The Widths of Strict Outerconfluent Graphs

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Introduction. *Confluent drawing* permits certain non-planar graphs to be drawn without crossings [5–8, 12, 14, 16]. In these drawings curved *tracks* meet at vertices and *junctions*; vertices are adjacent when connected by a smooth curve connects in the union of tracks. Applications include syntax diagrams [1] and the Hasse diagrams of posets [10]. *Strict confluent drawing* forbids multiple connections between vertices, or smooth curves from a vertex to itself [9, 11]. *Outerconfluent drawings* place tracks in a disk with the vertices on its boundary. *Strict outerconfluent graphs* are somewhat mysterious: they can be recognized efficiently given the vertex ordering around the disk but without it this remains open [9].

Strict outerconfluent graphs can be dense, with unbounded treewidth. They include distance-hereditary graphs [7] and *tree-like* strict outerconfluent graphs [11], both of bounded clique-width. In this work, we show that strict outerconfluent graphs have unbounded clique-width but bounded twin-width.

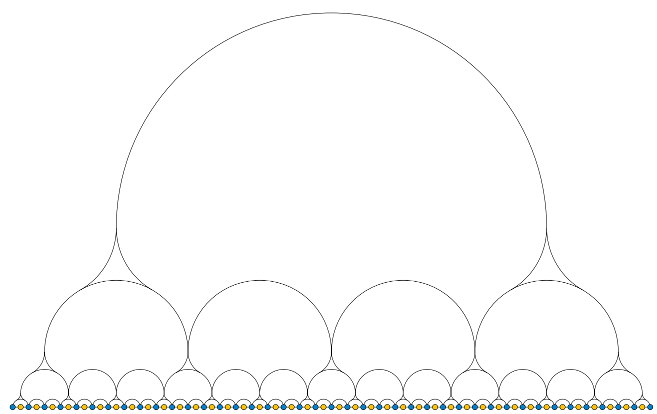

Fig. 1. Construction for strict outerconfluent graphs of unbounded clique-width

Unbounded clique-width. We prove that the recursively-constructed graphs of Fig. 1 have unbounded clique-width. It is convenient to use *rank-width*, derived from ternary trees with the graph vertices as leaves. Each tree edge defines a partition of the vertices into two subsets; the rank-width is the maximum rank of the biadjacency matrix of such a partition, for a ternary tree that minimizes this maximum. Rank-width r and clique-width c are related by $r \leq c \leq 2^{r+1} - 1$ [15].

Proof sketch: Some edge of any ternary tree partitions the vertices roughly evenly; we show that this partition has high rank, through the following steps.

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– The partition splits the vertices into contiguous intervals; selected interval endpoints induce a matching of rank proportional to the number of intervals.
– If there are only $O(1)$ intervals, then some two intervals on opposite sides of the partition are both much longer than the gap between them.
– For these two intervals, $\omega(1)$ nested semicircles in the drawing each connect a vertex in one interval to a vertex in the other.
– From these nested semicircles we construct a square submatrix of the biadjacency matrix, of size equal to the number of semicircles, of full rank.

Bounded twin-width. In contrast, we prove that strict outerconfluent graphs have bounded *twin-width*. Graphs of bounded twin-width include planar graphs and k-planar graphs, important in graph drawing [2, 13]. Twin-width is defined by merging clusters of vertices in pairs, starting with one-vertex clusters, until only one cluster is left. At each step, two clusters are connected by a *red edge* if some but not all adjacencies exist between their vertices. The twin-width is the minimum d so that this merging process can limit the degree of the graph of red edges to at most d [4]. Twin-width can also be characterized imprecisely by counting *ordered graphs*, graphs assigned a linear order on vertices. A family of ordered graphs is *hereditary* if induced subgraphs with induced orders remain in the family. It is *small* if it has singly-exponentially many n-vertex ordered graphs. Every hereditary family of graphs of bounded twin-width can ordered as a small family of ordered graphs, and every small family of ordered graphs has bounded twin-width [3]. We prove that (with their natural vertex orderings around the disk on which they are drawn) strict outerconfluent graphs form a small hereditary family of ordered graphs, and therefore have bounded twin-width.

Proof sketch: This family is hereditary: removing any vertices and unused tracks produces another strict outerconfluent graph. From any n-vertex strict outerconfluent drawing of a graph G, form a plane graph D, by reinterpreting the junctions of the drawing as vertices of D and adding one more vertex o, outside the disk of the drawing, connected to the vertices of G by edges external to the disk. An equivalent ordered drawing to the one we started with can be recovered from the plane embedding of D by specifying which vertex is o, which neighbor of o is the start of the linear vertex ordering, and (for each non-neighbor of o) how to partition the incoming edges into tracks meeting smoothly to form a junction. Thus, the number of ordered strict outerconfluent graphs is at most the number of possible combinations of a diagram and this specification of extra information. There are $O(n)$ junctions in the drawing [9], from which it follows that D has $O(n)$ vertices. The number of maximal planar graphs with n vertices is singly exponential [17], from which it follows from the uniqueness of their plane embeddings (up to choice of outer face) and by counting subgraphs that the number of plane graphs is also singly exponential. The numbers of ways to specify o and the start of the linear ordering are $O(n)$, and the number of ways to turn vertices of D into smooth junctions is singly exponential in n. Multiplying these choices gives a singly-exponential bound on strict outerconfluent drawings, showing that the family of strict outerconfluent graphs is small.

For details, see the full version of this paper, arXiv:2308.03967.

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