On the Biplanarity of Blowups

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Graph Drawing 2023

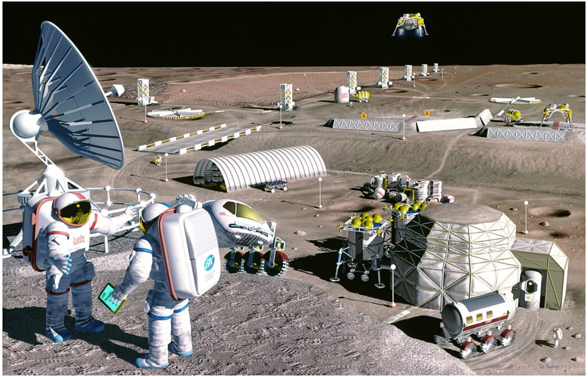
Ringel's Earth–Moon problem (1959)

Countries build moon colonies, with different borders than Earth

We must color the maps of the Earth and Moon, so

- Each country gets a single color for both maps
- Adjacent countries on either map get different colors
- We use few colors

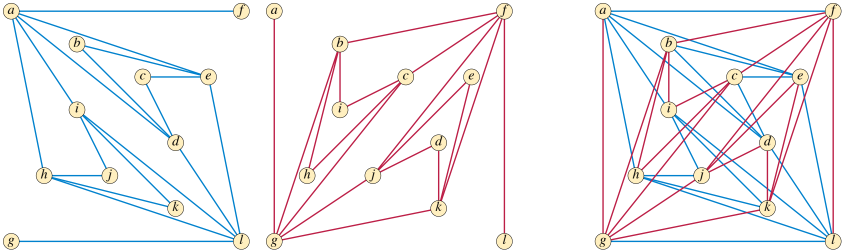
How many colors do we need?



Earth–Moon maps as graph drawings

Two equivalent versions:

- Draw graph as two planar subgraphs, covering all edges, in two separate planes (the dual graphs of the countries)
- Draw graph in one plane, with two colors of edges, so that no two edges of the same color cross (may need curved edges)

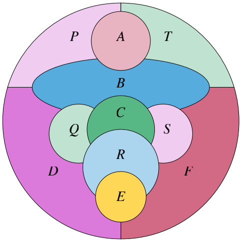


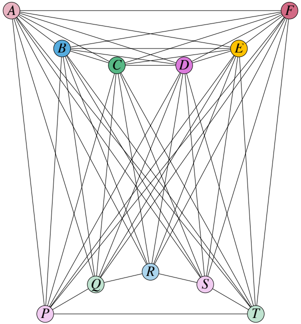
separate planes = # edge colors for no monochrome crossing = thickness

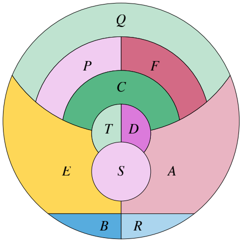
Graph is biplanar if its thickness is 2, or equivalently if it is dual to an Earth–Moon map

Sulanke's 9-chromatic Earth–Moon map (1974)

Connect all vertices of a 5-cycle to all vertices of a 6-clique







Requires 9 colors — 3 for cycle, 6 for clique

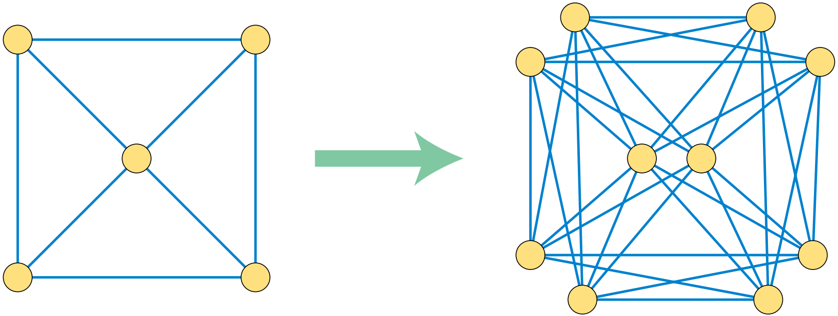
Best upper bound for arbitrary biplanar graphs is 12 colors

Kicked off a line of research: which simple combinations of smaller graphs are biplanar?

Blowups

k-blowup: replace every vertex by a k-vertex independent set

Replace every edge by a complete bipartite subgraph connecting two independent sets



Conjecture [Gethner 2018]: 2-blowups of planar graphs are biplanar

Weak evidence for the conjecture

Count the edges!

