Locked and Unlocked Smooth Embeddings of Surfaces

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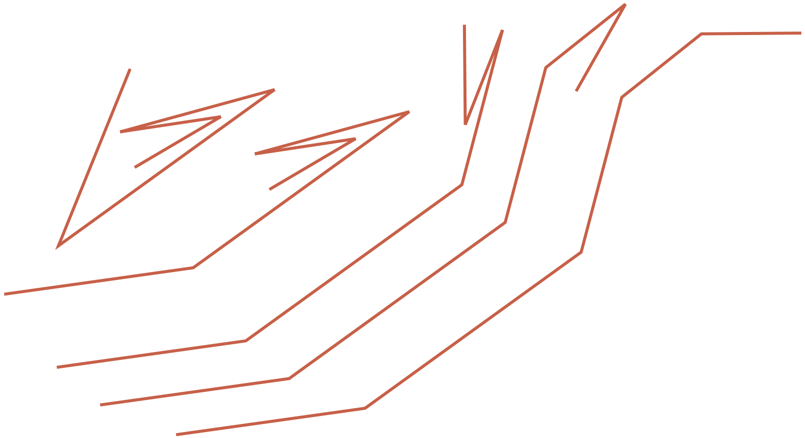
Carpenter's rule

Every polyline in the plane can be straightened!

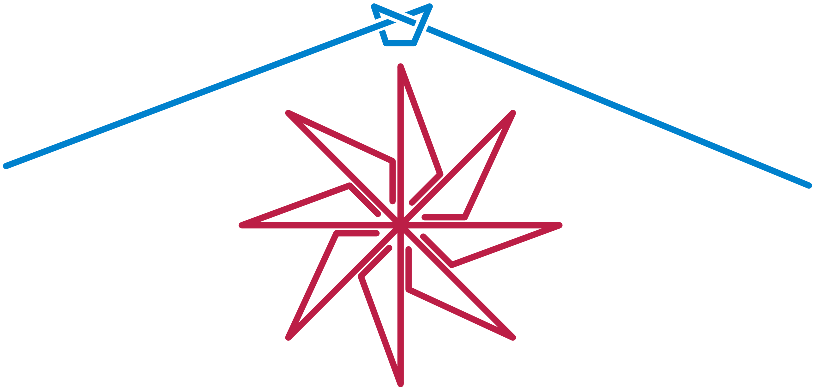
Line segments are rigid bars, connected at hinged vertices



[Connelly et al. 2003; Streinu and Whiteley 2005]

Locked linkages

2D trees and 3D polylines can be unstraightenable
even though they are topologically unlinked

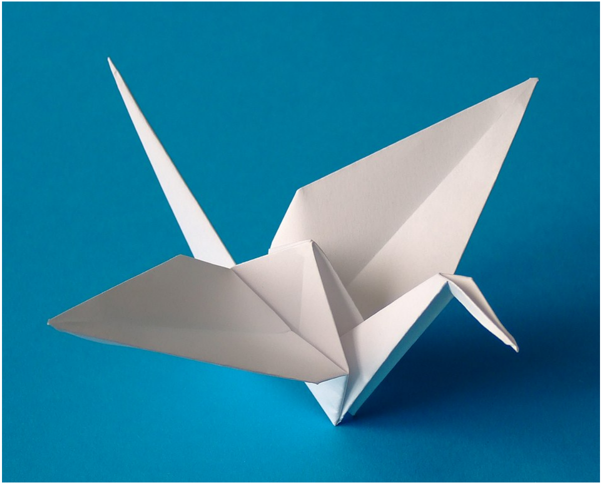


[Cantarella and Johnston 1998; Biedl et al. 2002]

Non-smooth 3D surfaces can be extremely flexible

[Borrelli et al. 2012]

Folded disks are never locked



[Bauer 2006]

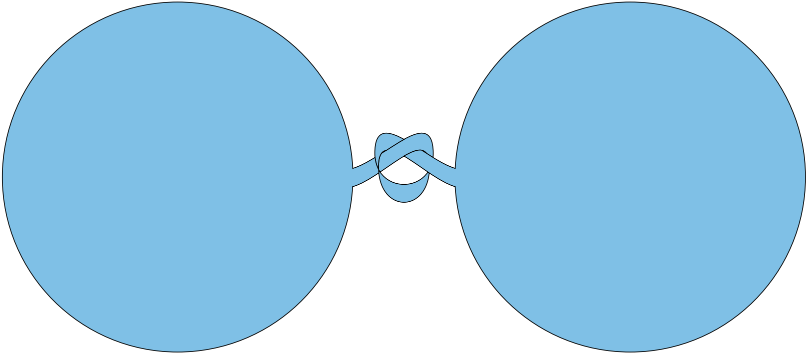
...but flattening may involve "rolling" folds across surface
[Demaine and Mitchell 2001; Demaine et al. 2004]

Smooth surfaces can be quite rigid

Familiar examples: rolled posters, bent pizza slices

[Simonson 2010]

Is this locked as a smooth surface?

