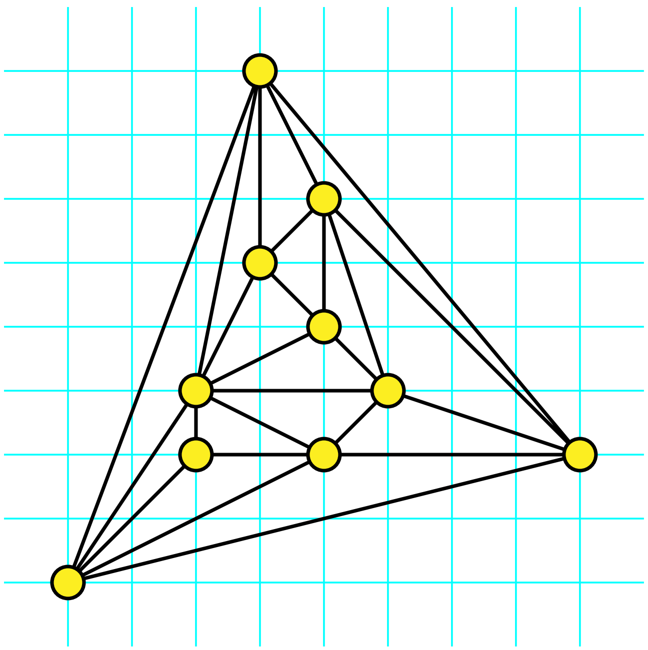
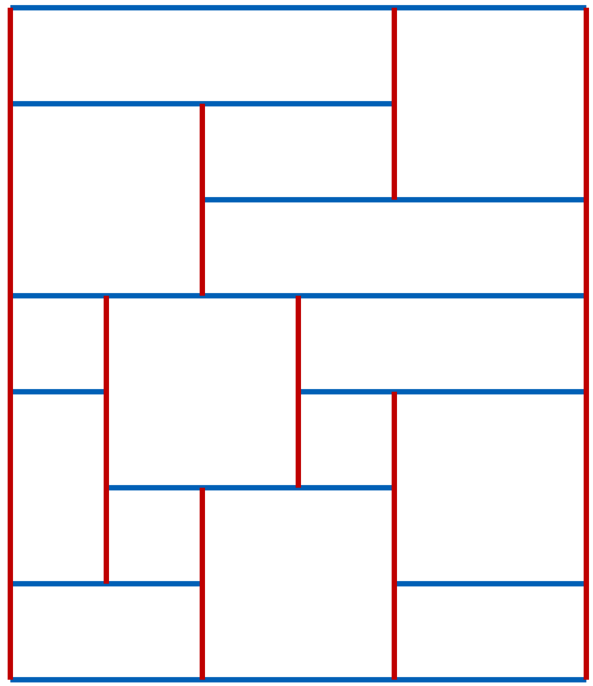
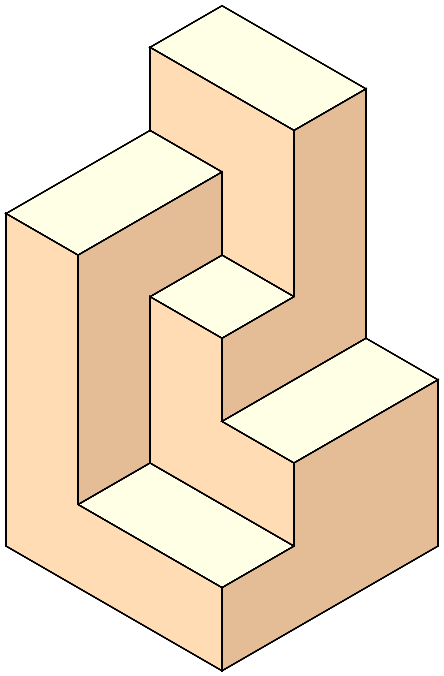
Regular Labelings and Geometric Structures







David Eppstein
Computer Science Dept.
Univ. of California, Irvine

What's in this talk?

Three problems of constructing a geometric output
from a combinatorial input:

grid embeddings of planar graphs

partitions of rectangles into smaller rectangles

polyhedra with axis-parallel faces

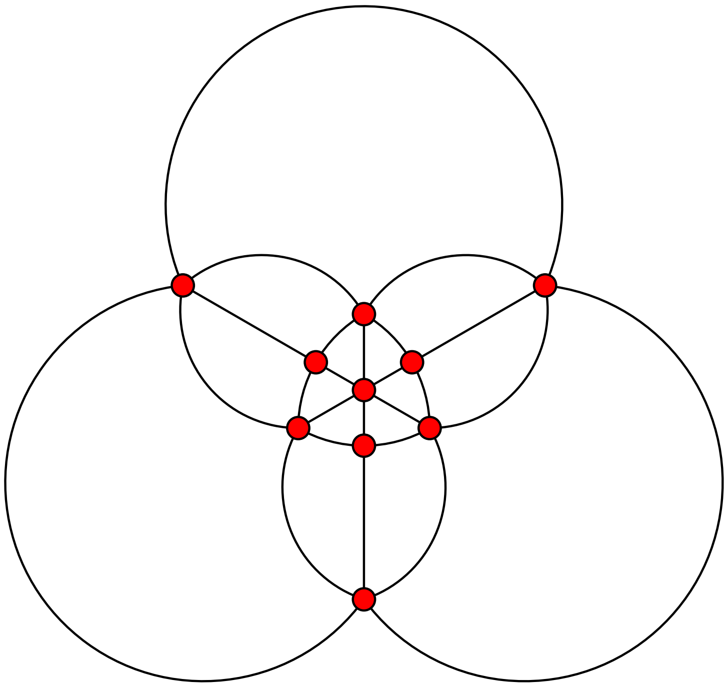
Combinatorial descriptions of these objects
as "regular labelings" of (near-)maximal planar graphs
by edge orientations and colors

Unexpected analogies between these types of labeling
and underlying structure as a distributive lattice

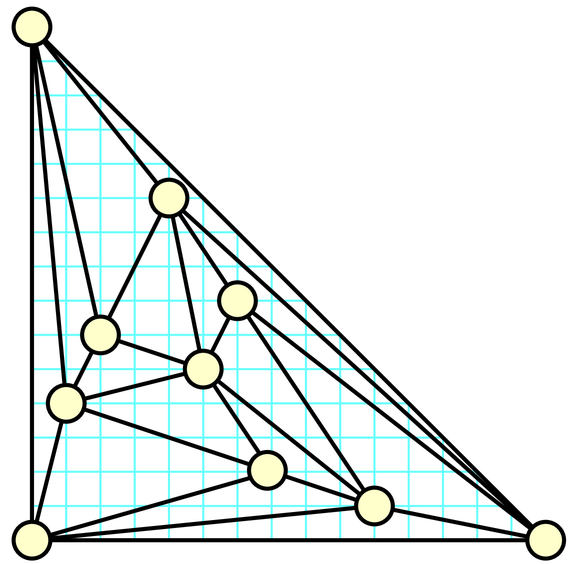
...leading to efficient algorithms for constructing these structures

Grid embedding problem

Given as input a planar graph



Produce as output integer coordinates for the vertices, describing a planar embedding of the graph



Combinatorial part (not today): cyclically order edges around each vertex

Geometric part: place the vertices in a grid

Preliminaries on planar embedding

The cyclic ordering of edges around each vertex uniquely defines faces of a surface embedding (cycles of edges that are consecutive in the ordering)

The embedding might be nonplanar, but is planar iff $V - E + F = 2$ (Euler's formula)

