Computational Complexities of Folding

David Eppstein

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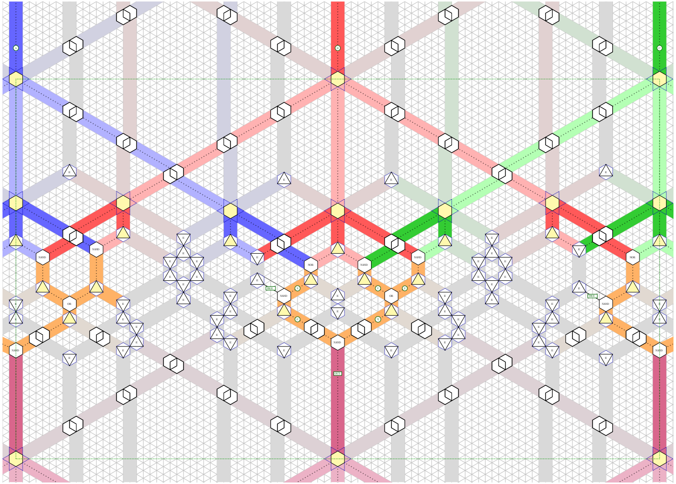
The big picture

Tour of some recent developments in the computational complexity of paperfolding

- Parameterized complexity
- Fine-grained complexity
- Galois complexity
- Space complexity
- Counting complexity
- Undecidability

The main idea

We can design origami structures that perform computations



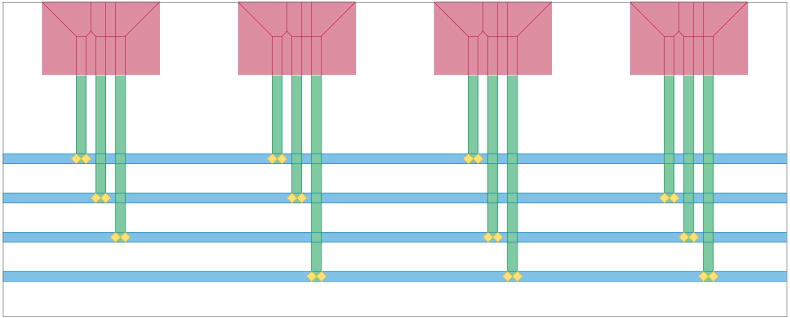
Folding these structures cannot be easier than doing the computation!

... because if it were, that would be a way of doing the computation

Can it fold?

Flat foldability is NP-complete

Outline of idea by [Bern and Hayes 1996]: convert Boolean circuit into folding pattern



Blue and green lines: pleats can fold in two ways, act like wires + binary signals

Red shaded regions: logical "gates" that test whether three incoming signals are unequal

Original construction is incorrect but was patched by Akitaya et al. [2016]

Two variations for labeled and unlabeled crease types, both hard

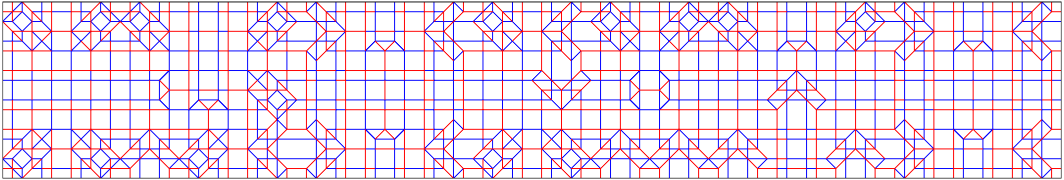
All fold lines on octagonal grid

Parameterized complexity:
Analyze difficulty by more than input size

Quantify features of an input that make it difficult by numerical parameters

Separate time bound into function of parameters $\times$ polynomial of input size

Can solve very large problems as long as the parameters stay small

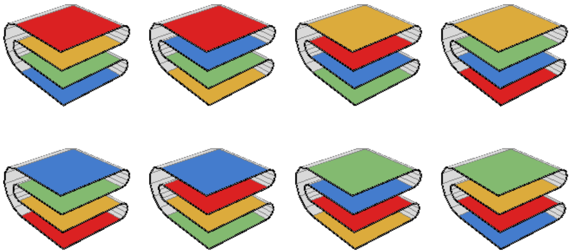


Maybe it's easy to fold grid-like crease patterns in which
one of the two grid dimensions is small?

