Patterns of Tangent Spheres

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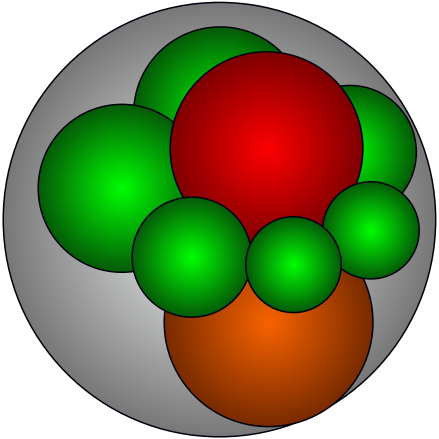
JCDCG^3, Tokyo, September 2026

Soddy's hexlet



For any three mutually tangent spheres (gray, red, orange)

∃ 6-cycle of consecutively-tangent spheres (green), tangent to given spheres

Recorded on an 1822 sangaku tablet from Samukawa Shrine, Kanagawa, Japan

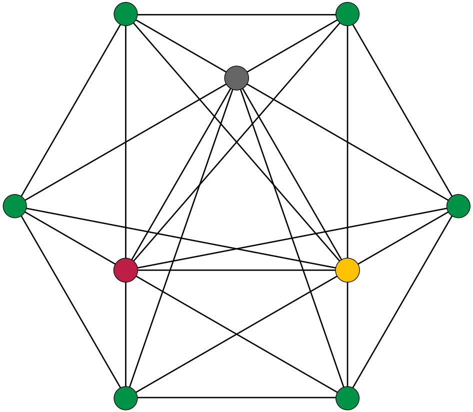
Rediscovered by Soddy [1937]

Replica of the original tablet

Dedicated in May 1822 (Bunsei 5) by Irisawa Shintarō Hiroatsu, a student of Uchida Itsumi

Reconstructed in August 2009 based on the *Kokon Sankan* (Ancient and Modern Mathematical Mirror) held by the Samukawa Shrine Hotoku Museum

Graph of tangencies



Represent each sphere by a graph vertex

Two vertices are the endpoints of an edge when their spheres are tangent

What graphs are possible?

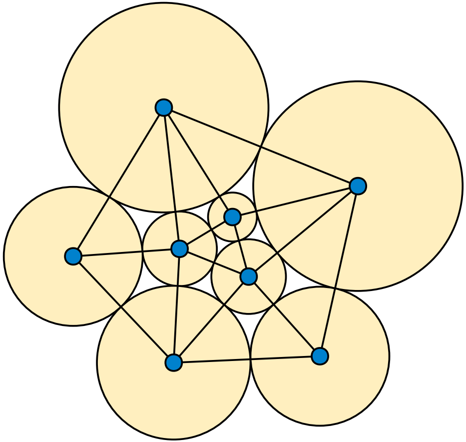
What does their structure imply about other problems in discrete and computational geometry?

Warmup: Tangency graphs in two dimensions

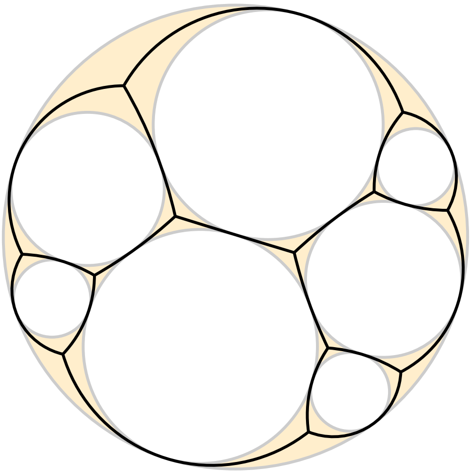
Koebe–Andreev–Thurston circle packing theorem: tangency graphs of non-crossing circles = all planar graphs

For maximal (triangulated) planar graphs, representation is unique up to Möbius transforms, has efficient numerical approximation [Dong et al. 2020]

