Decremental Greedy Polygons and Polyhedra Without Sharp Angles

David Eppstein

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A classical algorithm for graph degeneracy

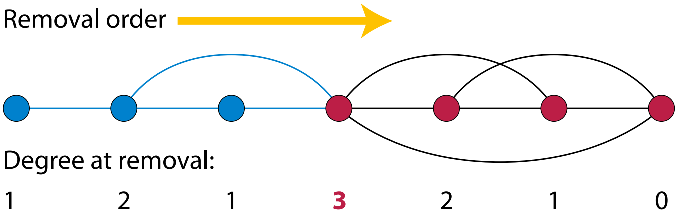
$$\text{Degeneracy} = \max_{\text{subgraph } S} \min_{v \in S} \text{degree}(v)$$

Repeat: remove the minimum degree vertex

The degeneracy is the largest degree d encountered along the way

The subgraph S just before the first degree-d removal is optimal

[Matula and Beck 1983]



General abstract nonsense

Measure quality of an element in a set, $x \in S$, by a function $q(x, S)$ that decreases as we remove elements from S (like degree of vertices in subgraphs)

Then, to find max-min quality set, $\max_{S \subset U} \min_{x \in S} q(x, S)$:

Repeatedly remove any element whose quality is $\leq$ best quality of any set seen so far

Each removal is safe
(removed element cannot be part of better set)

The best set seen along the way is optimal

More general abstract nonsense

Any set S of quality $Q(S) = \min_{x \in S} q(x, S)$ that is better than earlier sets
is the unique largest set of quality $\geq Q(S)$

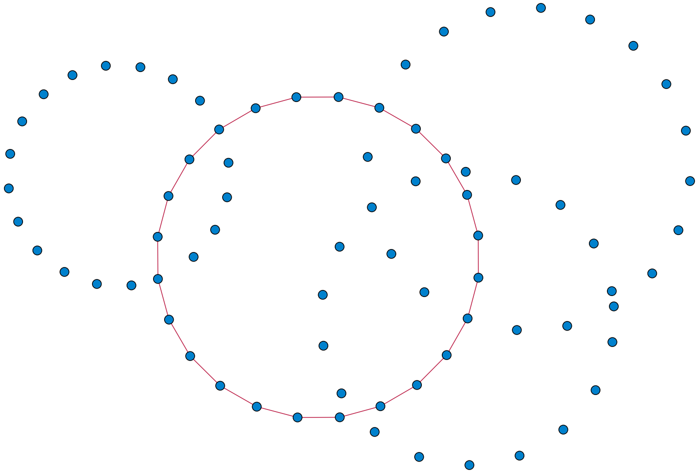
(It is the union of all subsets of quality $\geq Q(S)$)

Once an element becomes available to be removed (its quality is $\leq$ best seen so far)
it remains available until it actually is removed

Becoming available depends only on removed set and not removal order
(x available $\iff$ have already reached largest optimal set among sets containing x
and x's quality has dropped to the quality of this set or less)

(These are the defining properties of an *antimatroid*)

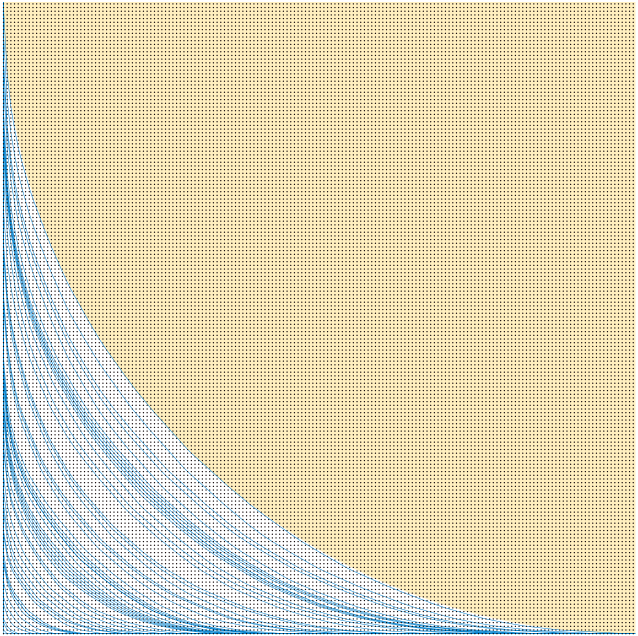
Max-min-angle polygon with vertices in a given set of points



Quality of a point = its angle as a convex hull vertex, or 2π if interior to hull

Repeatedly remove sharpest convex hull vertex, return best hull $\Rightarrow$ time $O(n \log n)$

Successive max-min-angle curves in a quarter-infinite grid



Some of these look like a quarter-circle!