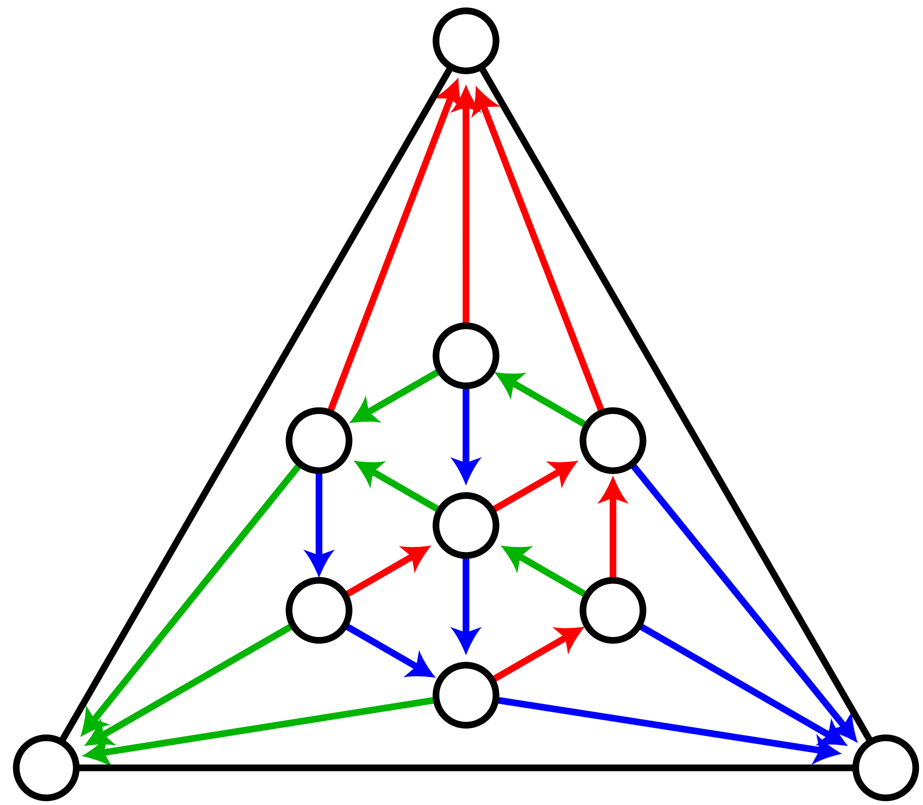
A planar embedding, represented by cyclic orderings, can be constructed in linear time [Hopcroft & Tarjan]

Without loss of generality, all faces are triangles (if larger faces exist, split them by adding edges). Equivalently, the graph is maximal planar.

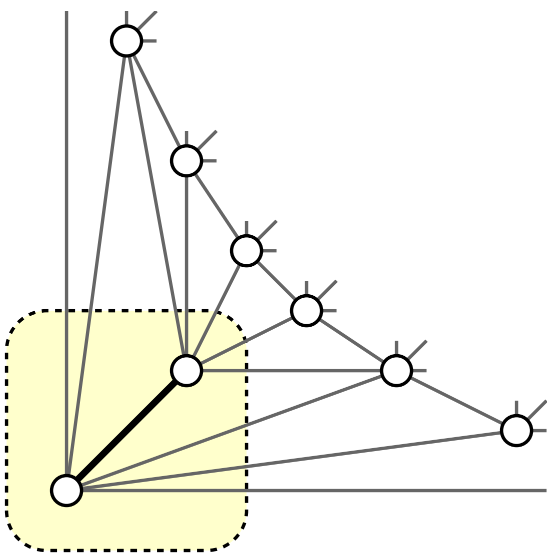
Schnyder's regular labeling [Schnyder, SODA 1990]

Given a maximal planar graph, cyclically ordered at each vertex:

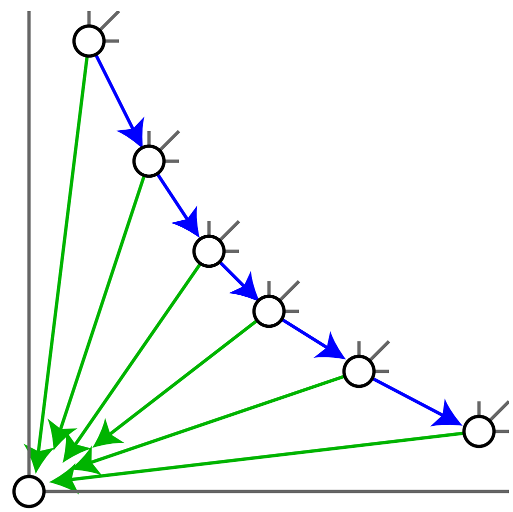
- Choose the outer triangle
- Orient each remaining edge and color it red, green, or blue
- Outer vertices have only incoming edges, in one color
- Inner vertices have cyclic order out-red, in-green, out-blue, in-red, out-green, in-blue (one out in each color)



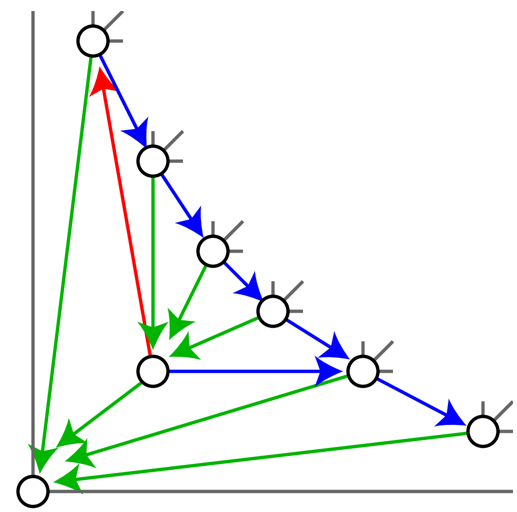
Existence and construction of regular labelings



Contract an edge adjacent to an outer vertex



Label the contracted graph recursively

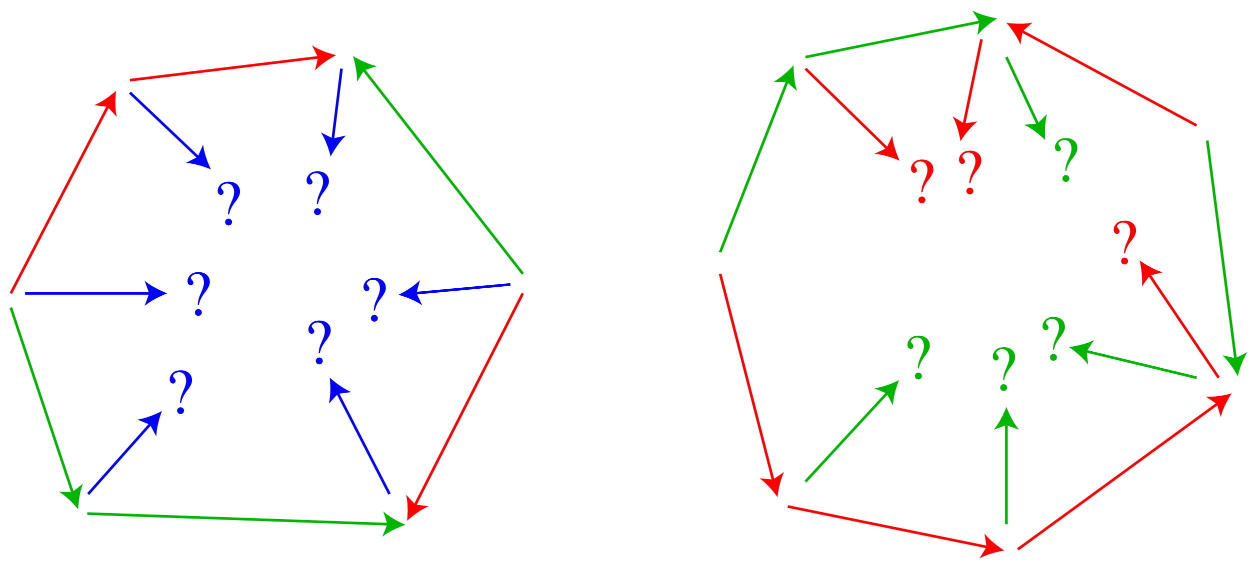


Uncontract and locally relabel

Acyclicity

Form a subgraph from the edges of two colors in a regular labeling
but reverse the orientations of the edges of one color

The resulting subgraph is st-planar
(acyclic, with a single source and sink, both on the outer face)



Proof idea: a cycle would lead to inward paths with nowhere to go

Corollary: edges of a single color form a tree (acyclic, outdegree = 1)

