Tangent Spheres and Integer Distances

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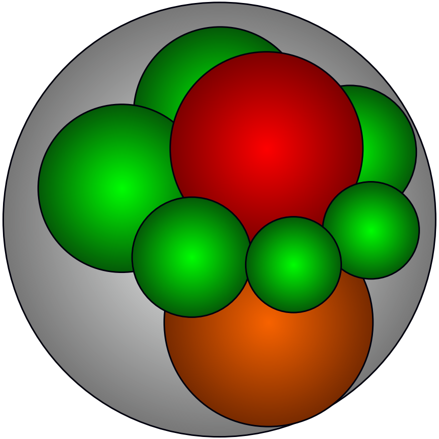
Soddy's hexlet



For any three mutually tangent spheres (gray, red, orange)

∃ 6-cycle of consecutively-tangent spheres (green), tangent to given spheres

Recorded on an 1822 sangaku tablet from Samukawa Shrine, Kanagawa, Japan

Rediscovered by Soddy [1937]

Replica of the original tablet

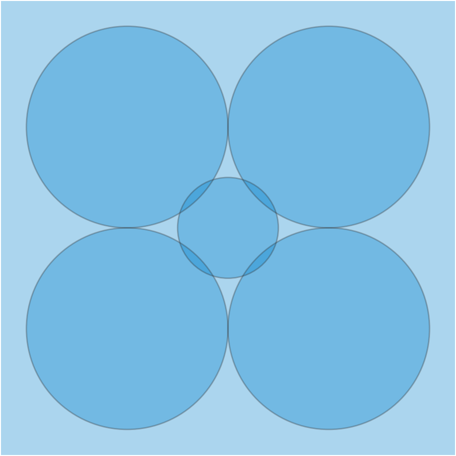
Dedicated in May 1822 (Bunsei 5) by Irisawa Shintarō Hiroatsu, a student of Uchida Itsumi

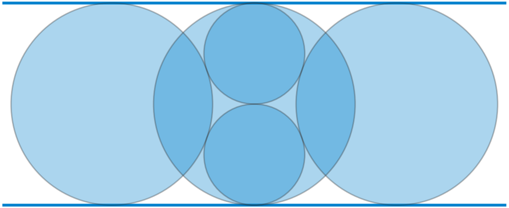
Reconstructed in August 2009 based on the *Kokon Sankan* (Ancient and Modern Mathematical Mirror) held by the Samukawa Shrine Hotoku Museum

Two four-cycles of spheres

Make a square of congruent spheres between two parallel planes (degenerate spheres)

Two half-height spheres fit into the hole in the center of the square





Bigger bicliques of sphere tangencies

Ball bearings are commonly arranged as congruent spheres with centers on a circle

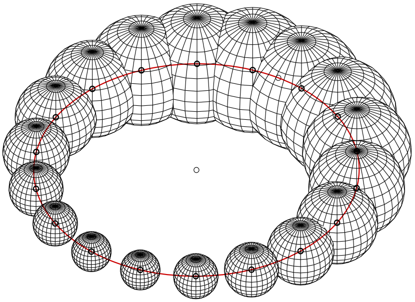
Any point on the perpendicular axis of the circle is the center of another sphere, tangent to all the ball bearings

The tangency pattern is a biclique $K_{a,b}$

The number of circles a in the ball bearing race and the number of circles b centered on the axis can both be arbitrarily large (allowing non-tangent circles to overlap)

More general bicliques

Dupin cyclide: inversion of a torus



Arbitrarily many tangent interior spheres

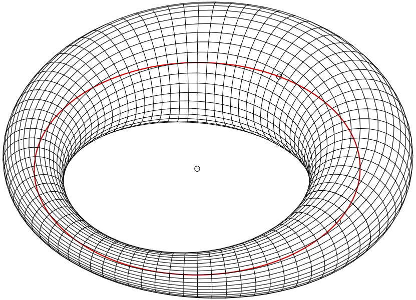
+ arbitrarily many tangent exterior spheres