A planar graph with n vertices has $\leq 3n - 6$ edges

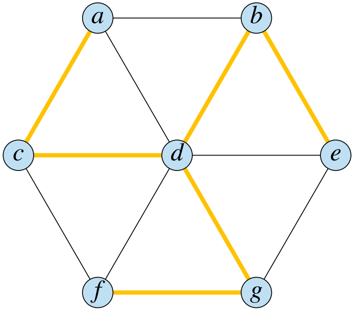
Its blowup has $2n$ vertices and $\leq 12n - 24$ edges

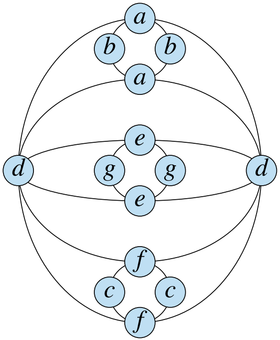
But a biplanar graph with $2n$ vertices can have up to $12n - 12$ edges, twelve more edges

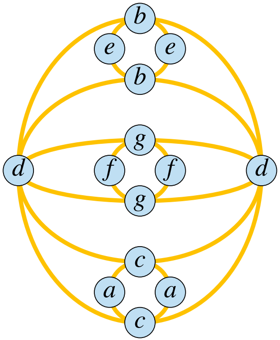
When counting is strong enough

Hereditary family with $\leq 2n - 2$ edges $\Rightarrow$ cover edges by two trees [Nash-Williams 1964]

2-blowup of a tree is planar $\Rightarrow$ 2-blowup of whole graph is biplanar [Albertson et al. 2010]





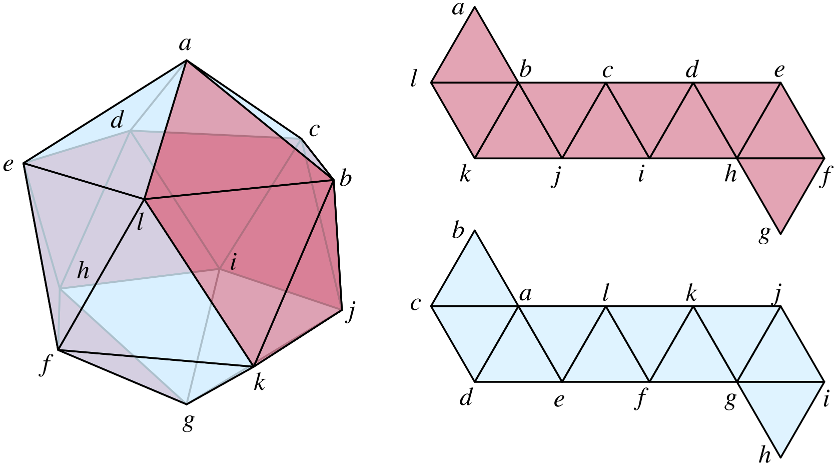


Works for triangle-free planar ($\leq 2n - 4$ edges) but not maximal planar ($3n - 6$ edges)

Proves 2-blowup thickness ≤ 3 for all planar graphs

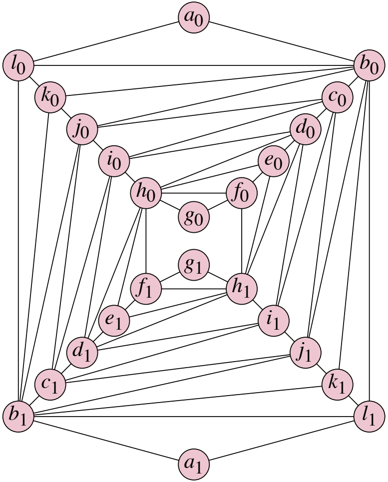
Some maximal planar graphs with biplanar blowups

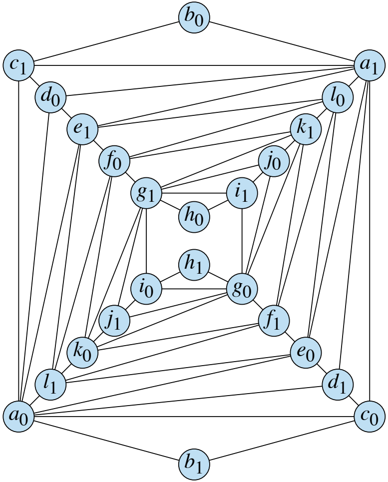
Partition dual graph into two induced paths ⇒ outerpaths in primal



Some maximal planar graphs with biplanar blowups

For each outerpath, draw all four copies of each diagonal edge and two of the four copies of each boundary edge as a planar graph





The story so far

Gethner: Are 2-blowups of planar graphs biplanar?