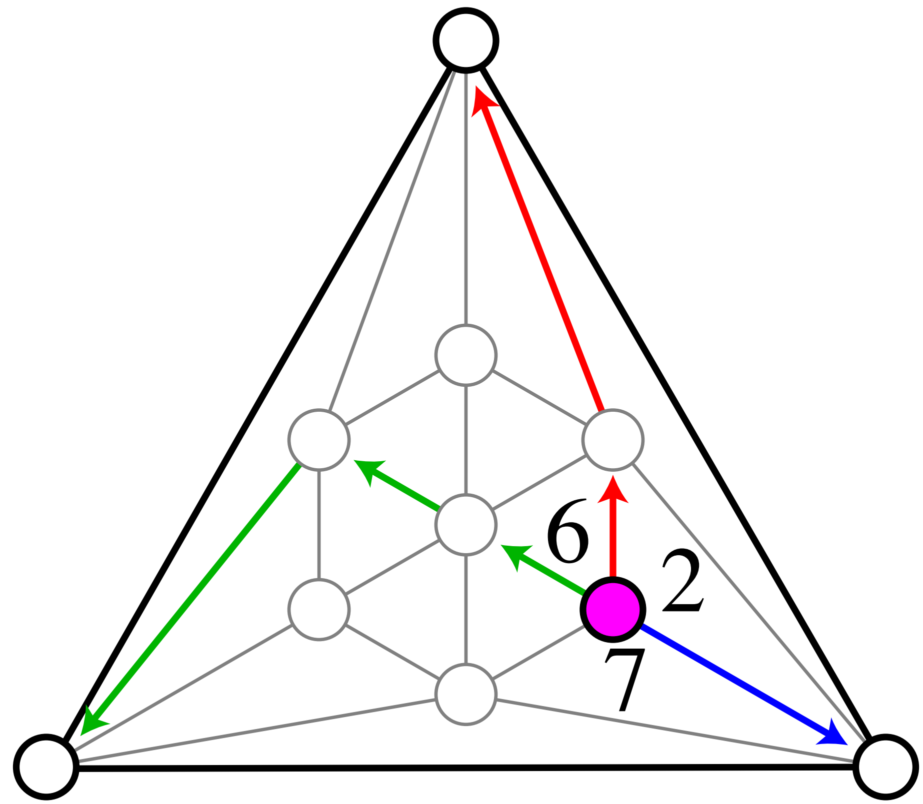
Root paths and face counts

Each inner vertex has one-color paths to each outer vertex
(one-color subgraphs are trees)

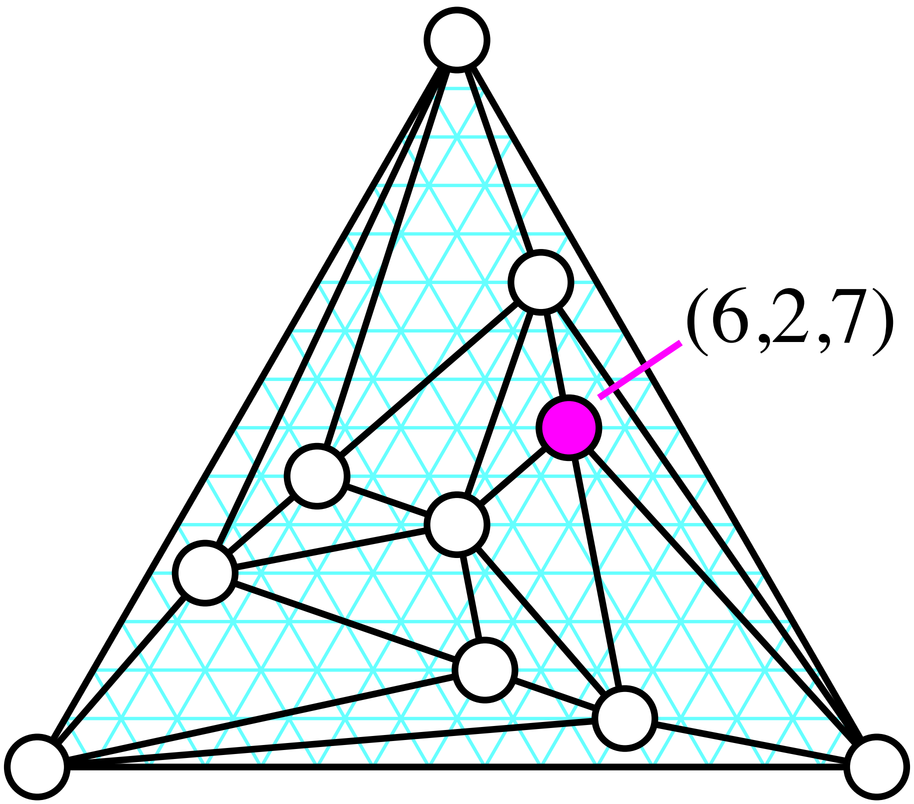
These paths don't cross
(a crossing would form a two-color cycle with one color reversed)

So they partition the faces of the graph into three regions

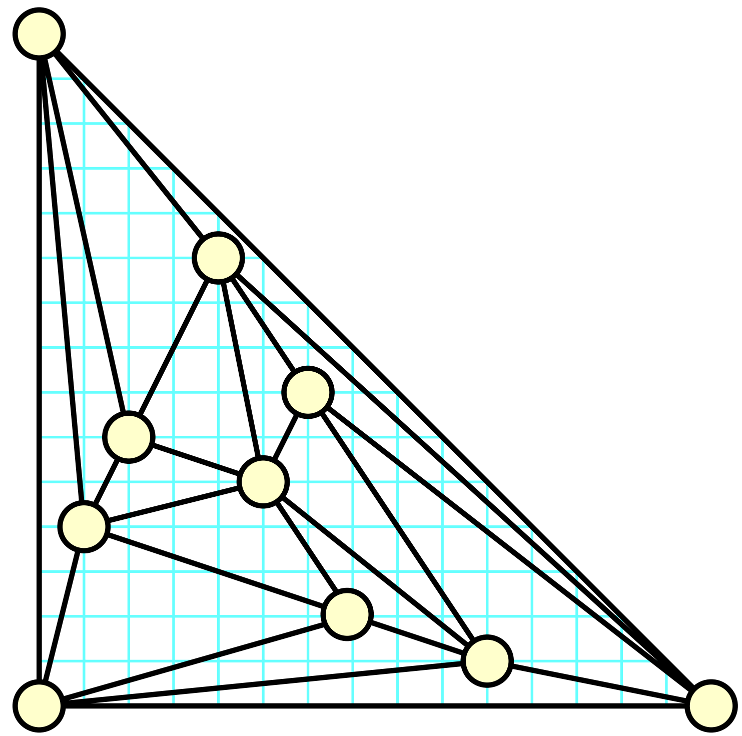
The counts of faces in these regions add up to F – 1



Schnyder's grid embedding

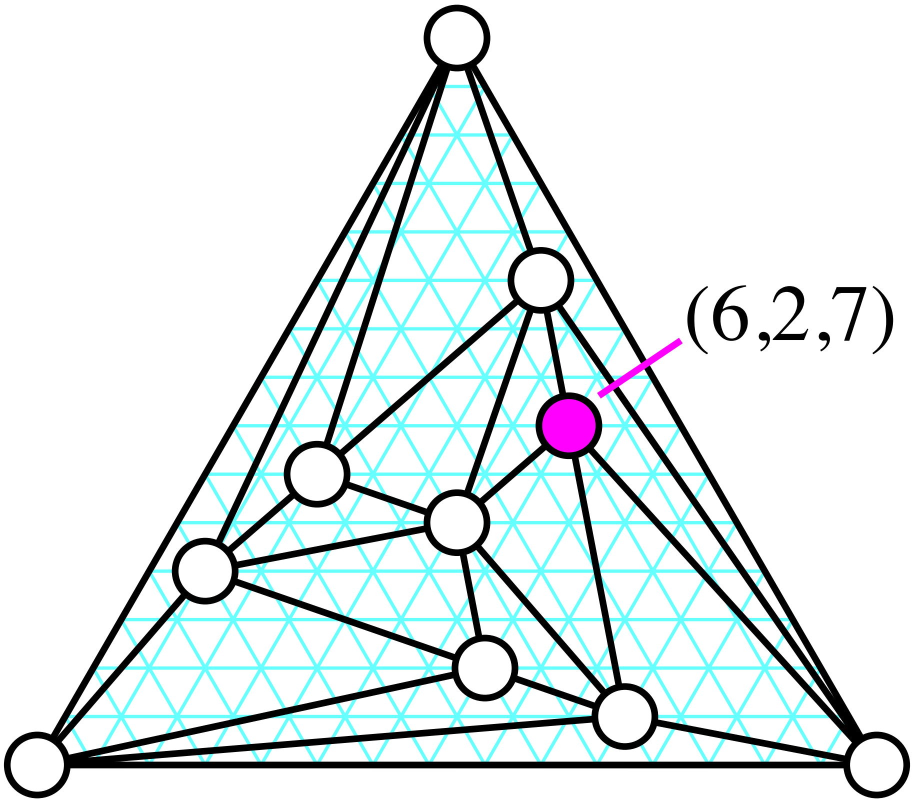


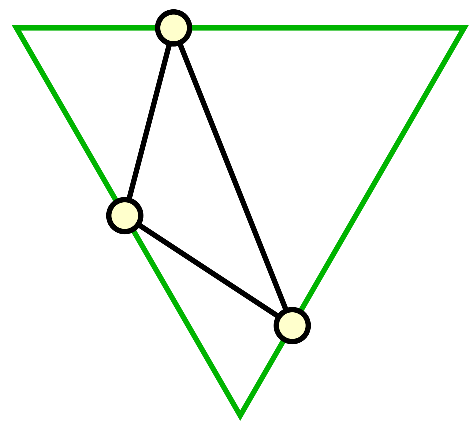
Use face counts as barycentric coordinates (distances from edges) in an equilateral triangle grid



Or, use two face counts as Cartesian coordinates and forget the third one

Why is the embedding planar?