But exact algorithms are not possible [Bannister et al. 2015]



Some applications of the circle packing theorem



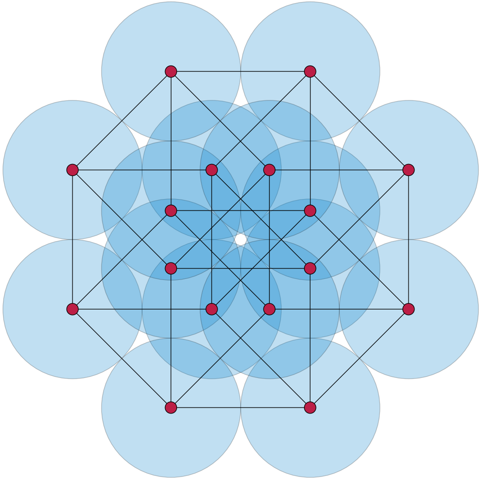
Approximate conformal and quasiconformal mapping [Rodin and Sullivan 1987]

Geometric routing schemes for computer networks [Yu et al. 2011]

Graph drawings with degree-bounded angular resolution [Malitz and Papakostas 1994] and slope number [Keszegh et al. 2013]

Graph-theoretic characterization of 2d soap bubble complexes [Eppstein 2014]

What if we allow overlapping circles?



Tangency graphs $\supset$ unit distance graphs

(vertices = points,
edges = pairs of points at distance one)

using circles of radius $\frac{1}{2}$

Number of edges in planar unit distance graphs

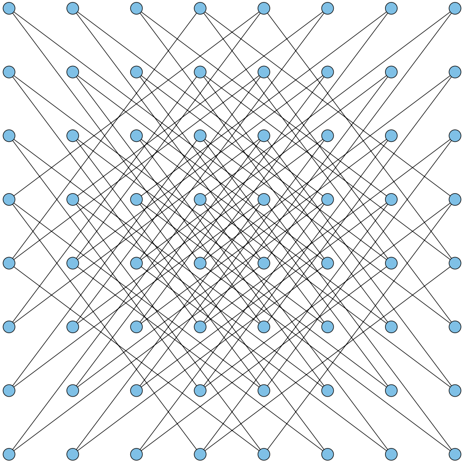
Erdős [1946] proved $O(n^{3/2})$ (later improved to $O(n^{4/3})$ and $\Omega(n^{1+c/\log\log n})$)

Recent target of LLM-assisted research $\Longrightarrow \Omega(n^{1.03583})$ (https://teorth.github.io/optimizationproblems/constants/84a.html)

Lower bounds are scaled grids and projected lattices $\Longrightarrow$ many collinearities

If we forbid three points in line, how many edges can there be?

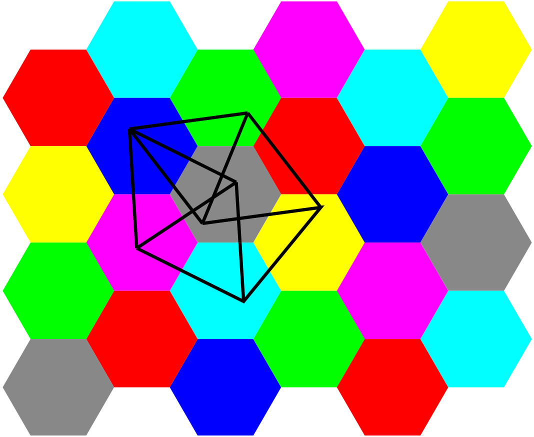
Projected hypercube $\Longrightarrow \Omega(n \log n)$; better?



Coloring unit distance graphs

Hadwiger–Nelson problem: How many colors are needed?