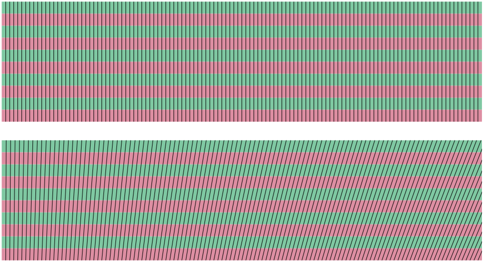


Appears to be: If you leave the disks flat, they don't fit through the knot, and if you roll them, they become rigid tubes, resembling the 3D locked polyline

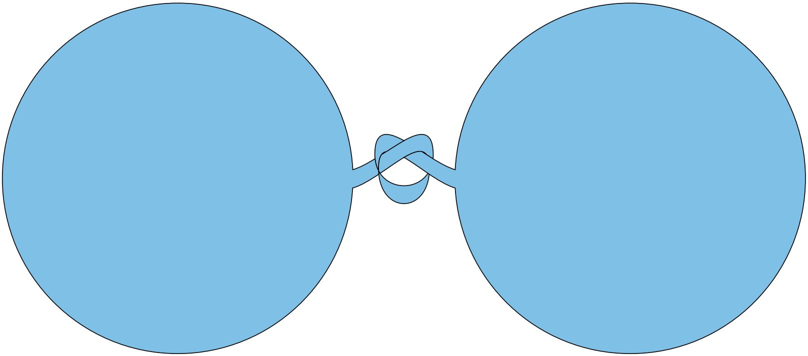
Rolled tubes can still twist

Extend paper yo-yo $\Rightarrow$ bend lines rotate on its surface

[Zhonghua88 2016]



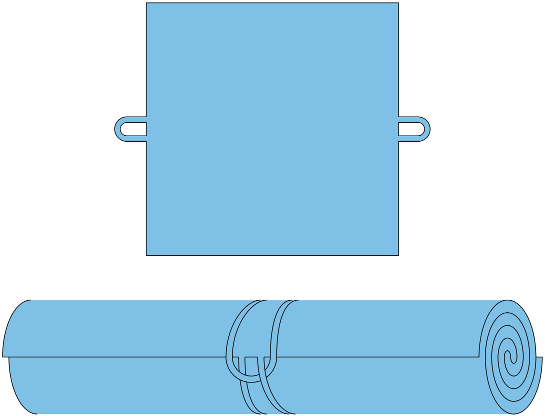
Is this smooth surface locked? NO!



Roll one disk, poke end into knot, then twist the roll

Twisting causes rolled disk to extend through knot until unknotted

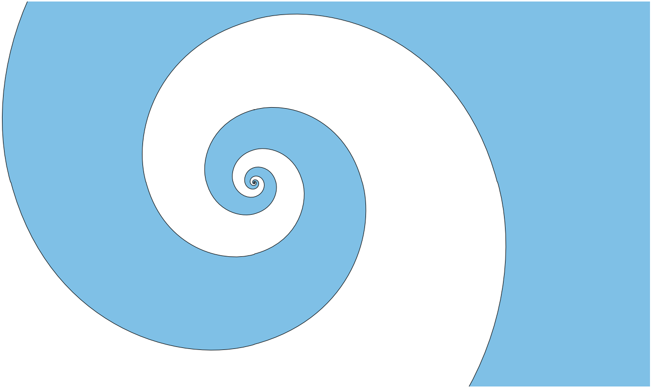
A surface that actually is locked



Proof idea: Rolled tube forms Borromean rings with two small loops, forcing them to stay near each other near center of tube; tangled loops prevent tube from unrolling; tube is too long to pull loops around ends and cannot twist far enough to make shorter

Main result

Let K be a compact subset of the plane with a continuous shrinking motion into itself (for example, a star-shaped polygon)



All smooth embeddings of K can be smoothly flattened

Main idea of proof

Let f embed K into $\mathbb{R}^3$

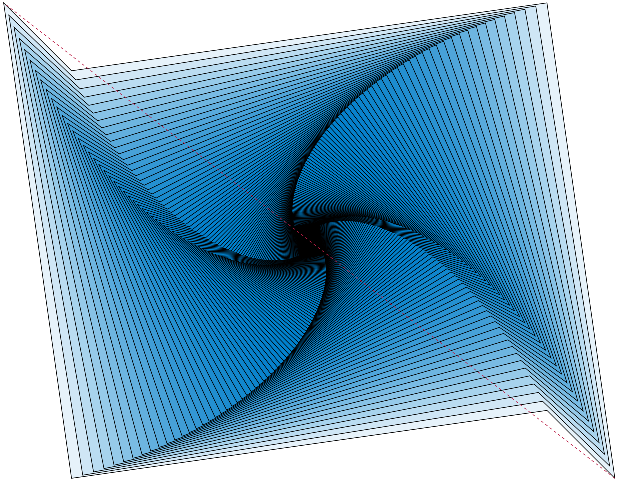
Let s_x, $x \in [0,1]$ be continuous shrinking by a scale factor of x

Compose and re-scale back to original size: $\dfrac{1}{x} f(s_x(K))$

(With a little care to prevent spinning in the limit as $x \to 0$)

Open problem

How fast to recognize polygon with continuous shrinking motion?



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