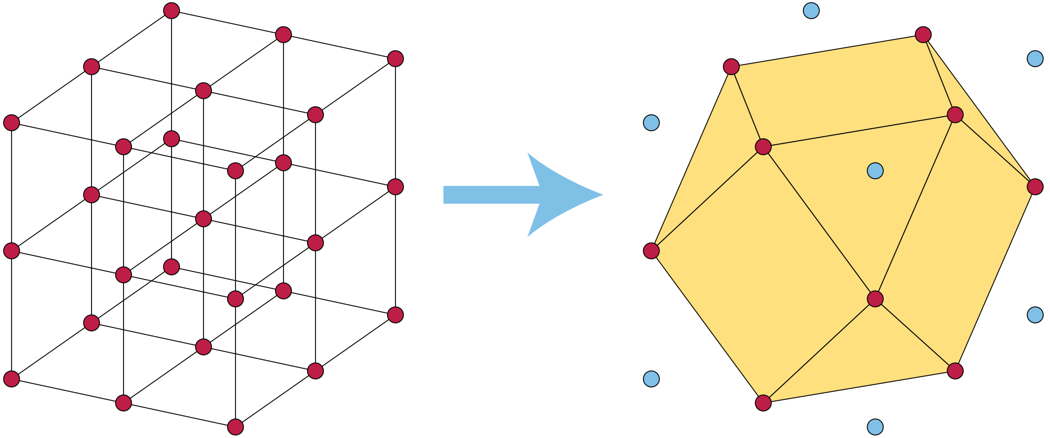
. . . but not all of them

. . . and spacing is irregular

Why??

Max-min solid angle polyhedron

Same algorithm



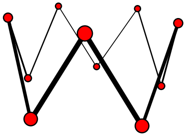
$O(n^2)$ using decremental maintenance of explicit 3d convex hull [Overmars 1983]

What about polygons in 3d?

Elements: triples of points pqr defining angles

Quality(pqr, S)

- The angle of triple pqr, if S also includes angles with p as middle point and with r as middle point
- 0 otherwise



Any nonzero-quality subset $\Rightarrow$ set of angles that includes a closed polygonal curve

Decremental greedy algorithm is $O(n^3 \log n)$, but we can do better!

Graph formulation

Directed line segment $p \to q \Rightarrow$ graph vertex

Directed angle $p \to q \to r \Rightarrow$ graph edge, weighted by angle

Problem becomes: Find bottleneck directed cycle

Decremental greedy algorithm:

Elements: graph edges

Quality of edge $p \to q$ in S:

- 0, if p has no incoming edges
- 0, if q has no outgoing edges
- Its weight, otherwise

$O(m \log n)$, but $O(m)$ if edges are already sorted

Faster bottleneck cycles & 3d polygons

Follow a partitioned search strategy used for other bottleneck optimization problems by Gabow and Tarjan [1988]:

Partition graph edges into too small, too big, and undetermined
(initially all edges are undetermined)

For $i = 0, 1, \ldots \log^* m$:

- Find 2^{2^i} k-medians among $O(m/2^{2^{i-1}})$ undetermined edges
- Partially sort undetermined edges into blocks between k-medians
- Use presorted algorithm to find block containing the optimal weight
- Leave that block undetermined; the rest are too small or too big

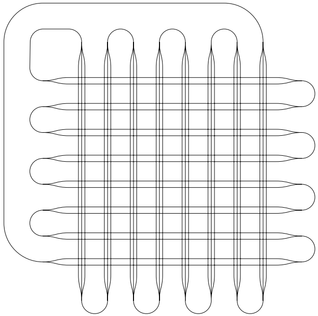
Time $O(m \log^* n)$ for bottleneck cycle in directed graph

Time $O(n^3 \log^* n)$ for max-min angle closed polygonal walk in 3d points

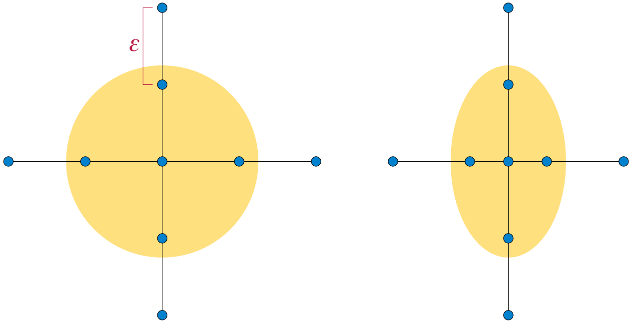
But the result might not be a simple polygon!

It can repeat points and use the same segment in both directions

With more care and more time we can avoid the repeated segments
(but still allowing repeated points)



Reduction from 3SAT,
mapped onto sphere



Two kinds of crossing gadget

Finding the max-min angle simple (or unknotted) 3d polygon is NP-hard

Conclusions

Greedy decremental algorithms for:

Max-min angle (simple, convex) polygon in 2d

Max-min solid angle polyhedron in 3d

Max-min angle closed polygonal walk in 3d

Bottleneck cycle in directed graphs

General framework of abstract optimization including graph degeneracy

But:

3d max-min angle simple polygon is hard

Image credits and references

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Regular skew decagon, CC-BY-SA image by Wikimedia Commons editor A2569875, July 27, 2016,
`https://commons.wikimedia.org/wiki/File:Regular_skew_decagon.svg`

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