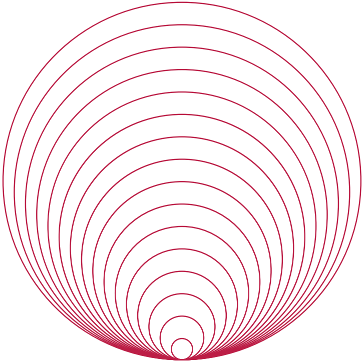
By the de Bruijn–Erdős theorem, same answer for finite graphs or whole plane



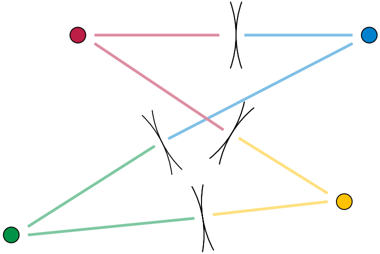
Seven colors suffice; at least five are needed [de Grey 2018]

Common generalization: external tangencies of arbitrary circles

Prevents “earrings” of nested tangent circles $\Rightarrow$ large complete graphs



Crossed 4-cycle not possible



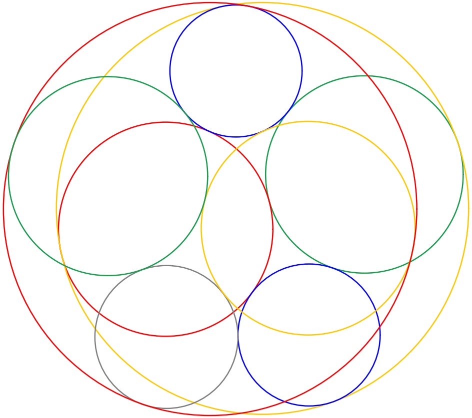
Uncrossed and crossed edge pairs would both have total length = $\sum$ radii

Triangle ineq. $\Rightarrow$ crossed edges are longer

No $K_{3,3}$ subgraph $\Rightarrow$ $O(n^{5/3})$ edges

Ringel's circle coloring problem

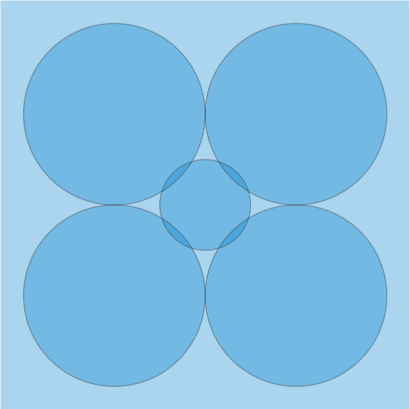
Jackson and Ringel [1984]: Are $O(1)$ colors enough for tangency graphs of circles?
(no three having a common tangent)

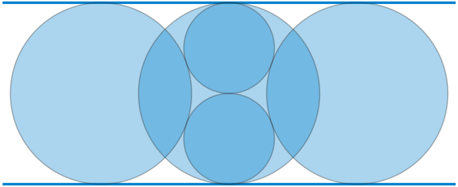


Answer: No! Not even if we allow only external tangencies [Davies et al. 2022]

Disjoint spheres

Hexlet is $C_6 + C_3$; we can also realize $C_4 + C_4 = K_{2,2,2,2}$



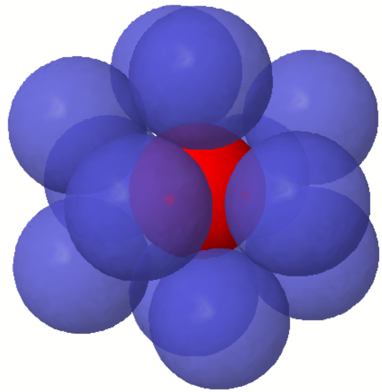


Make a square of congruent spheres between two parallel planes (degenerate spheres)

Two half-height spheres fit into the hole in the center of the square

Average kissing number

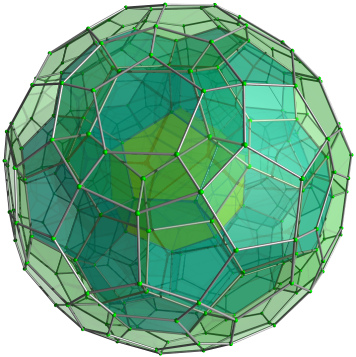
At most twelve congruent spheres can touch one central sphere