Each face can be inscribed in an upside-down equilateral triangle

With all faces consistently oriented, there can be no local nonplanarity

Consequences of Schnyder's grid embedding

Every n-vertex planar graph can be drawn in an (n – 2) x (n -2) grid

Uses vertex counts in place of face counts

Still the current record

Embedding can be constructed in linear time

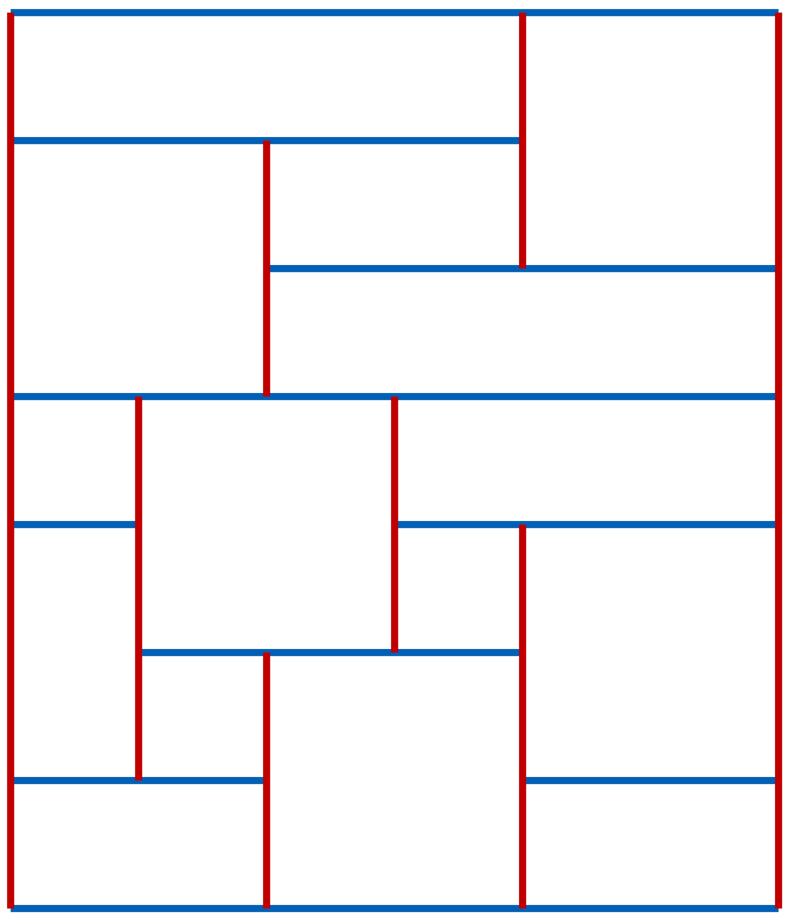
Every maximal planar graph has a greedy embedding [Dhandapani, DCG 2010]

There exists a distance-decreasing path between every two vertices

Found using sums of weights of faces in place of face counts

Existence of good weights proven non-constructively using fixed-point theorems

Rectangular subdivisions

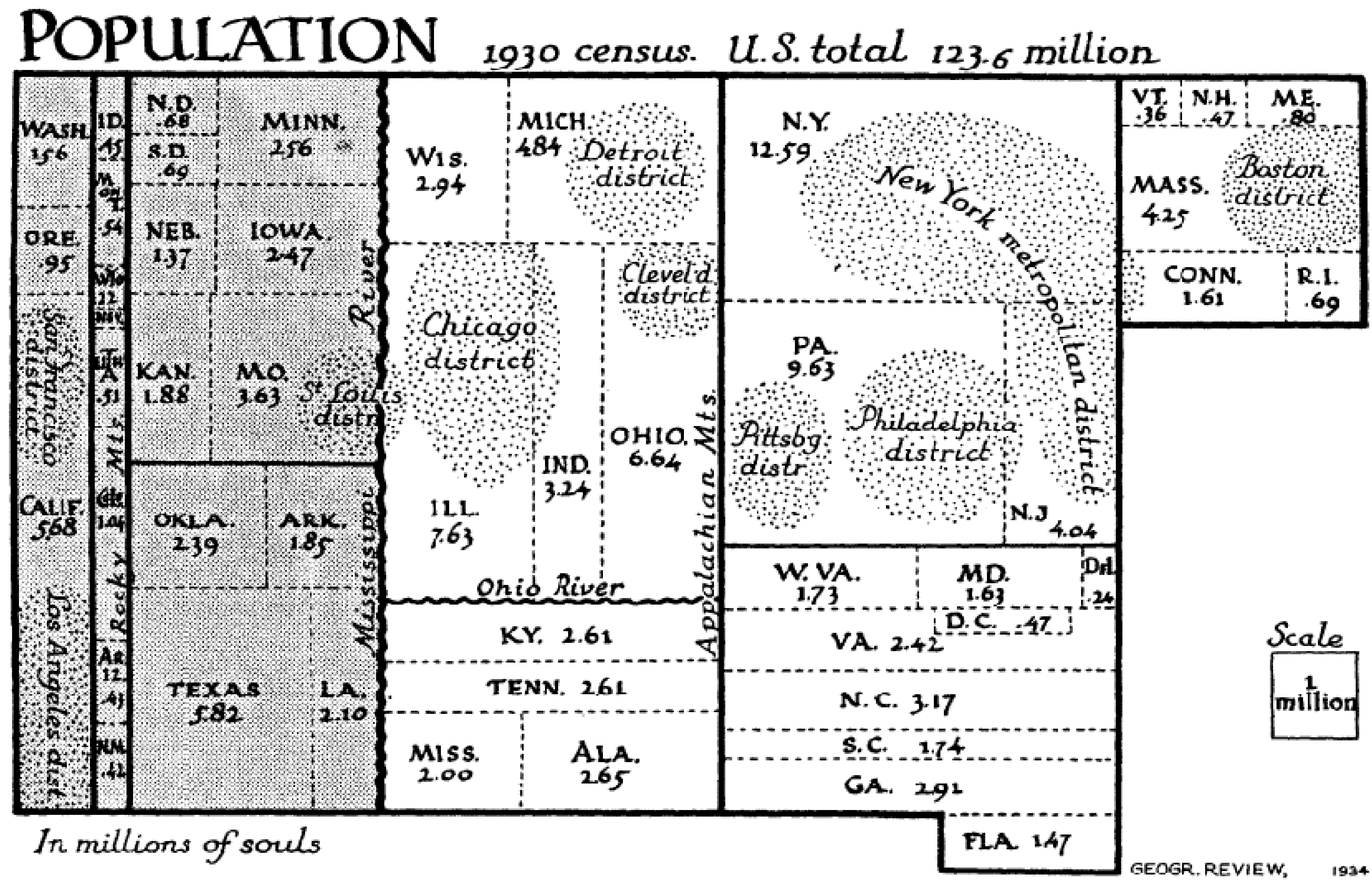


Partition an outer rectangle into smaller rectangles

Ubiquitous in many application areas as a way of subdividing space into useful sub-areas

In general, three rectangles meet at each vertex (four-rectangle vertices are a degenerate case)

