Planar Lombardi Drawings for Subcubic Graphs

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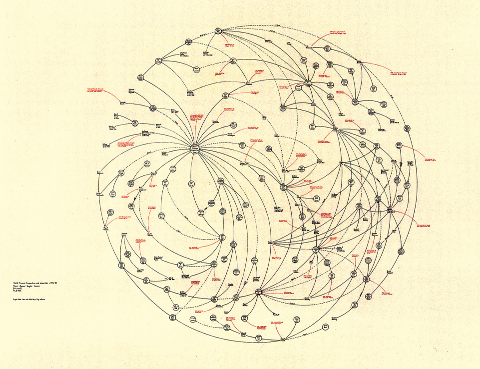
Mark Lombardi



World Finance Corporation and Associates, ca 1970–84: Miami, Ajman, and Bogota–Caracas (Brigada 2506: Cuban Anti-Castro Bay of Pigs Veteran), 7th version, Mark Lombardi, 1999, from *Mark Lombardi: Global Networks*, Independent Curators, 2003, p. 71

American neo-conceptual fine artist (1951–2000)

"Narrative structures", drawings of social networks relating to international conspiracies, based on newspapers and legal documents

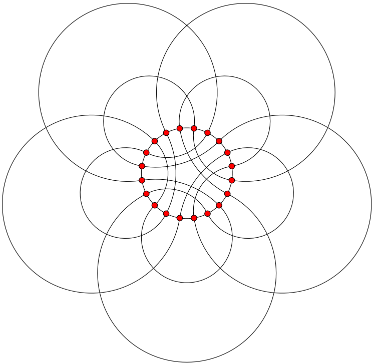
Unlike much graph drawing research, used curved arcs instead of polylines

Lombardi Drawing

A style of graph drawing inspired by Lombardi's art
[Duncan, E, Goodrich, Kobourov, & Nöllenburg, *Graph Drawing* 2010]

Edges drawn as circular arcs

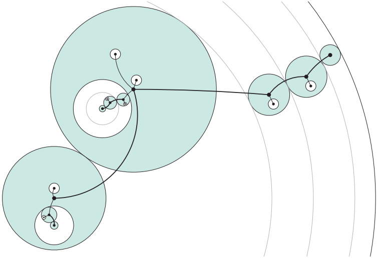
Edges must be equally spaced around each vertex



The Folkman Graph

Smallest edge-transitive but not vertex-transitive graph

Past results from Lombardi drawing



All plane trees (with ordered children) may be drawn with perfect angular resolution and polynomial area [Duncan et al, *GD* 2010]

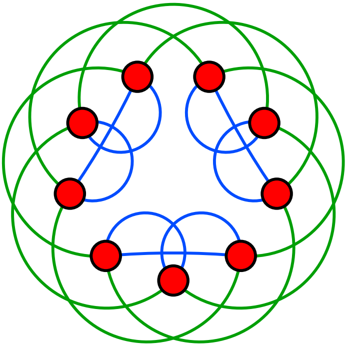
(Straight line drawings may require exponential area)

Past results from Lombardi drawing

k-Regular graphs have drawings with circular vertex placement if and only if

- $k = 0 \pmod 4$,
- k is odd and the graph has a perfect matching,
- the graph has a bipartite 2-regular subgraph, or
- there is a Hamiltonian cycle

[Duncan et al, *GD* 2010]

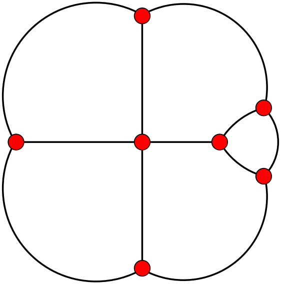


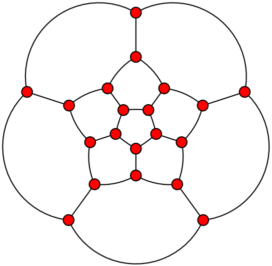
The 9-vertex Paley graph

Past results from Lombardi drawing

Halin graphs and the graphs of symmetric polyhedra have planar Lombardi drawings

[Duncan et al, *GD* 2010]