Grid dimension is the wrong parameter

Reason 1: It doesn't apply to non-grid folding patterns

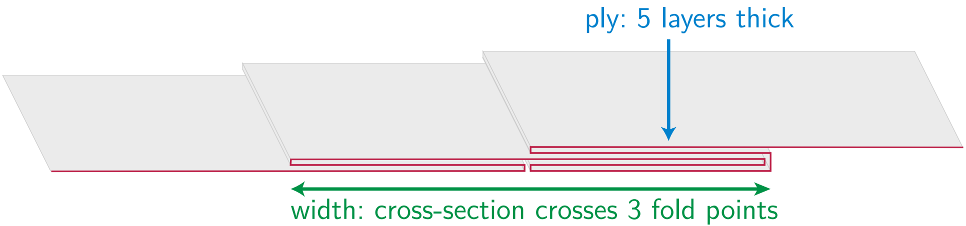
Reason 2: We don't even know how to fold $3 \times n$ square grids



The still-unsolved "map folding problem": flat-fold a grid given as input a mountain–valley assignment on all grid edges

Better parameters

Parameterize by cutting flat-folded state (whatever it is) by vertical lines, and looking at the geometry of the cross-sections



- Ply: how many layers are there, for the cross section with the most layers?
- Width: what is the maximum # fold points that a cross-section passes through?

  (Actual definition uses treewidth of planar graphs, similar but more complicated, does not require cross-sections to be parallel lines)

Fine-grained complexity

Focus on tightness of time bounds rather than the cruder distinction between polynomial and exponential time

Algorithm: Sweep vertical cross-section line left to right

Track cross-sections of folds of left side of line

Fewer cross-sections than folds, $O\big((\text{ply!})^{\text{width}+1} n\big)$

"Fixed-parameter tractable": bad function of parameters but linear in crease pattern size

Exponential time hypothesis $\Rightarrow$ exponential dependence on width cannot be improved

Proof idea: For NP-complete crease pattern, width is proportional to # circuit variables

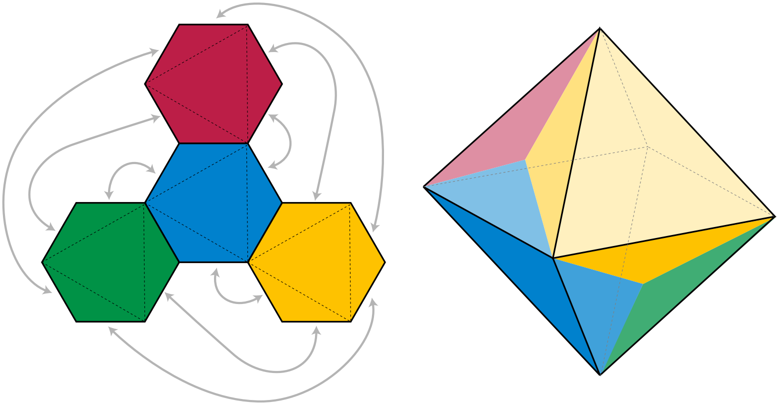
(But to understand dependence on ply, we need progress on map folding!)

[Eppstein 2023]

What does it fold to?

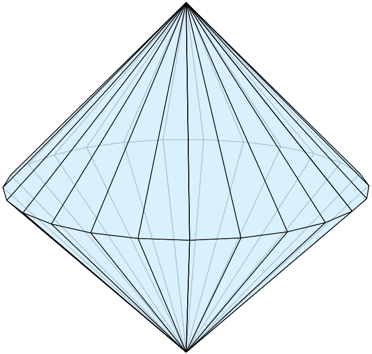
Alexandrov's theorem

Fold paper to form a surface that is topologically spherical, with finitely many "cone points" of total angle $< 360^\circ$ $\Rightarrow$ exactly one way for it to form a convex polyhedron



But how does the shape of the polyhedron depend on the folding pattern?
And where do the creases go?

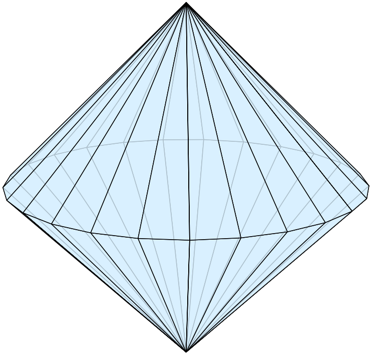
An easy(?) special case



Bipyramid with

- All faces isosceles triangles
- All edge lengths integers
- Same length for the two equal sides of each face
- Base lengths allowed to vary

One dimension lower



Equator must be a cyclic polygon!