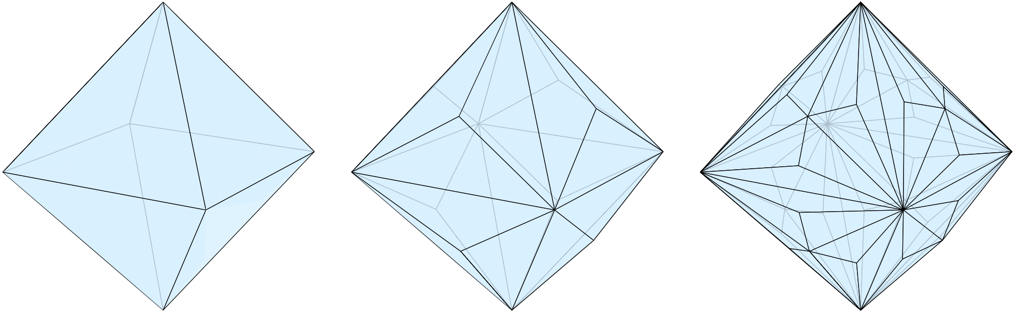
- Yes for triangle-free planar graphs [known]
- Yes for graphs that can be decomposed into two outerpaths [new]
- In general, they have thickness ≤ 3 [known]

Our main result

No!

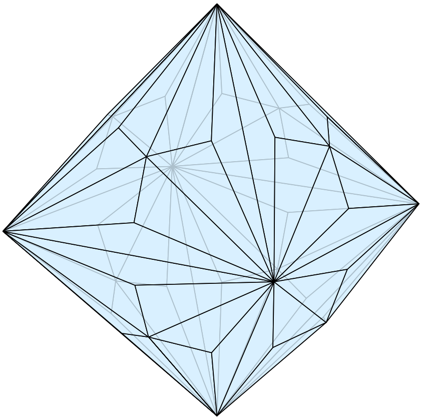
Our counterexamples

Kleetope of a polyhedron: glue a pyramid onto each face



Iterated Kleetope: do the same thing repeatedly, some number of times

Why the outerpath decomposition doesn't work



When we glue in a pyramid, each dual path can only visit two of the three new faces

k-level Kleetope $\Rightarrow$ # faces expands as 3^k

Longest path length expands only as 2^k

Dual paths too short for two to cover all faces

Why iterated Kleetopes can be hard to draw

The original graph is a subgraph of its Kleetope $\Rightarrow$
Drawings of Kleetope contain drawings of original

If drawing a graph is difficult,
drawings its Kleetope can be more difficult

Iterating can make difficulties pile up enough to
make drawing become impossible

Counting again

Blowups of maximal planar have only twelve fewer edges than max biplanar

$\Rightarrow$ In any drawing, most faces are triangles

$\Rightarrow$ $\exists$ vertex whose four images ($\times 2$ from blowup, $\times 2$ from biplanar) are surrounded entirely by triangular faces

Piling on constraints

There exists a vertex whose four images . . .

- . . . are each surrounded by triangular faces (max planar)
- . . . are each surrounded by three triangles (Kleetope)
- . . . are each surrounded by three triangles, sharing ≤ 1 edge with triangles around other images (Kleetope^2)
- . . . are each surrounded by three triangles, edge-disjoint with triangles around other images (Kleetope^3)

$\Rightarrow$ impossible to draw

Generalization: split thickness

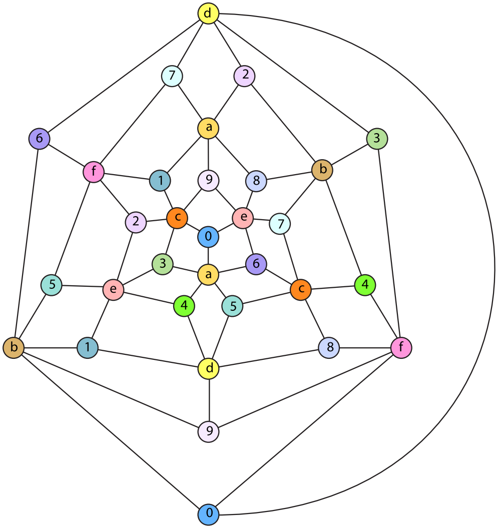
Allow each country to have a colony on the Earth, not on the moon

⇒ drawing with two copies of each node, in a single plane (here: $K_{6,10}$)
[Heawood 1890; Eppstein et al. 2018]

∃ drawings of 2-blowups of more graphs

E.g.: Kleetopes of outerpath decompositions

Same proof shows that 2-blowups of iterated Kleetopes of large maximal planar graphs do not have split thickness two



Summary

Some planar graphs (iterated Kleetopes of large polyhedra) have non-biplanar 2-blowups

Same proof shows that 2-blowup does not have split thickness two

However, graphs with two-outerpath decompositions have biplanar blowups

Their Kleetopes have split thickness two blowups

Some open problems

Are 2-blowups of 3-colorable planar graphs biplanar?

(Kleetopes are not 3-colorable)

Are 2-blowups of 4-vertex-connected planar graphs biplanar?

(Kleetopes are not 4-vertex-connected)

Biplanarity is NP-complete [Mansfield 1983]; is it hard on 2-blowups of planar graphs?

How hard is it to find our two-outerpath decompositions?

Can our two-outerpath drawing be strengthened to geometric thickness?

(Straight-line plane drawing with two edge colors, no monochromatic crossings)

References and image credits, I

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