together form a biclique of tangencies

Red curve through the inner sphere centers

+ corresponding curve through outer centers

are each conics in a plane

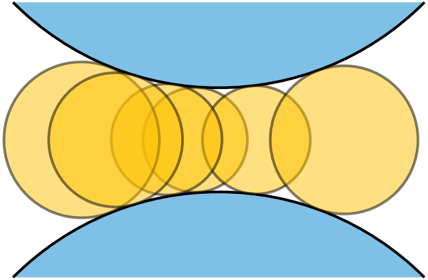


In other dimensions

In 2d: largest bicliques of circles are $K_{2,n}$

In 3d: can form bicliques of spheres with centers on two plane conics

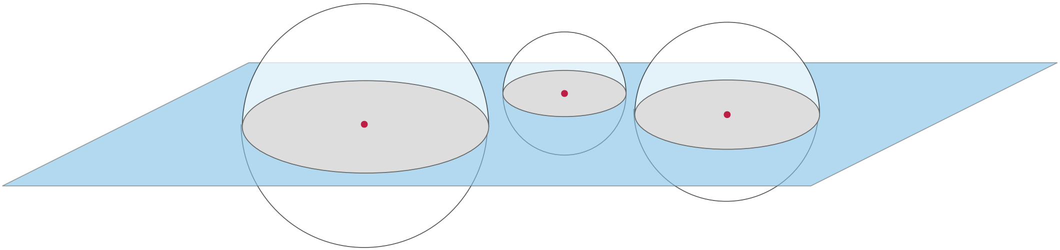
Maybe higher dimensions even more flexible?



Higher-dimensional bicliques remain constrained

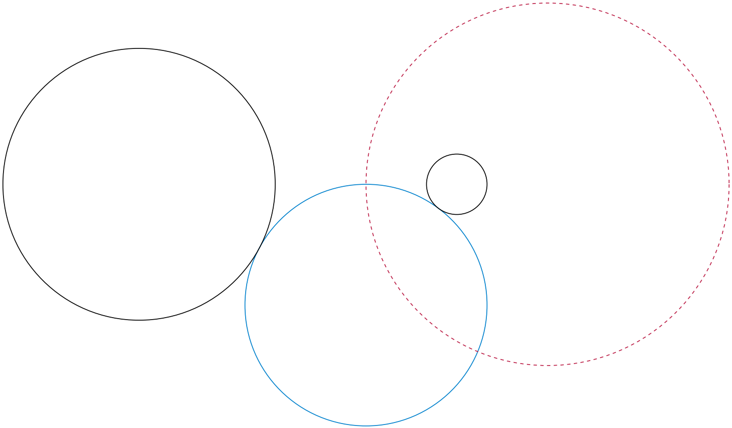
Let A and B be two sets of ≥ 3 hyperspheres in d-dimensional Euclidean or hyperbolic space, so that every pair of one sphere from A and one sphere from B are externally tangent

Then the centers of the spheres in A (and symmetrically in B) lie on a hyperplane



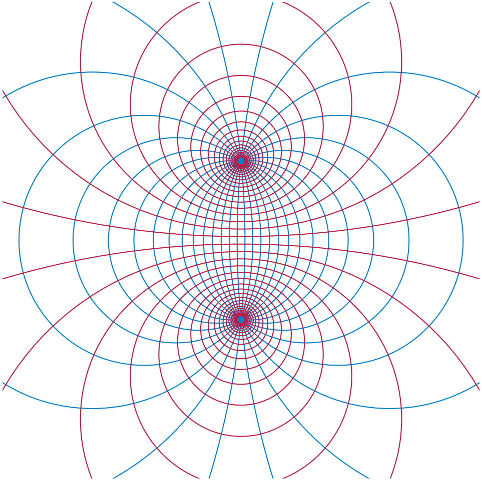
Proof outline, I: An orthogonal midsphere

- Each A_i and A_j can be inverted to each other through a sphere of antisimilitude M_{ij}
- Inversion preserves tangency so inverse(B_k) still tangent to A_i and A_j, must equal B_k
- Therefore M_{ij} is orthogonal to B_k