Not necessarily formed by recursive bisection

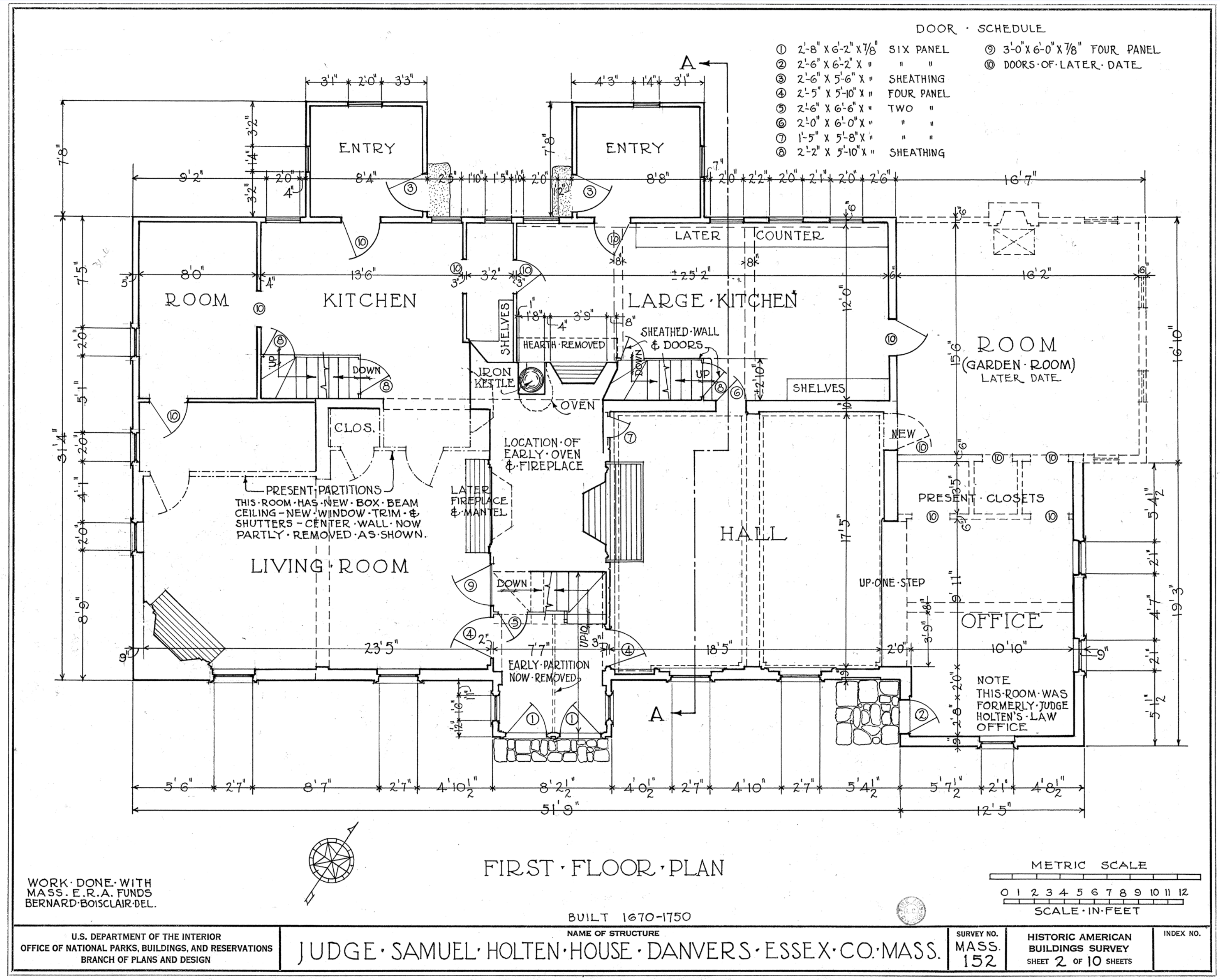


Rectangular cartogram by Raisz 1934

Furniture design

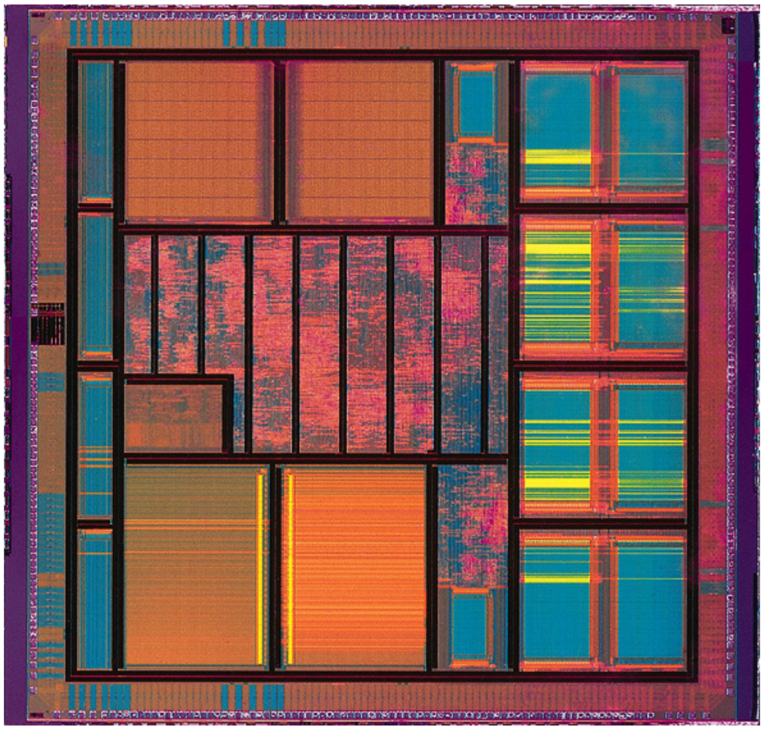
Tema Domino and Seletti shelves;
typecase by SuzetteSuzette on flickr, http://www.flickr.com/photos/suzettesuzette/4846983081/