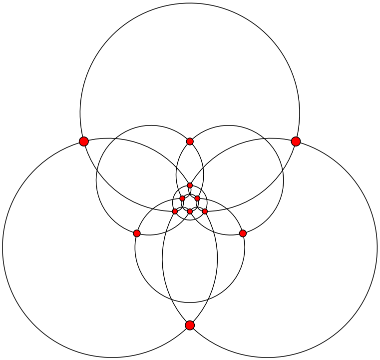


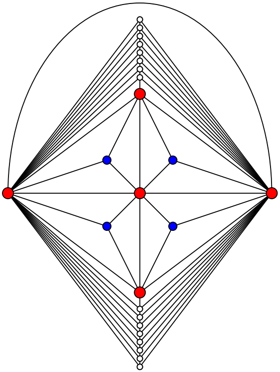


Past results from Lombardi drawing

Not every planar graph has a planar Lombardi drawing

[Duncan et al, *GD* 2010; Duncan, E, Goodrich, Kobourov, Löffler, *GD* 2011]





What we still don't know

Which planar graphs have planar Lombardi drawings?

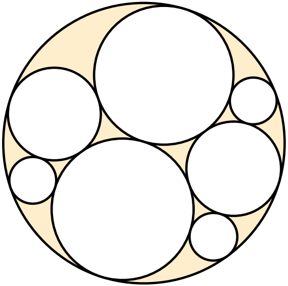
Which regular planar graphs have planar Lombardi drawings?

Do all outerplanar graphs have planar Lombardi drawings?
What about series-parallel graphs, or treewidth ≤ 2?

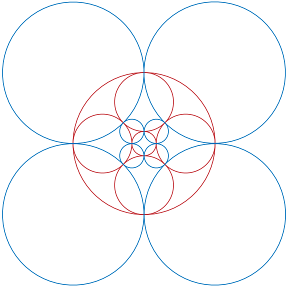
What is the complexity of finding (planar) Lombardi drawings?

Today: Progress on regular and low-degree
planar Lombardi drawings

A key tool: Koebe–Andreev–Thurston circle packing



The vertices of every maximal planar graph may be represented by interior-disjoint circles such that vertices are adjacent iff circles are tangent



The vertices of every 3-connected planar graph and its dual may be represented by circles that are perpendicular for incident vertex-face pairs

Both representations are unique up to Möbius transformations

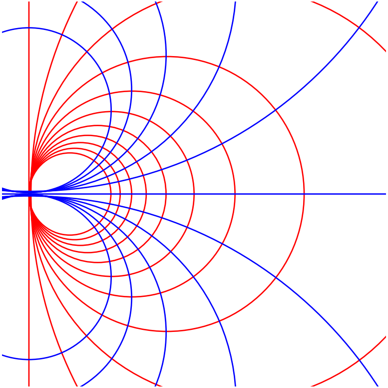
A second key tool: Möbius transformations

If we represent each point in the plane by a complex number, the Möbius transformations are exactly the fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d}$$

and their complex conjugates,

where a, b, c, and d are complex numbers with $ad - bc \neq 0$



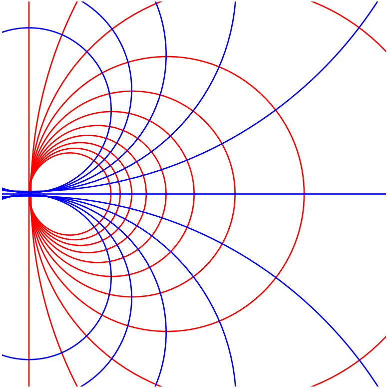
Properties of Möbius transformations

They include the translations, rotations, congruences, and similarities

Conformal (preserve angles between curves that meet at a point)

Preserve circularity (counting lines as infinite-radius circles)

Therefore, a Möbius transformation of a Lombardi drawing remains Lombardi.

