# The Widths of Strict Outerconfluent Graphs

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## What are they?

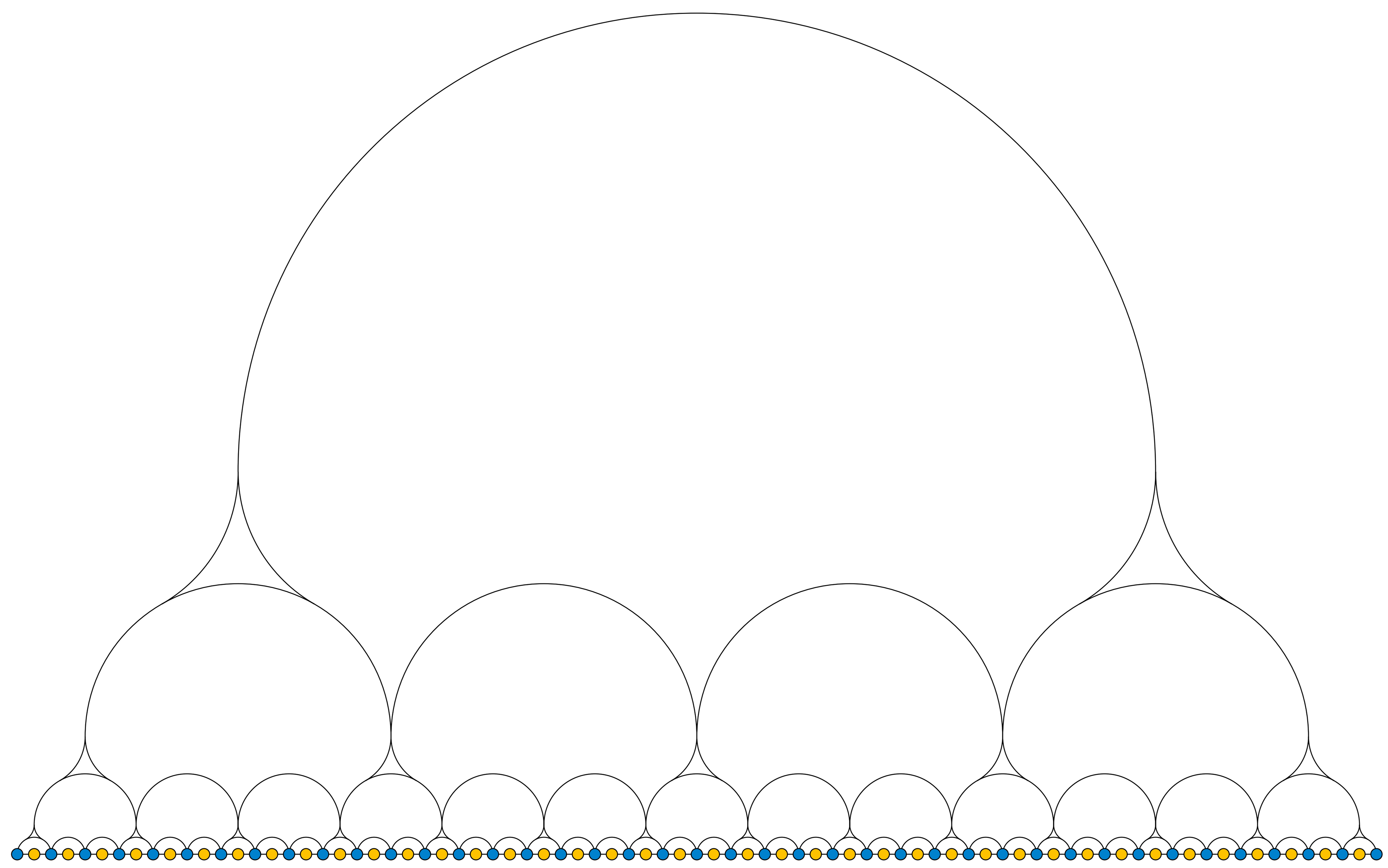
**Outer:** Vertices are placed on the boundary of the drawing area (in this example, at bottom)

**Confluent:** Adjacency is indicated by smooth curves through a collection of tracks meeting at junctions

**Strict:** Each two adjacent vertices have only one smooth curve, and there are no smooth loops

**Can be dense:** Recursively constructed example below has $n = 3^k$ vertices, $\Theta(4^k) \approx n^{1.26}$ edges

**Recognition complexity:** Polynomial for given vertex ordering, otherwise unknown [Eppstein et al. 2016]



# Outerplanar graphs have bounded treewidth. What about outerconfluent graphs?

They can be dense, but bounded-treewidth graphs are sparse ⇒ we must consider other kinds of width