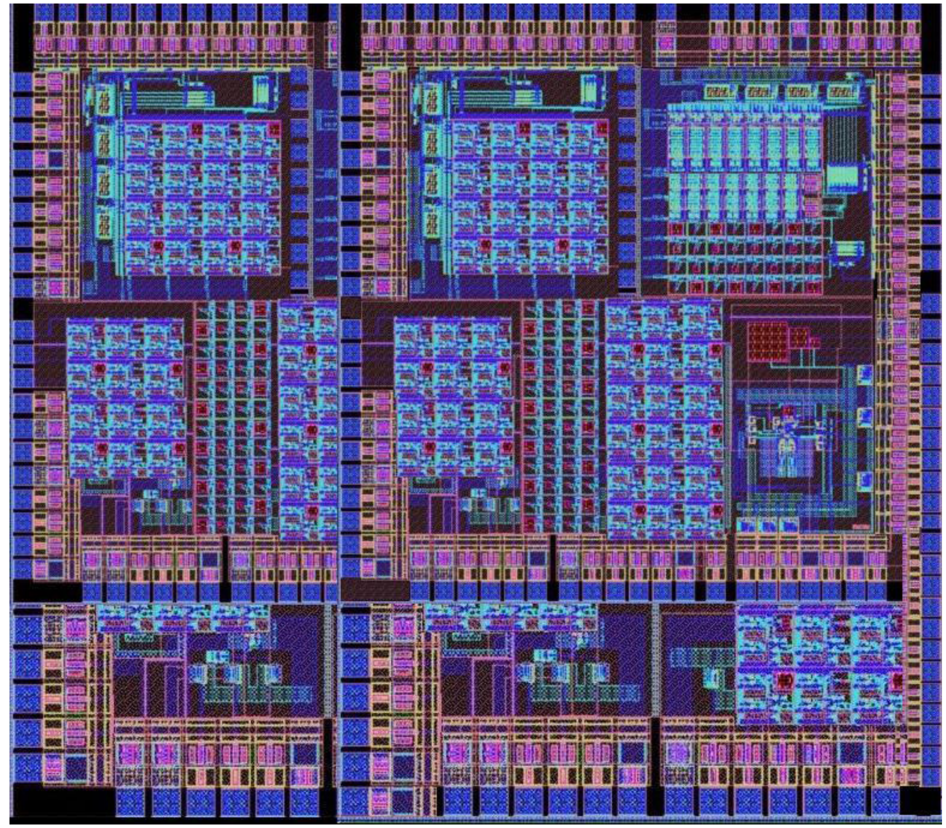


Floor plan of Judge Samuel Holten House, Danvers, Massachusetts

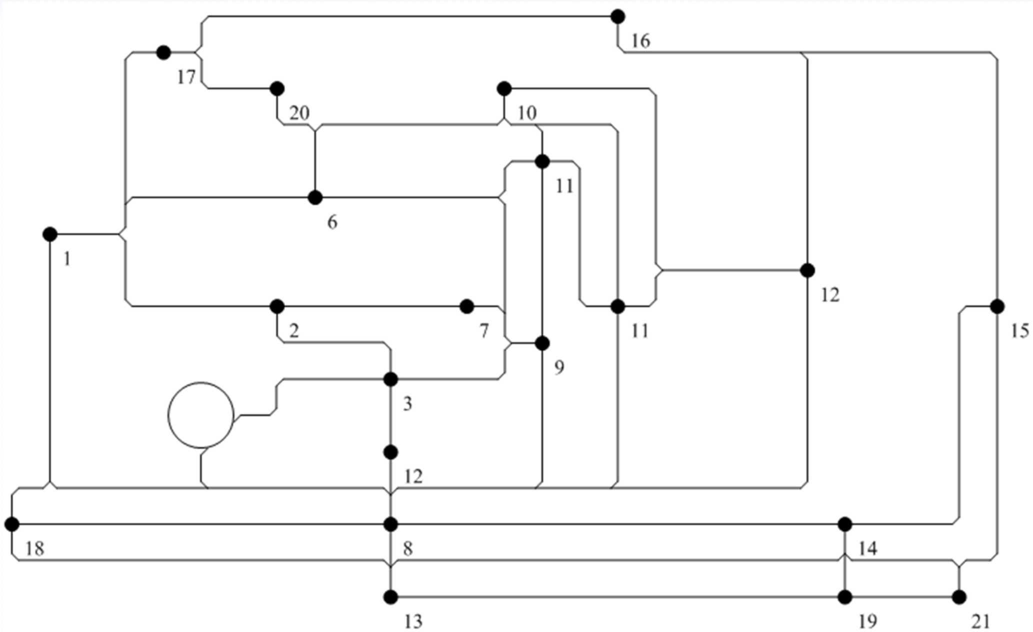
Treemap-based stock market visualization (Wattenberg, 1999)

http://commons.wikimedia.org/wiki/File:Smart_Money_magazine%E2%80%99s_Map_of_the_Market.jpg

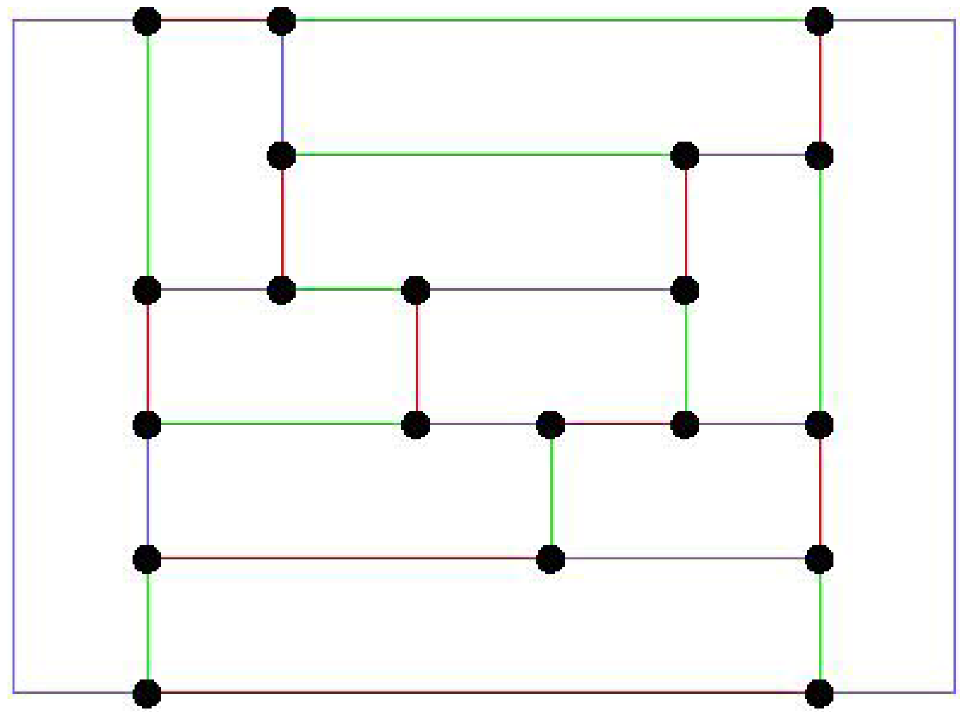




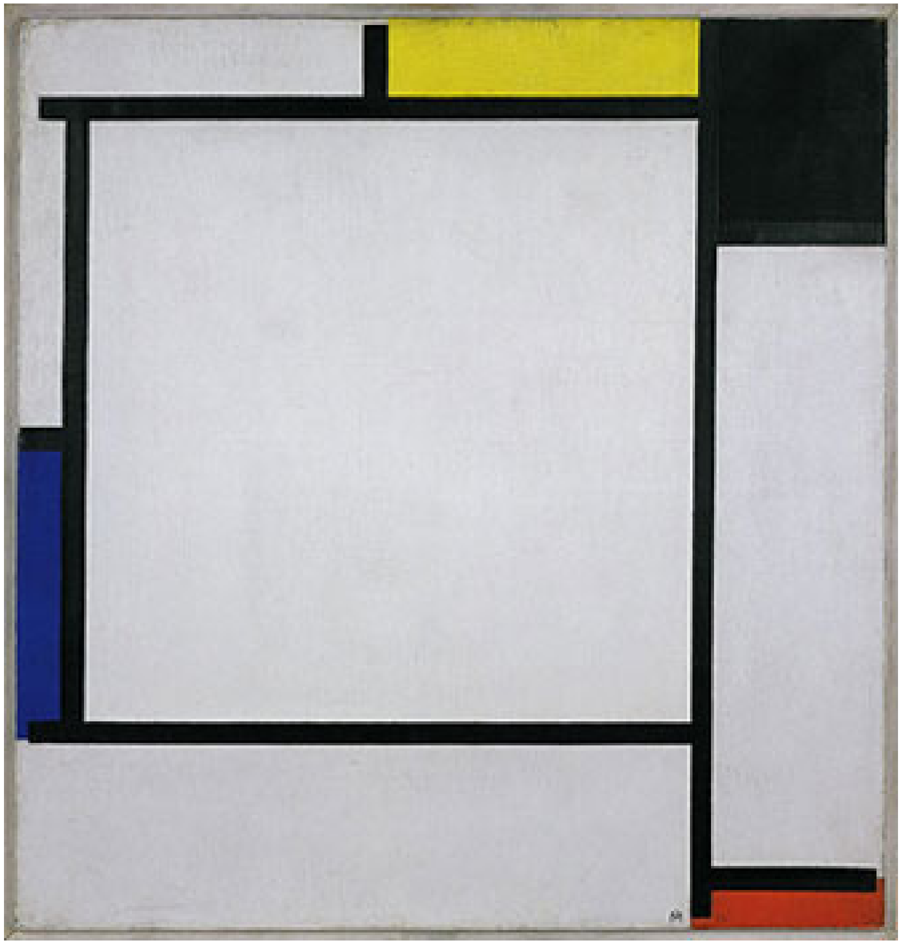
Subdivision of VLSI circuits into functional subunits
http://commons.wikimedia.org/wiki/File:Diopsis.jpg
http://commons.wikimedia.org/wiki/File:InternalIntegratedCircuit2.JPG