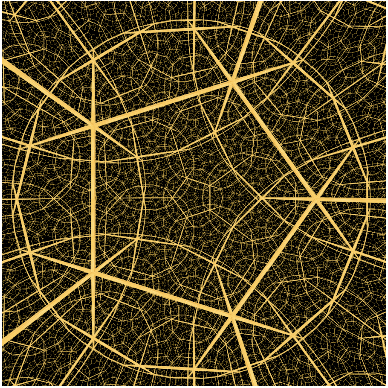


Third key tool: 3d hyperbolic geometry

3d hyperbolic geometry can be modeled as a Euclidean halfspace

Hyperbolic lines and planes are modeled as semicircles and hemispheres perpendicular to the boundary plane of the halfspace

In this model, congruences of hyperbolic space correspond one-for-one with Möbius transformations of the boundary plane



PD image "Hyperbolic orthogonal dodecahedral honeycomb.png" by Tomruen from Wikimedia commons

Hyperbolic Voronoi diagrams of circle packings

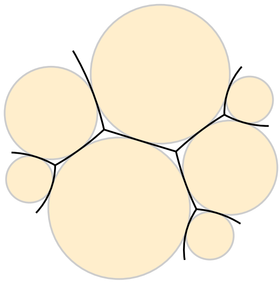
Given circles in the Euclidean plane

View plane as boundary of hyperbolic space

Each circle bounds a hyperbolic plane

Construct the 3d hyperbolic Voronoi diagram of these hyperbolic planes (if circles may cross, use *signed distance* from each plane)

Restrict the Voronoi diagram to the boundary plane of the model

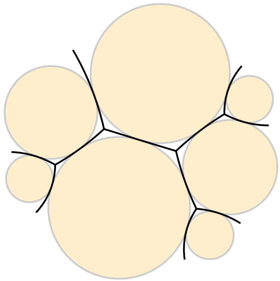


Properties of this hyperbolic Voronoi diagram

Bisector of disjoint 3d hyperbolic planes is a plane ⇒ bisector of disjoint circles is a circle

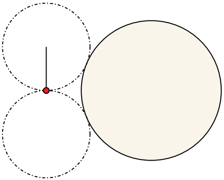
Voronoi diagram is invariant under hyperbolic congruences ⇒ planar diagram is invariant under Möbius transformations

Three tangent circles can be transformed to equal radii ⇒ their diagram is a *double bubble* (three circular arcs meeting at angles of $2\pi/3$ at the two isodynamic points of the triangle of tangent points)

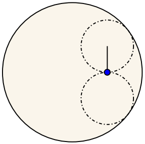


Is this a planar Voronoi diagram? For what distance?

Radial power distance:



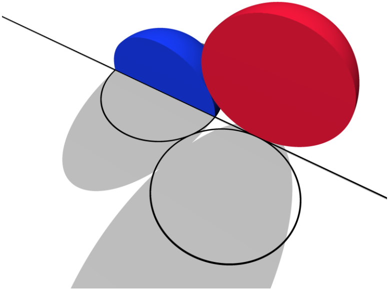
For points outside circle, power = (positive) radius of equal circles tangent to each other at point and tangent to circle



For points inside circle, power = negative radius of equal circles tangent to each other at point and tangent to circle

In either case, it has the formula $\dfrac{d^2 - r^2}{2r}$

Why are Voronoi diagrams for this distance the same as diagrams from 3d hyperbolic geometry?

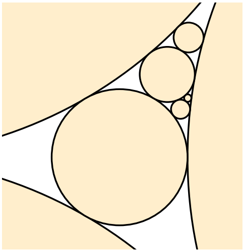


For points in (Euclidean or hyperbolic) 3d space, nearest neighbor = point that touches smallest concentric sphere

For boundary points of hyperbolic space, replace concentric spheres by *horospheres* (Euclidean spheres tangent to boundary plane)

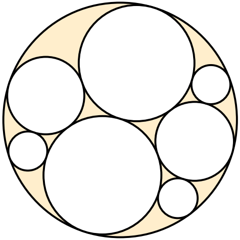
Tangent circles for radial power = cross-sections of horospheres

Lombardi drawing for 3-connected 3-regular planar graphs



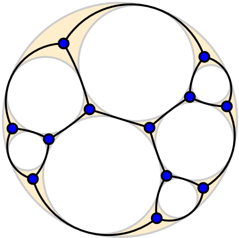
Find a circle packing for the dual (a maximal planar graph)