Realizable on average for tangency graphs by standard close-packing

Allowing varying radii, average degree can be ≥ 12.612 [Eppstein et al. 2003] and is ≤ 13.606 [Dostert et al. 2022]

Lower bound inscribes spheres in faces of 4-polytope combining multiple 120-cells



Bigger bicliques when spheres can cross

Ball bearings are commonly arranged as congruent spheres with centers on a circle

Any point on the perpendicular axis of the circle is the center of another sphere, tangent to all the ball bearings

The tangency pattern is a biclique $K_{a,b}$

The number of circles a in the ball bearing race and the number of circles b centered on the axis can both be arbitrarily large (allowing non-tangent circles to overlap)

$\Rightarrow$ no non-trivial graph density bounds

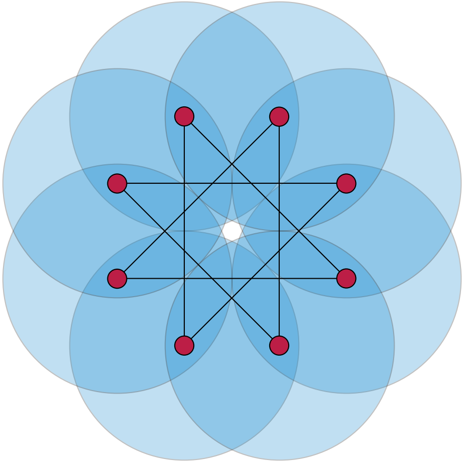
Additional cycle–cycle joins

Hexlet is $C_6 + C_3$
and we have already seen $C_4 + C_4$

Eight spheres in pattern shown here leave enough space in center for a column of six disjoint spheres between two parallel planes $\Rightarrow C_8 + C_8$

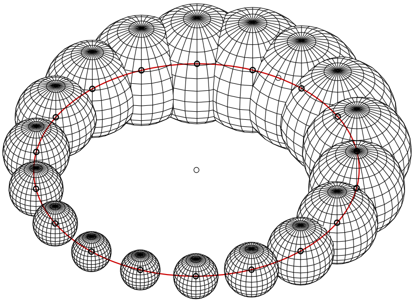
More generally we get $C_{4k} + C_{4k}$ and $C_{2k+1} + C_{4k+2}$ for any integer $k \geq 1$

(Construction involves placing centers on two perpendicular regular polygons on sphere in $\mathbb{R}^4$ + stereographic projection)



More general bicliques

Dupin cyclide: inversion of a torus



Arbitrarily many tangent interior spheres

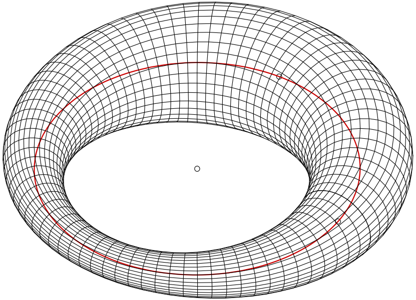
+ arbitrarily many tangent exterior spheres