## Clique-width

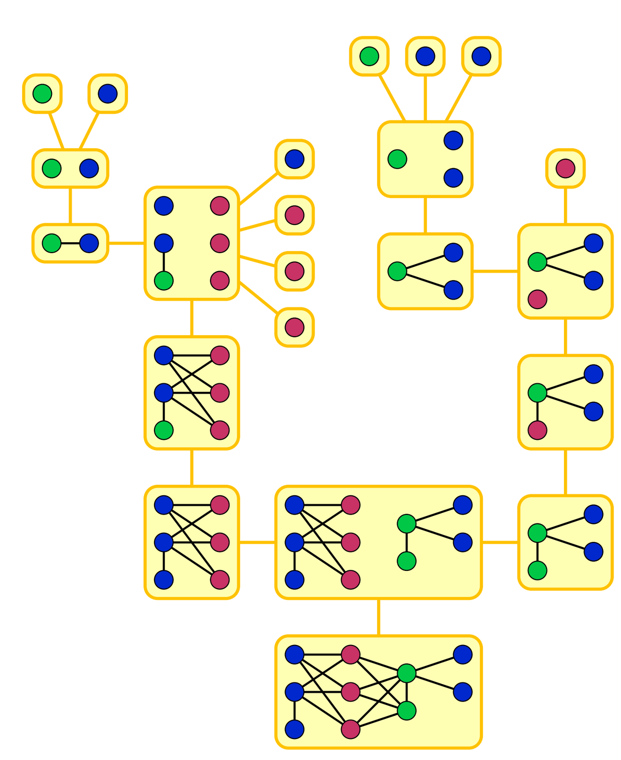
\# colors to construct using disjoint unions, recoloring, and adding 2-color bicliques

Bounded for tree-like, distance-hereditary subclasses of strict outerconfluent graphs [Eppstein et al. 2005; Förster et al. 2021]

Equivalent to treewidth in sparse graphs

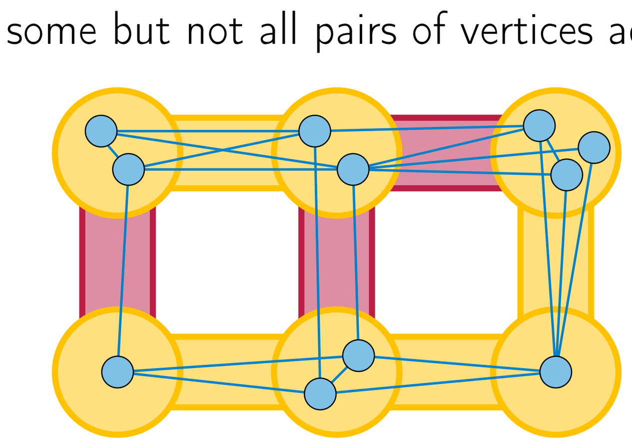
Equivalent to *rank-width* for all graphs

Low rank-width: hierarchical clustering where each split has a low-rank biadjacency matrix [Oum and Seymour 2006]



## Twin-width

For any vertex clustering, define *red graph* of pairs of clusters with some but not all pairs of vertices adjacent



Repeatedly merge from $n$ clusters down to one, keeping red degree small
Twin-width = max degree for merge sequence that minimizes this max

Bounded for planar, $k$-planar, bounded genus, etc. [Bonnet et al. 2021]

## Unbounded clique-width

**Theorem:** Recursive construction above has unbounded clique-width

*Proof ideas:*

- Find balanced vertex split from optimal rank-width clustering
- If split forms many blocks of contiguous vertices, induced matching from pairs of vertices on block boundaries ⇒ high rank
- Otherwise split separates two large blocks with a narrow gap
- Many nested semicircular arches above these two blocks ⇒ pairs of vertices adjacent via distinct arches ⇒ high-rank submatrix

## Bounded twin-width

**Theorem:** Strict outerconfluent graphs have bounded twin-width

*Proof ideas:*

- Ordered graph = graph + linear ordering on vertices
- Small class of graphs: # $n$-vertex graphs $\leq c^n$ for some $c$
- For hereditary classes of ordered graphs (i.e. closed under induced subgraphs & orders), bounded twin-width = small [Bonnet et al. 2022]
- Outerconfluent graphs, ordered by boundary position, are hereditary
- Strict confluent ⇒ $O(n)$ junctions [Eppstein et al. 2016] ⇒ small

## References

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