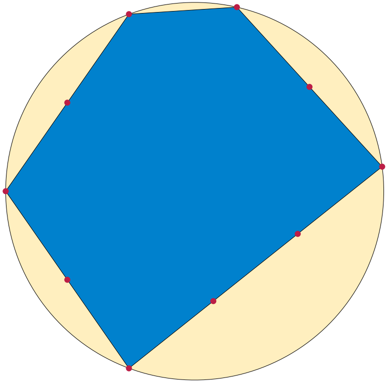
- Symmetry of face dimensions + uniqueness of realization $\Rightarrow$ top and bottom are mirrored $\Rightarrow$ equator is a plane polygon
- All vertices at equal distances from top and bottom apex $\Rightarrow$ they lie on a circle
- Equator edge lengths must be as specified from the folding pattern

Galois complexity

Coordinates for a cyclic polygon with given edge lengths are roots of polynomials with:

High algebraic degree (for regular p-gon with p prime, the polynomial is $x^p - 1$)

Unsolvable Galois groups $\Rightarrow$ no closed-form formula [Varfolomeev 2004]



The same difficulties extend to finding the shape of a bipyramid from its folding pattern

$\Rightarrow$ numerical approximations rather than exact symbolic descriptions may be necessary

How to fold it?

A mismatch between theory and practice

Previous hardness proofs:

Difficult to find folded state of crease pattern

Many simulation tasks:

We know both crease pattern and folded goal

Seek motion from one to the other

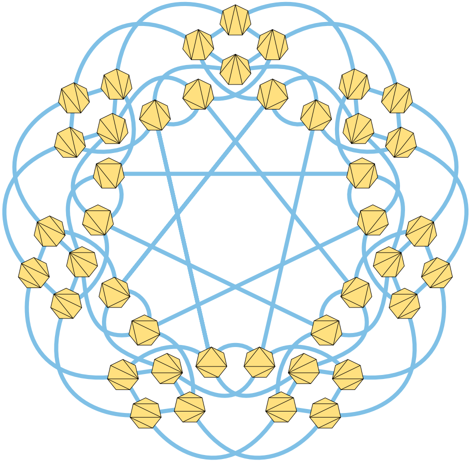
Reconfiguration complexity

Define combinatorial system of states+moves

Study complexity of problems:

- Can I reach one state from another?
- Are all states connected?
- How to reach goal in fewest moves?

Often PSPACE-complete (harder than NP)



Face flips in origami tessellations

States = locally valid mountain-valley assignments on a tessellation:

- Satisfies Kawasaki's & Maekawa's theorems
- Not required to have a global flat folding

Move = reverse the assignment on a single face

On triangular tiling, $O(n)$ flips always suffice, but it is NP-complete to minimize length

[Akitaya et al. 2020b]

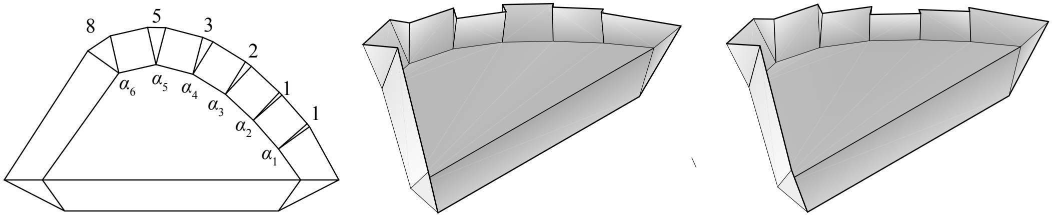
Weaknesses: NP vs PSPACE; local vs global folding; do moves make sense?

When even a single step is hard to find

For flat-foldable unlabeled crease patterns, starting from a completely unfolded state, rigidly flexing every crease simultaneously is weakly NP-complete

Must subdivide certain angles into two subsets with equal sums

Weakly NP-complete ⇒ high numerical precision is necessary for accurate results



Finding a subset of a given crease system that can flex rigidly, starting from a completely unfolded state, is strongly NP-complete

[Akitaya et al. 2020a]

Flaps and flips

A toy problem for origami reconfiguration

State: Equal-sized squares of paper lie flat on a flat table, attached to the table by a hinge along one side

Move: flip one square to new flat position with hinged edge attached in same place

All other squares remain where they were

Moving square can be pulled from or inserted into pockets (non-rigidly); can flip across hinge even if other squares lie above hinge

Example: The green square can flip across its hinge but must remain under the lower red square