together form a biclique of tangencies

Red curve through the inner sphere centers

+ corresponding curve through outer centers

are each conics in a plane

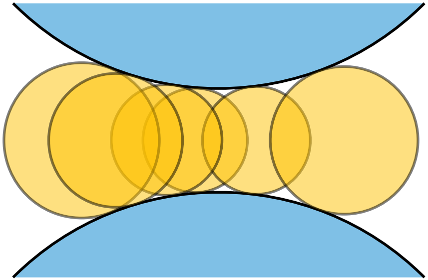


In other dimensions

In 2d: largest bicliques of circles are $K_{2,n}$

In 3d: can form bicliques of spheres with centers on two plane conics

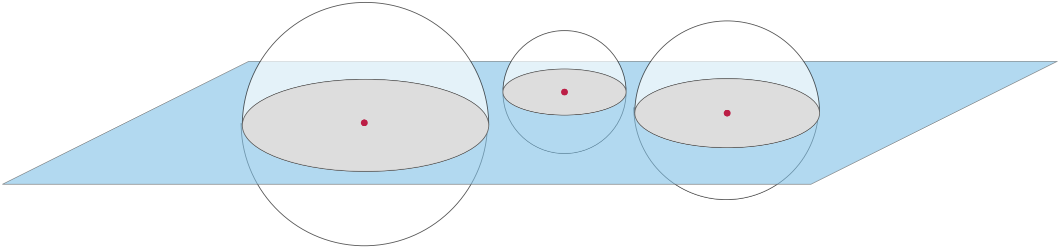
Maybe higher dimensions even more flexible?



New key lemma on high dimensional bicliques

Let A and B be two sets of ≥ 3 hyperspheres in d-dimensional Euclidean or hyperbolic space, so that every pair of one sphere from A and one sphere from B are externally tangent

Then the centers of the spheres in A (and symmetrically in B) lie on a hyperplane



[Eppstein 2026a]

Proof ingredients, I: Pencils and orthogonality

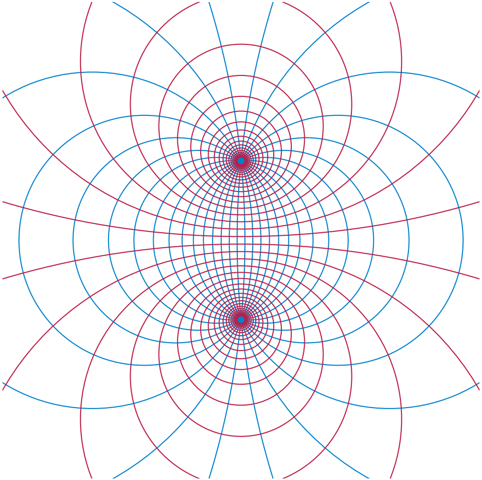
Pencil = 1-dimensional family

Orthogonal = cross at right angles

Five types of pencils of circles:

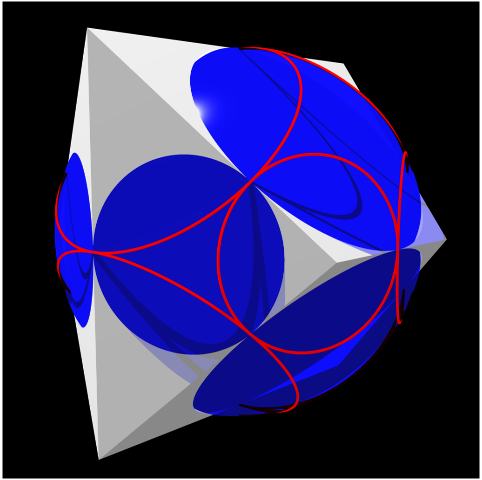
- Elliptic (through two points, blue)
- Hyperbolic (orthog. to elliptic, red)
- Parabolic (tangent to a line at a point, orthog. parabolic)
- Concentric
- Lines through a point (degenerate circles, orthog. concentric)

In dimension ≥ 3, pencils are not orthogonal to other pencils



Proof ingredients, II: Lifting

Stereographic transformation: spheres in $\mathbb{R}^d \Rightarrow$ spheres on a hypersphere in $\mathbb{R}^{d+1}$



Two ways of generating a sphere:

- Horizon from a viewpoint (red)
- Slice by a hyperplane (blue)

