs. As is well known, the planar graphs are exactly those for which neither $K_{3,3}$ nor K_5 is a minor. We first investigate graphs for which some complete bipartite graph $K_{b,c}$ (with $b \geq c$ and $c \leq 4$) is not a minor. We show that the subgraphs occurring $O(n)$ times in this family of graphs are exactly the c -connected graphs. We extend this to other minor-closed families of graphs where some complete bipartite graph is excluded. Our main result for planar graphs appears as a corollary of this generalization.

Our proofs use a simple Ramsey-theoretic lemma that may be of independent interest. In English, it states that if a set is formed as the union of a large number of k -tuples, then we can find a smaller set of tuples for which each member of the set always appears in the same tuple position. In the language of graph theory, if many copies of a subgraph occur in a larger graph, we can find a smaller number of copies of the subgraph, so that among those copies each vertex of the larger graph appears in only one way as a vertex of the subgraph.

Finally, we discuss algorithmic enumeration of subgraphs. In some cases, we are able to make our results constructive, so that the copies of a given subgraph can be enumerated in $O(n)$ time, matching the previous results for cliques in planar graphs.

2 A Ramsey-theoretic lemma

Given a set S , denote the ordered k -tuples of members of S by S^k . We do not allow any tuple to contain the same element in more than one position. Each tuple can be interpreted as a function from the integers $1 \dots k$ onto the set S , such that for any tuple t , and $1 \leq i < j \leq k$, $t(i) \neq t(j)$.

Suppose we have a collection T of k -tuples, $T \subset S^k$. We say T is *coherent* if each member of S appears in at most a single position of the k -tuples. In other words, for any distinct tuples t and t' , and for all integers $1 \leq i < i' \leq k$, $t(i) \neq t'(i')$.

As we now show, any set of tuples has a large coherent subset.

Lemma 1. *Suppose we are given a set T containing at least $k!^2 m^k$ k -tuples. Then there is a coherent subset $T' \subset T$ containing at least m tuples.*

Proof. We use induction on k . Any set of 1-tuples is coherent.

First assume some element $x \in S$ is contained in at least $k!^2 m^{k-1}/k$ of the k -tuples. Then there is a collection of $(k-1)!^2 m^{k-1}$ of those tuples for which x always appears in some particular position, say position k . By

ignoring this shared position, we construct a collection of $(k - 1)$ -tuples which by induction must have a coherent subcollection of size m . Restoring the shared positions gives our coherent subset T' .

Otherwise, each element is in fewer than $k!^2 m^{k-1}/k$ of the tuples. Then we can repeatedly pick a tuple t , add it to T' , and throw away the tuples that have an element in common with t . By the time we run out of tuples to pick, we will have added m tuples to T' . \square

3 Complete bipartite minors

We first show that not too many copies of a highly connected subgraph can share the same low-degree vertex, unless the graph contains a large complete bipartite graph as a minor. The bound on the number of times the subgraph can occur will then follow from the sparseness of minor-free graphs.

Lemma 2. *Let G be a c -connected graph with k vertices. Suppose that $k!^2 b^{k^2} d^{ck^2}$ copies of G are subgraphs of some larger graph H , and that some vertex x (with degree d in H) is contained in each of these copies of G . Then H contains a complete bipartite graph $K_{b,c}$ as a minor.*

Proof. By Lemma 1, we can find $b^k d^{ck}$ copies of G such that each vertex of H is used in at most one way as a vertex of G . We throw away the remaining copies of G .

We now have a one-to-many correspondence between the vertices of G and those of H . Suppose each vertex of G corresponds to fewer than bd^c different vertices of H . Each copy of G could then be found by choosing one such vertex of H for each of the k vertices of G , so there would be fewer than $(bd^c)^k$ possible copies of G in the coherent set, a contradiction. Therefore we can find some vertex y of G that corresponds to at least $\binom{d}{c}(b-1) + 1 < bd^c$ vertices of H .

Since G is c -connected, we can find c vertex-disjoint paths in G from y to x . Partition the remaining copies of G into equivalence classes according to the edges in H adjacent to x used by these paths. There are $\binom{d}{c}$ such classes, so some class uses at least b copies of vertex y . Unless $b = 1$, y will not be adjacent to x in G ; if $b = 1$ the lemma is trivially solved by the neighbors of any vertex. We select the copies of G in this class, and throw away the remaining copies.