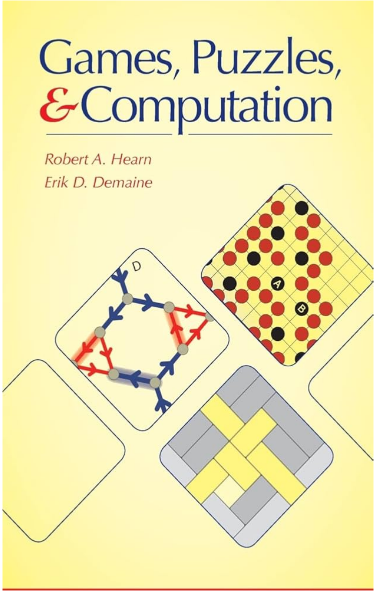
Nondeterministic constraint logic

Circuit-like reconfiguration problem used by [Hearn and Demaine 2009] to prove PSPACE-hardness of many puzzles and games

State: planar diagram of

- Blue and red arrows
- Gates: junction where three arrows meet
- OR (3 blue): ≥ 1 arrow must point inward
- AND (1 blue, 2 red): at least one blue or both red arrows must point inward

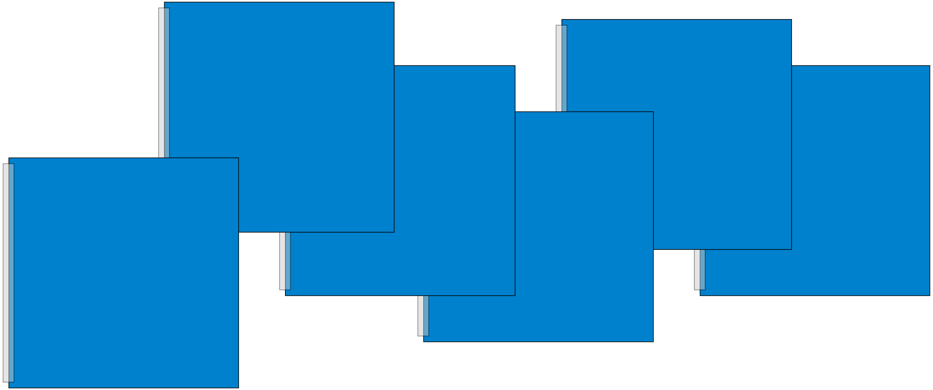
Move: Reverse an arrow!



Simulating an arrow

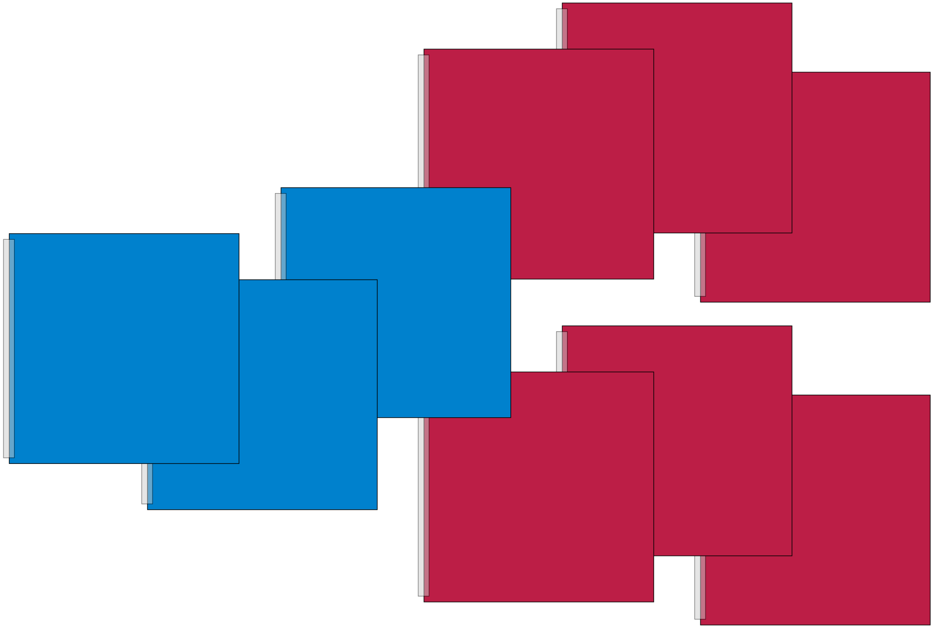
arrow points toward the hinged side of each square

to reverse arrow, flip squares one at a time starting from the arrowhead
(intermediate states have no arrowhead at either end, not problematic)