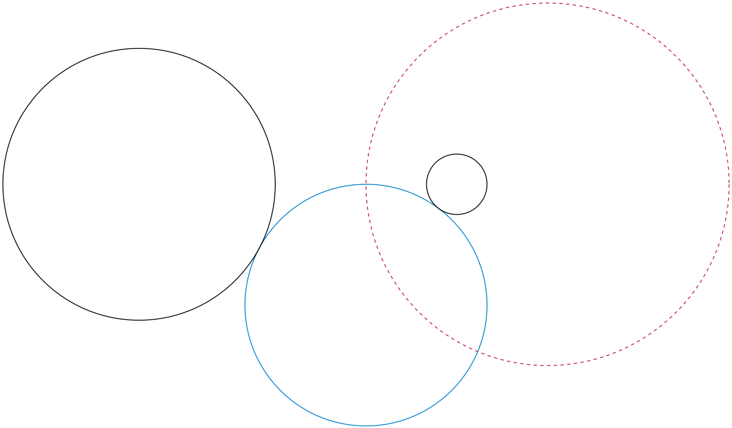
Orthogonal spheres $\Longleftrightarrow$ viewpoint of one sphere is on slice hyperplane of other

Pencil = line of viewpoints
= rotate slice around $(d-1)$-dim'l axis

Viewpoint on two slices of a pencil (orthogonal to two spheres) lies on axis and all slices (orthogonal to whole pencil)

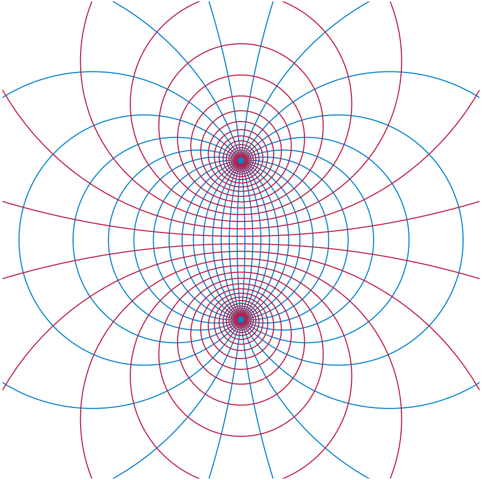
Proof outline, I: An orthogonal midsphere

- Each A_i and A_j can be inverted to each other through a sphere of antisimilitude M_{ij}
- Inversion preserves tangency so inverse(B_k) still tangent to A_i and A_j, must equal B_k
- Therefore M_{ij} is orthogonal to B_k



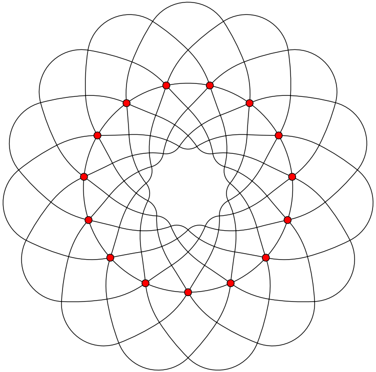
Proof outline, II: From one to many orthogonal spheres

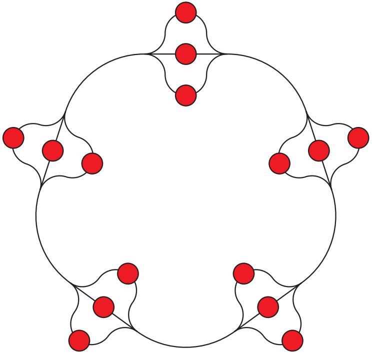
- M_{12} and M_{23} define a one-dimensional pencil of spheres
- When two spheres in a pencil are orthogonal to B_k, all of them are
- The spheres in the pencil cover space so one of them contains center of B_1
- This orthogonal sphere through the center of B_1 is a hyperplane
- This hyperplane, orthogonal to all B_k, contains all their centers
- QED



Consequence: An unrealizable triangle-free graph

The two drawings below are the same graph, from the 2011 Graph Drawing Contest





Perturb so no triple of spheres has collinear centers

Biclique lemma $\Rightarrow$ 6 centers of non-adjacent triples coplanar $\Rightarrow$ all centers coplanar

So this is really a system of 2d circle tangencies, impossible because many $K_{3,3}$ subgraphs

Global positioning system

You receive signals from multiple satellites telling you "I was at this location at this time"