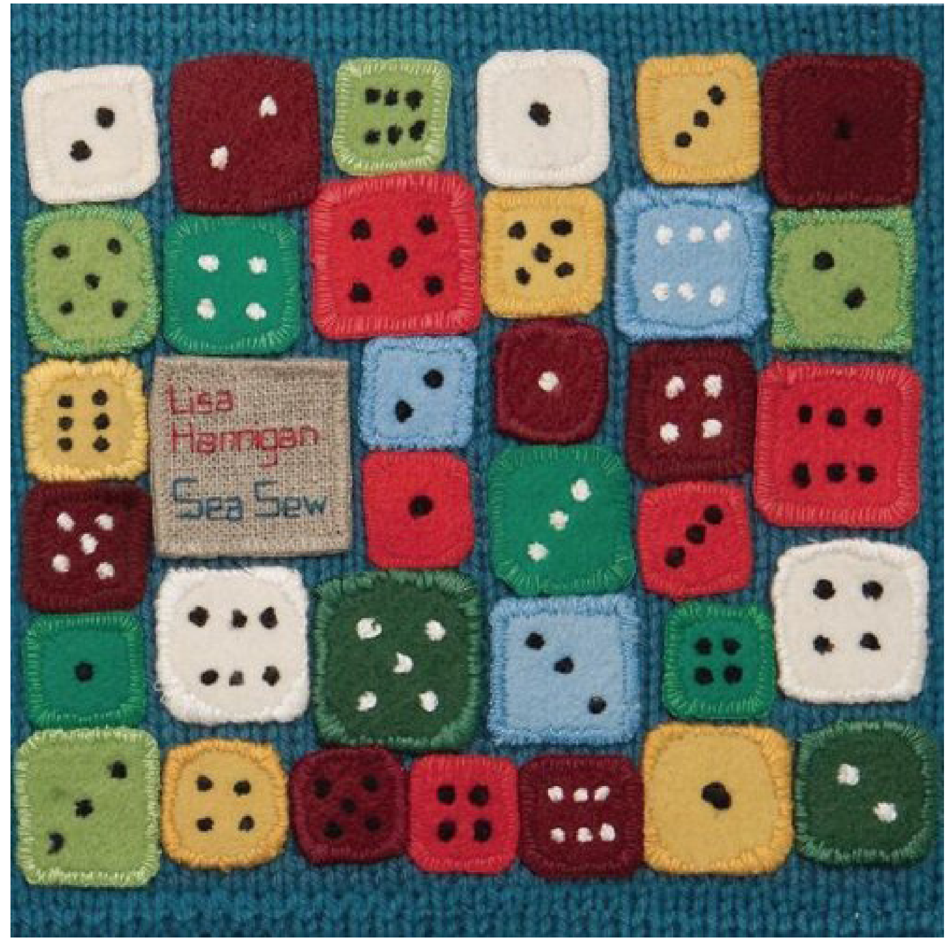
Confluent graph drawing
guided by a rectangular subdivision
(Quercini & Ancona, GD 2010)



Screenshot from the
Goblin graph library,
http://sourceforge.net/projects/goblin2/



Fine art:
Tableau 2, Piet Mondrian, 1922
Solomon R. Guggenheim Museum

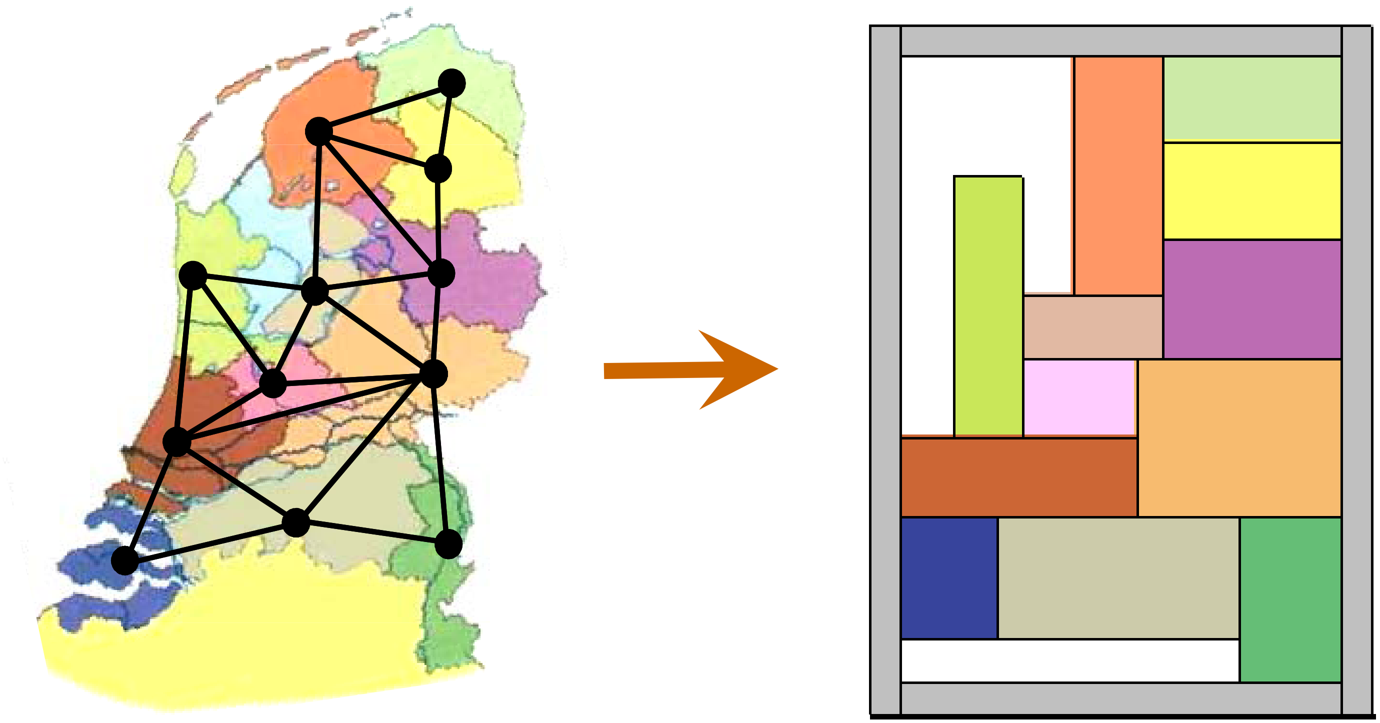


Commercial art:
CD cover for *Sea Sew*, Lisa Hannigan
(cover art also by Hannigan)

Algorithmic construction of rectangular subdivisions

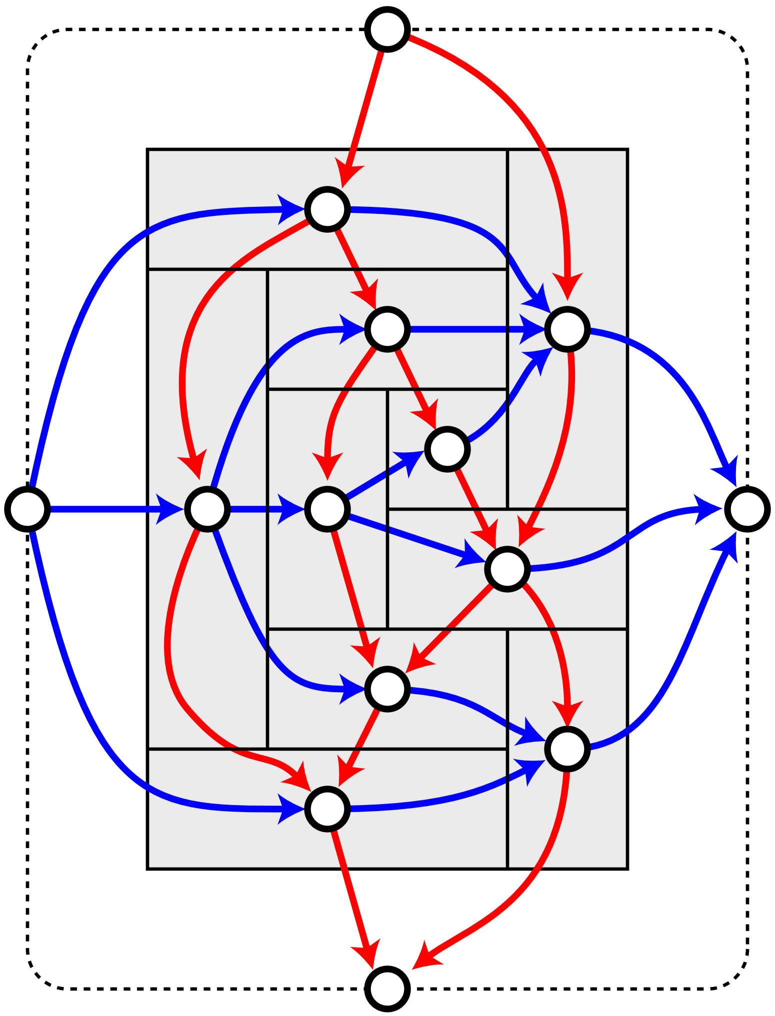
Input: a planar graph representing desired rectangle adjacencies

(possibly also additional information about orientation, size, etc)