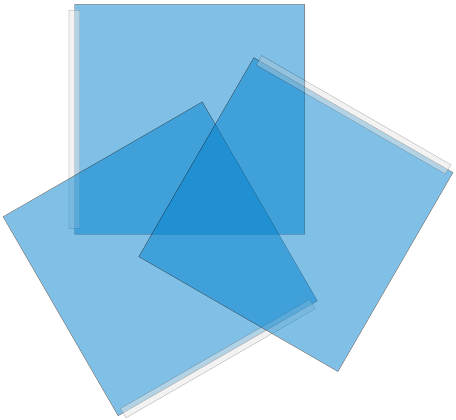


each square overlaps the hinges of its neighbors on both sides,
impossible for squares at both ends of arrow to flip inward (double arrowhead)

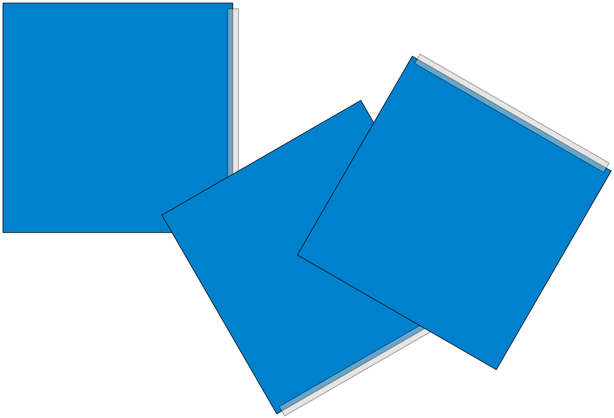
Simulating an and-gate



Simulating an or-gate



three outward-pointing arrows: impossible
cyclic above-below order in central triangle



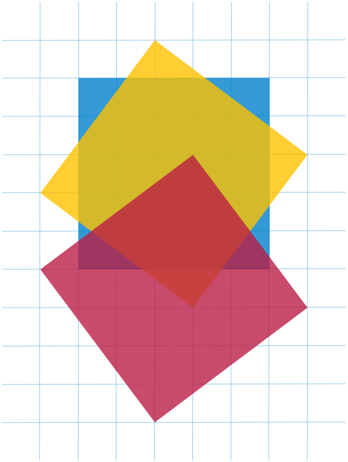
if one arrow points in, all squares lie flat

if two point in, either can be flipped out

Space complexity

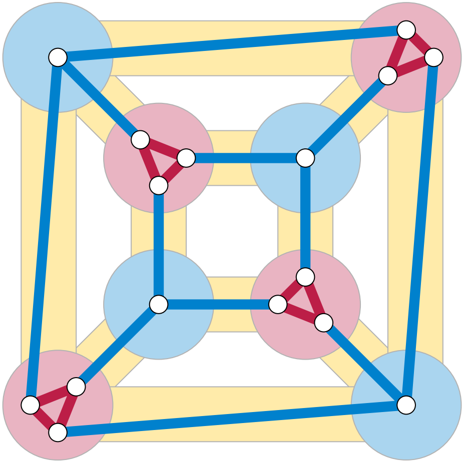
For the "flaps and flips" reconfiguration problem:

- Testing whether one state can reach another is PSPACE-complete
- Testing whether all states are connected is PSPACE-complete
- Completeness holds even for patterns of bounded ply and bounded width
- Hard configurations can have integer vertex coordinates
- Getting from one state to another may require exponentially many flips

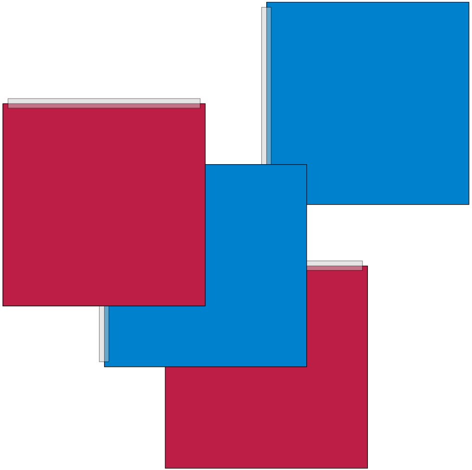


Counting complexity

Counting valid states of flaps-and-flips instance is #P-complete



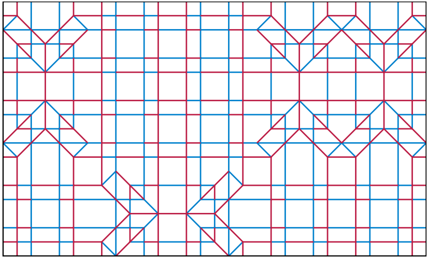
Reduce counting cubic bipartite
matchings to constraint logic
[E, unpublished, Barbados 2019]

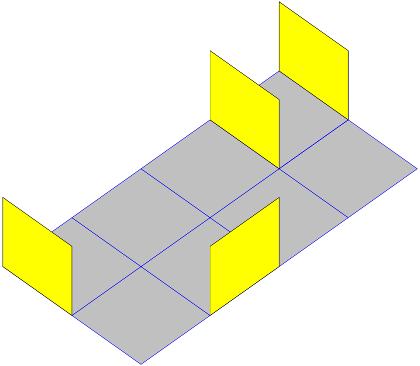


Crossover gadget

Single-sheet flat-folding

We can make single-sheet crease patterns that produce flaps!