Satellites have precise and synchronized clocks

Your clock is precise but unsynchronized

From the relative times you received the signals, you can determine the difference of distances between your location and any two satellite locations

From differences of distances to sphere tangencies

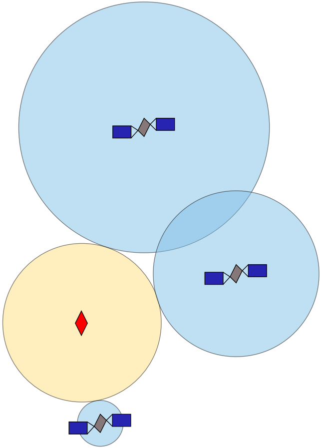
Among k (unknown) satellite distances d_i, let d_ℓ be min
(all other satellites have a positive difference of distances)

Choose small ε, and center a sphere around each satellite of positive radius $d_i - d_\ell + \varepsilon$

Uses only difference of distances $\Rightarrow$ you can construct it

Your location must be the center of a sphere of radius $d_\ell - \varepsilon$ tangent to all the other spheres

Biclique lemma $\Rightarrow$ four non-planar satellites cannot be part of $K_{4,3}$ $\Rightarrow$ at most two location spheres
(one of which is probably not near the Earth's surface)



Therefore: Every non-planar set of four satellite locations gives you ≤ 2 possible positions

The Erdős–Anning theorem

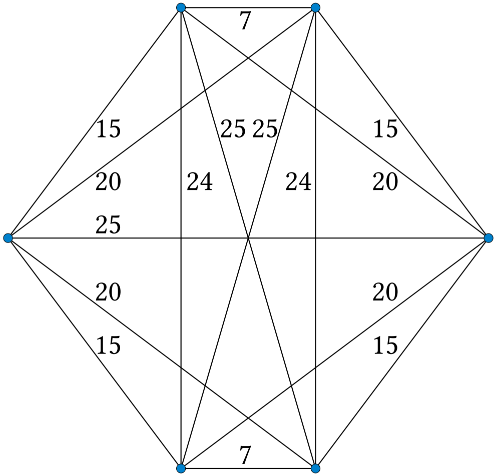
Points in the plane with integer distances are finite or collinear [Anning and Erdős 1945]

Original proof via trigonometric inequalities

Proof via crossings of hyperbolas [Erdős 1945] (missing case analysis of degeneracies), also found by Choquet & Kreweras

Handwavy claims to generalize to higher dimensional Euclidean space

Alternative proof via additively weighted Voronoi diagrams generalizes to many other two-dimensional metrics (e.g. hyperbolic) but not hyperbolic space [Eppstein 2026b]



Erdős and Anning in space!

Given points in any finite-dimensional Euclidean or hyperbolic space with integer distances

(dim > 1, allowing any scale for hyperbolic distance)

Choose $d + 1$ points that span the space

(≥ 3 of them, cannot be on a hyperplane)

These are your GPS satellites

Two GPS locations for $d + 1$ choices of closest satellite and $(\text{diam} + 1)^d$ integer difference vectors

$\Rightarrow$ Total points $\leq 2(d + 1)(\text{subset diameter} + 1)^d$

Therefore: Points in $\mathbb{R}^d$ or $\mathbb{H}^d$ with integer distances must be finite or collinear

Conclusions

Even in 2d, much about external tangency graphs remains to be proven

- Density and chromatic number of unit distance graphs still open
- Density of tangency distance graphs only know $O(n^{5/3})$
- Complexity of recognition

3d tangency graphs can be dense, but still have nontrivial forbidden subgraphs

Applications in Euclidean and hyperbolic distance geometry in arbitrary dimensions

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Samukawa tablet: Image from http://www.wasan.jp/kanagawa/samukawa.html

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Giant ball bearing race with Konrad Österström sitting inside it, at the the Göteborgs Jubilee exhibition, 1923: Public domain image from Wikimedia commons, https://commons.wikimedia.org/wiki/File:1923_vykort_kullager.jpg

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