We then form a minor of H by removing all edges not corresponding to portions of these vertex-disjoint paths, and contracting all remaining edges except those adjacent to x or to copies of y . Because the paths are

vertex-disjoint in G , and because we have selected a coherent collection of subgraphs, the images of the paths in H are also vertex-disjoint. Thus the contraction process described above finds a minor in which x is connected to c vertices, which are also each connected to $b - 1$ copies of y . This gives us $K_{b,c}$ as a minor of H . \square

Theorem 1. *Let $F_{b,c}$ be the family of graphs in which $K_{b,c}$ does not occur as a minor, and let $G \in F_{b,c}$ be c -connected. Then there is some constant $\kappa = \kappa(G)$ such that, for any n -vertex graph $H \in F_{b,c}$, there can be at most κn copies of G as a subgraph in H .*

Proof. Since $F_{b,c}$ is a minor-free family of graphs, the graphs in $F_{b,c}$ are sparse (any such graph has $O(n)$ edges). Thus we can find a vertex x in H which has degree at most some constant $d = O(1)$. By Lemma 2, there are at most $O(1)$ copies of G as a subgraph of H containing vertex x . Let κ denote this number of copies. Then by induction there are at most $\kappa(n - 1)$ copies of G in the graph H' formed by removing x from H . Thus there are at most κn copies in all. \square

4 Other forbidden minors

We have shown that c -connected graphs cannot occur more than linearly within the $K_{b,c}$ -free graphs. We now show that for small c this is an exact characterization: graphs with smaller connectivity will always be able to occur more than a linear number of times. We also generalize this characterization to some other minor-closed families of graphs.

Theorem 2. *Let S be a set of c -connected graphs containing some complete bipartite graph $K_{b,c}$, with $b \geq c$ and $c \leq 3$. Denote by F_S the family of graphs not containing any member of S as a minor. Then a graph $G \in F_S$ occurs $O(n)$ times as a subgraph of members of F_S if and only if G is c -connected.*

Proof. If G is c -connected, the result follows from Theorem 1 and the fact that F_S is a subset of $F_{b,c}$.

If G is in F_S but not c -connected, it can be separated by some $(c - 1)$ -tuple of vertices. Let A and B be two components of the separated graph. Then we can form a family of graphs in which G occurs $\Omega(n^2)$ times, by connecting many copies of A and B , together with a single copy of $G - A - B$, to the same $(c - 1)$ -tuple. We now show that no such graph G' contains a minor in S .

We assume for a contradiction that such a minor, M , exists. M is formed by contracting certain connected sets of vertices in G' , and then possibly removing some edges. First note that no two such sets can be entirely contained within components of G' on different sides of the separating $(c-1)$ -tuple, since the two corresponding vertices in $K_{b,c}$ would then be separated by a cut of at most $(c-1)$ vertices. So all vertices in the $K_{b,c}$ minor come from sets containing at least one vertex in some particular component C of G' . At most $(c-1) \leq 2$ vertices in M can come from sets involving points outside that component, because each such set must include a vertex in the separating $(c-1)$ -tuple. So the portion of G' outside C can only contribute a single edge in M , connecting those two vertices. But that same edge can be found in G by contracting one of components A and B (whichever is not equal to C), so G also contains M as a minor, contradicting the assumption that G is in F_S . \square

Corollary 1. *An outerplanar graph G occurs $O(n)$ times as a subgraph of all outerplanar graphs if and only if G is biconnected.*

Proof. Apply Theorem 2, with $S = \{K_{2,3}, K_4\}$. \square

Corollary 2. *A planar graph G occurs $O(n)$ times as a subgraph of all planar graphs if and only if G is 3-connected.*

Proof. Apply Theorem 2, with $S = \{K_{3,3}, K_5\}$. \square

We can extend the theorem to a more restricted class of $K_{b,4}$ -free graph families:

Theorem 3. *Let S be a set of triangle-free 4-connected graphs containing some complete bipartite graph $K_{b,4}$, with $b \geq 4$. Denote by F_S the family of graphs not containing any member of S as a minor. Then a graph $G \in F_S$ occurs $O(n)$ times as a subgraph of members of F_S if and only if G is 4-connected.*

Proof. Sufficiency follows from Theorem 1. For necessity, we follow the previous proof: we assume G has a separating 3-tuple, and repeat certain components around this tuple to construct a family of graphs G' in which G occurs $\Omega(n^2)$ times. Let M be a triangle-free minor of some such graph G' . As before, all vertices of M come from sets involving a single component C of G' , and at most three can cross the separating 3-tuple. Then since M is triangle-free, one of the three vertices is adjacent to all edges among the three. By collapsing all of G outside of C into that one vertex, we find M as a minor in G as well as in G' , so M cannot be in S . \square

Unfortunately the same proof technique runs into trouble for $K_{b,5}$ -free graphs: there are graphs containing no $K_{5,5}$ -minor, yet having a separating 4-tuple for which any repetition of components as above would form a $K_{5,5}$ -minor.

5 Algorithms

We now discuss algorithms for enumerating the occurrences of a given subgraph. Clearly, for any fixed subgraph G , there is a polynomial time algorithm for enumerating its occurrences: if there are k vertices in G , simply test all $\binom{n}{k} = O(n^k)$ possible choices for those vertices. We are interested in situations in which the occurrences can be listed more efficiently, in linear time (matching our bounds on the occurrences of G).

Such algorithms were already known for enumerating copies of the complete graphs K_3 and K_4 as subgraphs of planar graphs. More generally, in any family of graphs with bounded arboricity, all clique subgraphs can be listed in linear time [3, 4].

The family of $K_{1,b}$ -free graphs is also relatively easy. Since these graphs have maximum vertex degree $b - 1$, there are fewer than $b^k = O(1)$ vertices within distance k of any given vertex. The occurrences of any connected subgraph G can therefore be enumerated in time $O(n)$.