Hardness proof should extend to reconfiguring flat foldings of single-sheet patterns
with $O(1)$ changes of crease orientation per move
or only allowing refolding along one line segment at a time

Undecidability

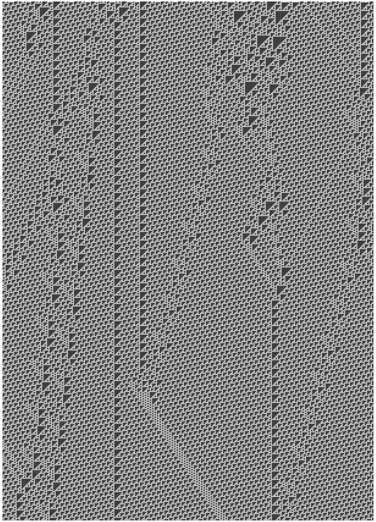
Does a finite perturbation stay finite?

Given as input:

- A repeating unlabeled crease pattern on an infinite half-plane with optional folds
- The folded state along the edge of the half-plane, repeating except for a finite perturbation

There is no algorithm to test the existence of a folded state of the entire half-plane in which the perturbation remains finite, even if each fold is uniquely determined and easy to find

[Hull and Zakharevich 2023]



Main idea: simulate "Rule 110" cellular automaton

Alternative proof of Turing completeness

Build computer circuits (CPU + memory) out of Boolean gates based on "not all equal" and crossover gadgets of Bern and Hayes [1996], without optional folds [Assis 2024]

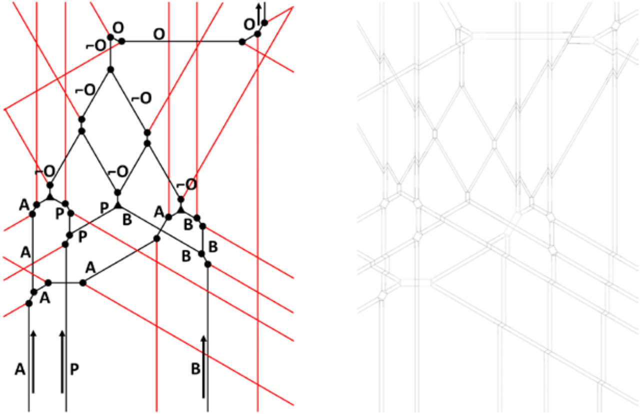


Figure 8: The crease pattern schematic (left) and full crease pattern (right) of the origami NAND gate.

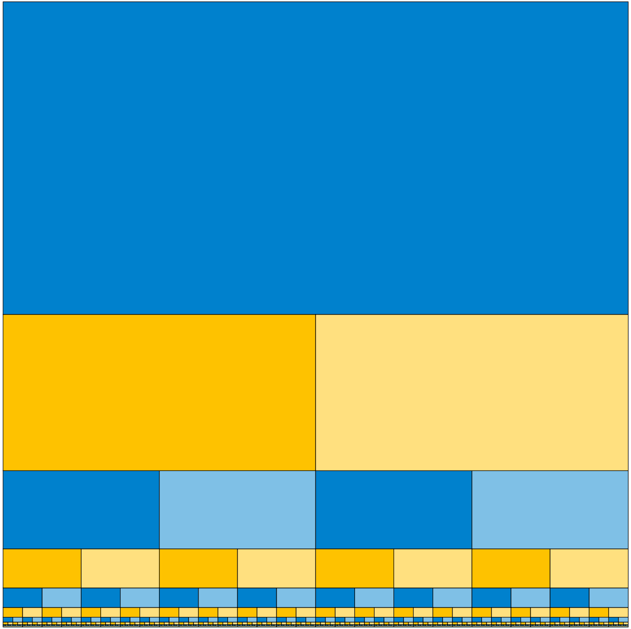
In presentation at 8OSME, Assis stated: like Hull and Zakharevich [2023], this needs an infinite sheet of paper. But does it?