Proof outline, II: From one to many orthogonal spheres

- M_{12} and M_{23} define a one-dimensional pencil of spheres
- When two spheres in a pencil are orthogonal to B_k, all of them are
- The spheres in the pencil cover space so one of them contains center of B_1
- This orthogonal sphere through the center of B_1 is a hyperplane
- This hyperplane, orthogonal to all B_k, contains all their centers
- QED



Global positioning system

You receive signals from multiple satellites telling you “I was at this location at this time”

Satellites have precise and synchronized clocks

Your clock is precise but unsynchronized

From the relative times you received the signals, you can determine the difference of distances between your location and any two satellite locations

From differences of distances to sphere tangencies

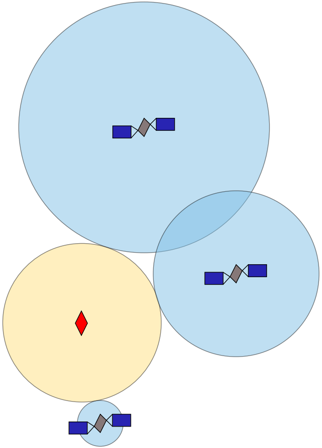
Among k (unknown) satellite distances d_i, let d_ℓ be min
(all other satellites have a positive difference of distances)

Choose small ε, and center a sphere around each satellite of positive radius $d_i - d_\ell + \varepsilon$

Uses only difference of distances $\Rightarrow$ you can construct it

Your location must be the center of a sphere of radius $d_\ell - \varepsilon$ tangent to all the other spheres

Four non-coplanar satellites $\Rightarrow$ cannot be the planar side of a $K_{4,3}$ $\Rightarrow$ only two such centers
(one of which is probably not near the Earth's surface)



Therefore: Every non-planar set of four satellite locations gives you ≤ 2 possible positions

The Erdős–Anning theorem

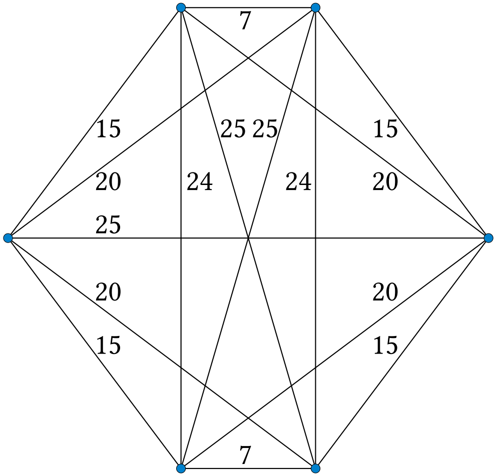
Points in the plane with integer distances are finite or collinear [Anning and Erdős 1945]

Original proof via trigonometric inequalities

Proof via crossings of hyperbolas [Erdős 1945] (missing case analysis of degeneracies), also found by Choquet & Kreweras

Handwavy claims to generalize to higher dimensional Euclidean space

Alternative proof via additively weighted Voronoi diagrams generalizes to many other two-dimensional metrics (e.g. hyperbolic) but not hyperbolic space [Eppstein 2026]



Erdős and Anning in space!

Given points in any finite-dimensional Euclidean or hyperbolic space with integer distances

(dim > 1, allowing any scale for hyperbolic distance)

Choose $d + 1$ points that span the space

(≥ 3 of them, cannot be on a hyperplane)

These are your GPS satellites

Two GPS locations for $d + 1$ choices of closest satellite and $(\text{diam} + 1)^d$ integer difference vectors

$\Rightarrow$ Total points $\leq 2(d + 1)(\text{subset diameter} + 1)^d$

Therefore: Points in $\mathbb{R}^d$ or $\mathbb{H}^d$ with integer distances must be finite or collinear

References

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Samukawa tablet: Image from `http://www.wasan.jp/kanagawa/samukawa.html`

Giant ball bearing race with Konrad Österström sitting inside it, at the the Göteborgs Jubilee exhibition, 1923: Public domain image from Wikimedia commons,
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