[Mohar, *Disc. Math.* 1993; Collins, Stephenson, *CGTA* 2003]



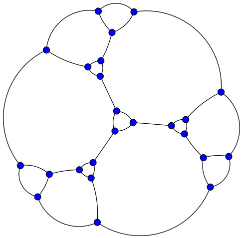
Use a Möbius transformation to make one circle exterior, maximize smallest radius

[Bern, E, *WADS* 2001]



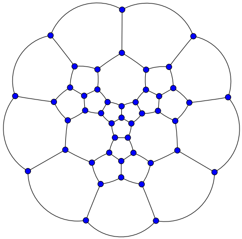
The Möbius-invariant power diagram is a Lombardi drawing of the original graph

Examples of 3-connected planar Lombardi drawings



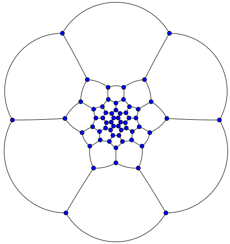
Smallest power-of-two cycle has length 16

[Markström, *Cong. Num.* 2004]



Non-Hamiltonian cyclically 5-connected graph

[Grinberg, *Latvian Math. Yearbook* 1968]



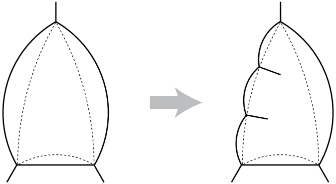
Buckyball (truncated icosahedron)

Lombardi drawing for arbitrary planar graphs of degree ≤ 3

For 2-connected graphs, decompose using an SPQR tree, and use Möbius transformations to glue together the pieces

For graphs with bridges:

- Split into 2-connected subgraphs by cutting each bridge
- Use SPQR trees to decompose into 3-connected components
- Modify 3-connected drawings to make attachments for bridges



- Möbius transform and glue back together

Lombardi drawing for (some) 4-regular planar graphs

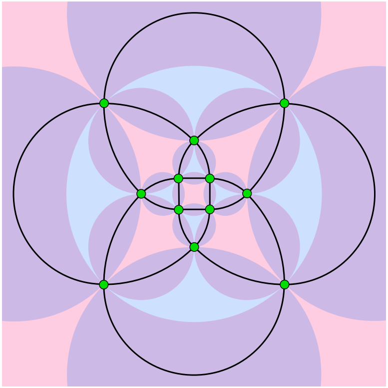
Two-color the faces of the graph G

Construct the incidence graph H of one color class

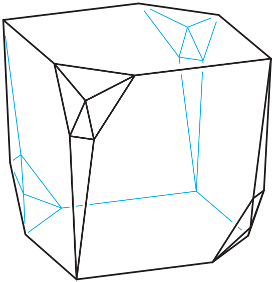
If H is 3-connected, then:

Find an orthogonal circle packing of H and its dual

The Möbius-invariant power diagram is a Lombardi drawing of G

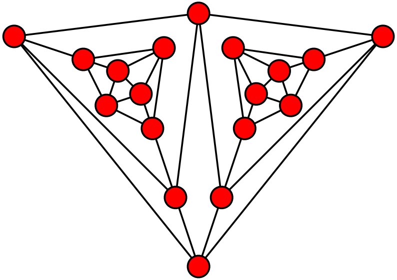


But it doesn't work for all 4-regular graphs



A 3-connected 4-regular graph for which H is not 3-connected

[Dillencourt, E, *Elect. Geom. Models* 2003]



A 2-connected 4-regular planar graph with no planar Lombardi drawing

Conclusions

Every planar graph of maximum degree ≤ 3
has a planar Lombardi drawing

Runtime depends on numerics of circle packing
but implemented for the 3-connected case

4-regular medial graphs of 3-connected planar graphs
have planar Lombardi drawings

But other 4-regular planar graphs
may not have a planar Lombardi drawing

Future work

Much more still remains unknown about Lombardi drawings

The same methods used here to find Lombardi drawings can also be used to understand the combinatorial structure of soap bubbles.