Tomoko Fuse's "infinite origami"

Origami art from fractal crease patterns

Presented by Fuse at 8OSME, same day as my talk (earlier version of this one) and Assis's talk

Recursive subdivision of origami square into rectangles



Main idea:

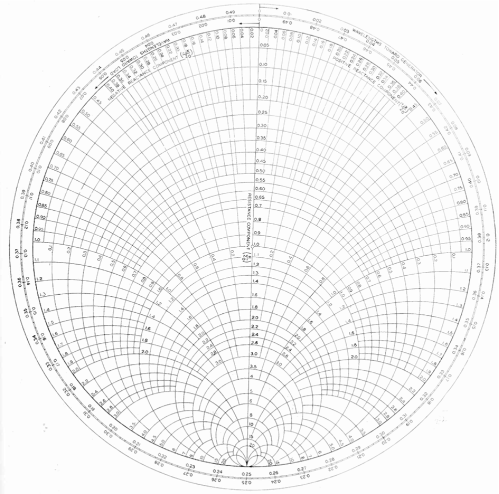
Use recursive tiling by similar rectangles

Use the same crease pattern within every tile, simulating a fixed (but arbitrary) Boolean circuit

Combine tiled circuits for universal computation

This is the binary tiling of the hyperbolic plane

(in the Poincaré half-plane model). Studied by Böröczky [1974]; earlier precedents:



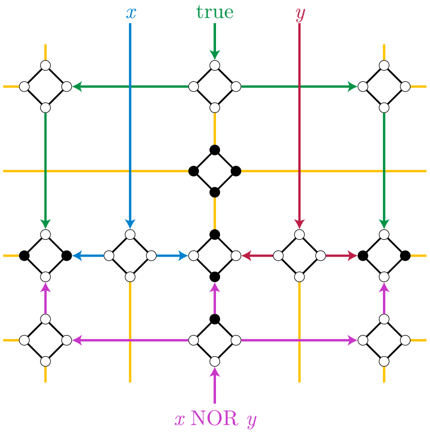
Smith chart in radio engineering
[Mizuhashi 1937; Smith 1939; Volpert 1940]

M. C. Escher, *Regular Division of The Plane VI*, May 1957

Circuitry within each tile

Octagonal-grid gadgets of Akitaya et al. [2016]

- Crossover
- White circle: split/merge
- Dark circle: not all equal

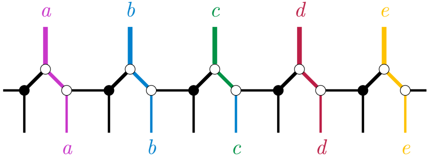


Grid layout

- Each grid row or column carries a (possibly unused) signal
- Each grid vertex can be a crossover or four-gadget cluster

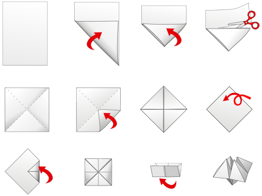
Grid scale transition at tile boundary

Signal width and gadgets automatically scale correctly!



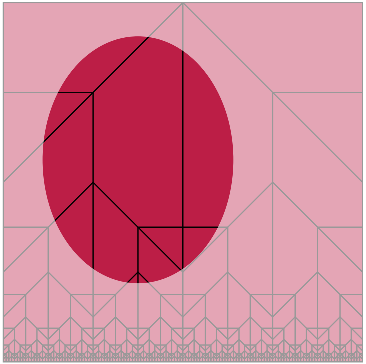
Local vs global

Intuition: Choosing folds step by step cannot produce inconsistencies in the limit



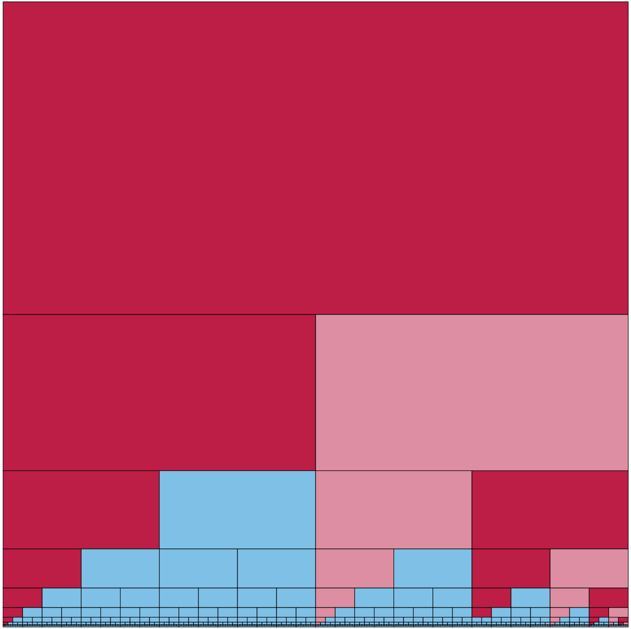
For a locally finite crease pattern, the following are equivalent:

- The whole crease pattern folds flat
- Every patch with finitely many creases folds flat



Main idea: Kőnig's infinity lemma [Kőnig 1927]

Turing machines on the binary tiling



Initial state enters on left side

Each tile can either compute single local update of TM, shifted down by one or copy local state from left to right

Tile tells its two children which of these two behaviors to follow

Global TM states ⇒ vertical chains of tiles nested down and to right

Setting a logic bomb

We can make our origami simulation explode (or at least fail to fold) whenever the Turing machine stops

(use not-all-equal gadget with equal inputs)

But testing whether it will stop is the undecidable halting problem

⇒ It is undecidable (co-RE-complete) to determine whether an infinite crease pattern on square paper (with a fractally repeating design + finitely many fixed choices) folds flat

(for either labeled or unlabeled creases)

End matter

Conclusions

Origami is hard!

. . . that's part of what makes it interesting

NP-hardness, ETH $\Rightarrow$ if you don't already know what you're folding, figuring it out from the crease pattern is non-obvious

Galois complexity $\Rightarrow$ for fully 3d folds, numerical approximation may be necessary

PSPACE-hardness $\Rightarrow$ repeated folding and unfolding may be necessary

Undecidability $\Rightarrow$ infinite origami is super hard, but maybe worth exploring more

Image credits I

Origami Dream Boat, art by by Sam E. Hues (@sam3hues), Clear Labs, St. Petersburg, Florida, depicting Macy Sun Kober (@kimchi.jpg). Photo by James Faircloth (@urbanmuralhunter), taken January 2019; uploaded under CC-BY-NC-ND to Flickr by Terence Faircloth, `https://www.flickr.com/photos/atelier_tee/49769232808`

Folding pattern for Rule 110 cell from [Hull and Zakharevich 2023]

"Origami" crease pattern and crease pattern for flaps from Origami Maze Font by Erik Demaine, Martin Demaine, and Jason Ku, 2010, `https://erikdemaine.org/fonts/maze/`

Eight ways to fold a 2x2 map along its creases, CC-BY-SA image by Robert Dickau, March 24, 2010, `https://commons.wikimedia.org/wiki/File:MapFoldings-2x2.png`

Sand through a magnifying glass created by Adobe Generative AI

Hexagons folding to an octahedron created by the author for Wikipedia, based on an example from [Khramtcova and Langerman 2017]

Bipyramid from [Eppstein 2021]

Image credits II

Rubik's cube, CC-BY-SA image by Famartin, January 14, 2021, `https://commons.wikimedia.org/wiki/File:2021-01-14_18_25_51_A_scrambled_Rubik%27s_Cube_in_the_Franklin_Farm_section_of_Oak_Hill,_Fairfax_County,_Virginia_(cropped).jpg`

Blue triangle mesh, cropped from CC-BY-NC image by Ryan Robinson, May 4, 2007, "Advanced Concept Car Prototype", `https://www.flickr.com/photos/infinite-origami/484625351/`

Single-step rigid folding hardness illustration from [Akitaya et al. 2020a]

Games, Puzzles, & Computation: cover of [Hearn and Demaine 2009]

Rule 110 pattern created with Golly, `https://golly.sourceforge.io/`

NAND gate schematic and folding pattern: figure 8 of [Assis 2024]

Tomoko Fuse: credited to M. Amena, from "Tomoko Fuse, la origamista que tiene al mundo en sus manos", *La Nacion*, October 16, 2016, `https://www.lanacion.com.ar/buenos-aires/tomoko-fuse-la-origamista-que-tiene-al-mundo-en-sus-manos-nid1947475/`

Smith Chart: [Smith 1939]

Image credits III

M. C. Escher, Regular Division of The Plane VI, from Escher in Het Palais, “The Regular Division of The Plane at the ‘De Roos’ foundation”, `https://www.escherinhetpaleis.nl/escher-today/the-regular-division-of-the-plane-at-the-de-roos-foundation/?lang=en`

How to fold a paper fortune teller 12 steps, public domain image by MichaelPhilip, September 2009, `https://commons.wikimedia.org/wiki/File:Fortuneteller_mgx.svg`

Cover art from *Speed* (1994), film from 20th Century Fox, as used by Amazon Prime Video

Other images created by the author

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