The first interesting cases are the families of $K_{2,b}$ -free graphs. Theorem 1 shows that in any such family, the subgraphs appearing $O(n)$ times are the biconnected graphs. The biconnected $K_{2,2}$ -free graphs are simply the triangles, which can be enumerated using the clique algorithms cited above. We next consider $K_{2,3}$ -free graphs. The biconnected components of these graphs are either outerplanar graphs, or copies of K_4 . Thus we can enumerate any biconnected subgraph of such a graph, if only we can do so for outerplanar graphs.

Theorem 4. *Let G be biconnected. Then there is a linear-time algorithm which enumerates the occurrences of G in any outerplanar graph.*

Proof. Any biconnected outerplanar graph is Hamiltonian, and adding extra edges to a graph only makes it easier to enumerate its occurrences, so we can assume without loss of generality that G is a simple cycle C_k . Let H be an outerplanar graph with n vertices, for which we are to enumerate the occurrences of G as a subgraph.

Any occurrence of G consists of the vertices in a union of facial cycles of H . To find an occurrence of G , we proceed as follows. We start by choosing

a single facial cycle in H . Then as long as the current subgraph is a cycle shorter than G , we choose an edge of the subgraph and replace it by the adjacent facial cycle. There are at most k edges to be replaced at each step, and each step increases the cycle length by at least one, so there are $O(k^k)$ ways to augment the initially chosen cycle into an occurrence of G . Thus the total time to enumerate the occurrences of G is $O(k^k n) = O(n)$. \square

Corollary 3. *Let G be biconnected. Then there is a linear-time algorithm which enumerates the occurrences of G in any $K_{2,3}$ -free graph.* \square

Corollary 4. *Let G be a wheel W_k . Then there is a linear-time algorithm which enumerates the occurrences of G in any $K_{1,2,3}$ -free graph, and in particular in any $K_{3,3}$ -free or planar graph.*

Proof. W_k consists of a single “hub” vertex connected to each vertex of a cycle C_k . Let H be a $K_{1,2,3}$ -free graph. We independently enumerate, for each vertex x of H , the occurrences of W_k for which x corresponds to the hub. This can be done by finding all occurrences of C_k in the neighbors of x . But the induced graph of these neighbors must be $K_{2,3}$ -free, as any $K_{2,3}$ minor together with the hub vertex would form a $K_{1,2,3}$ in the original graph. So by Corollary 3 we can find all such occurrences in time linear in the number of neighbors. The time to find all occurrences of W_k is proportional to the sum of the degrees of each vertex of H , which is $O(n)$. \square

This result generalizes the enumeration of cliques in planar graphs, since the only such cliques that can occur are K_3 and K_4 , which can alternately be interpreted as wheels W_2 and W_3 .

We would like to be able to extend our results further, at least to the enumeration of all 3-connected planar subgraphs. We have been unable to achieve such a result. However Corollary 4 is promising, as wheels figure prominently in Tutte’s characterization of triconnected graphs [7].

6 Conclusions

We have described a number of circumstances in which some subgraph occurs $O(n)$ times in the graphs of a minor-closed family. We have also described algorithms that, in a more limited class of situations, can efficiently enumerate the occurrences of a given subgraph.

A number of directions for possible generalization suggest themselves. First, we would like to extend our algorithms so that we can enumerate

the occurrences of a subgraph G whenever G is guaranteed to have $O(n)$ occurrences. This would entail a new, constructive proof of Theorem 1.

Second, we would like to characterize the subgraphs occurring few times in minor-closed families other than those described in Theorem 2. In particular, the forbidden minors for graphs of bounded genus include graphs that are not 3-connected, so our characterization does not apply to these families. We would also like to know what happens when some $K_{b,c}$ is excluded for $b, c \geq 5$.

Third, perhaps we should investigate families of graphs that are not closed under minors. Any family of sparse graphs (more precisely, graphs with bounded arboricity) will contain only $O(n)$ cliques. But perhaps some other subgraphs occur few times. For instance, what are the subgraphs occurring few times in the k -page graphs [2]?

Finally, it might be interesting to characterize graphs occurring a number of times which is a non-linear function of n . For the families for which Theorem 2 applies, we can demonstrate a gap in the possible such functions: any subgraph G occurs $\Omega(n)$ times, and the theorem tells us that G either occurs $O(n)$ or $\Omega(n^2)$ times. For other families, there may be graphs that occur only $O(1)$ times (indeed, if F excludes a minor consisting of k disjoint copies of G , then G occurs at most $k - 1$ times in graphs of family F). If G is a tree with k leaves (other than a single edge), then it occurs $\Theta(n^k)$ times as a subgraph in other trees. Perhaps for planar graphs, a similar characterization can be described in terms of the tree of triconnected components. For sparse graphs (graphs with $O(n)$ edges) there can be at most $O(n^{3/2})$ triangles [1], and more generally $O(n^{c/2})$ complete subgraphs K_c . Are there minor-closed families, and subgraphs in those families, such that the function describing the number of occurrences is similarly nonpolynomial?

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