Output: a rectangular subdivision with those adjacencies

Regular labelings from rectangular subdivisions



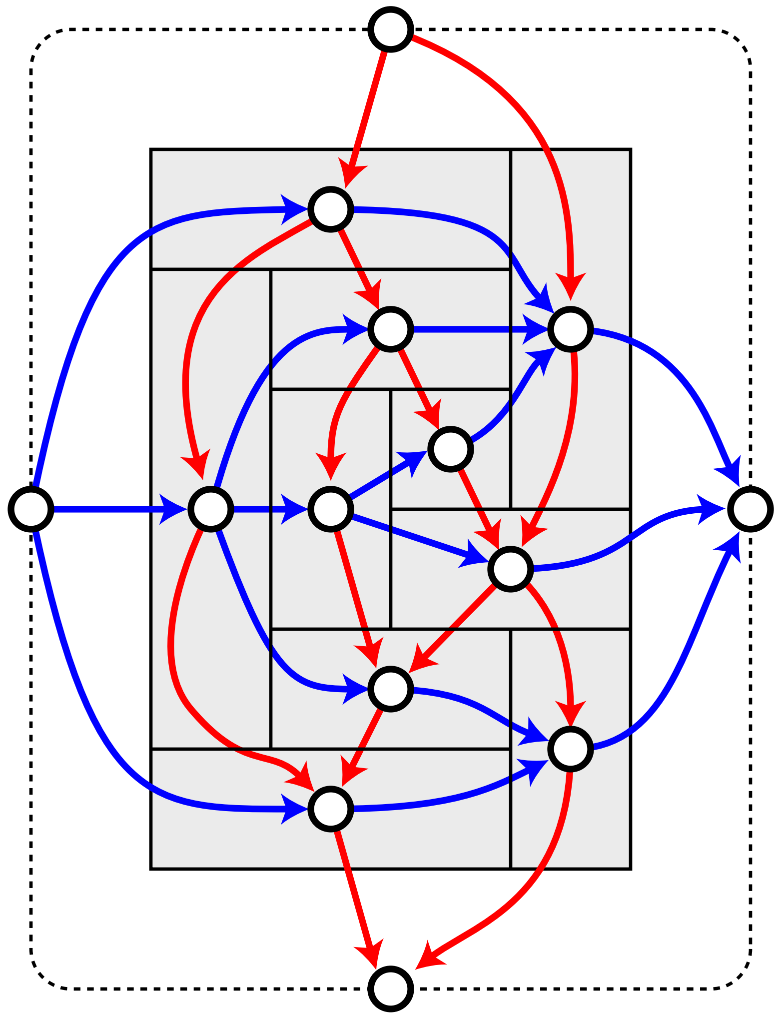
Augment adjacency graph with four extra vertices: one for each side of the outside rectangle

Color side-by-side adjacencies blue and orient them from left to right

Color above-below adjacencies red and orient them from top to bottom

[Koźmiński & Kinnen 1985; He 1993; Kant & He 1997]

Defining regular labelings for rectangular subdivisions



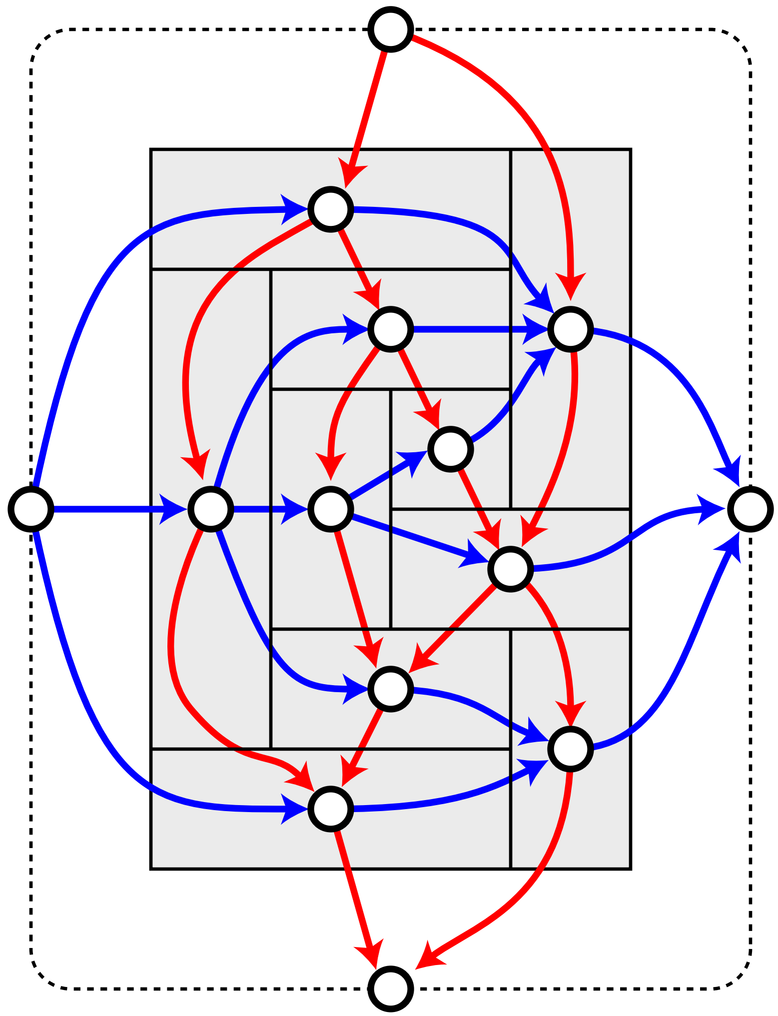
The "extended adjacency graph" is maximal planar except for a quadrilateral outer face

Outer four edges are uncolored

Each outer vertex has edges of a single color and orientation

Each inner vertex has edges of all colors and orientations in the cyclic order in-red, out-blue, out-red, in-blue

Existence and construction of labelings



Necessary and sufficient condition: no separating triangle

(Would lead to region surrounded by 3 rectangles, geometrically impossible)

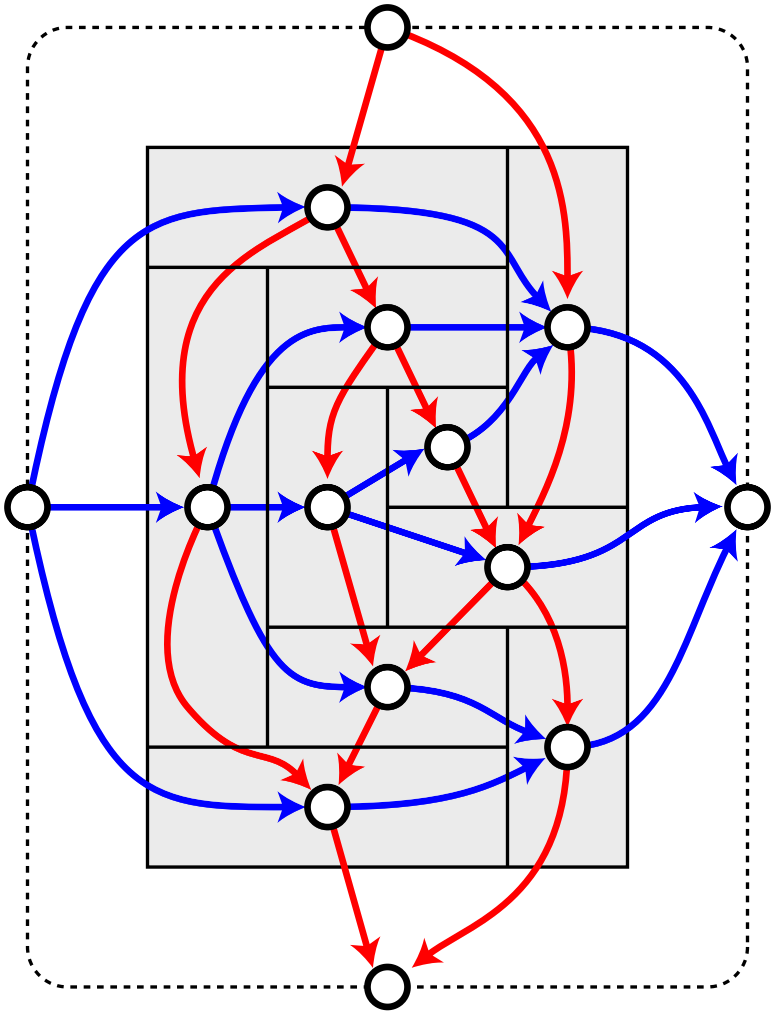
Proof and linear time algorithm use same idea as Schnyder:

contract an edge

recursively label smaller graph

uncontract and locally relabel

Rectangular subdivisions from labelings



Each one-color subgraph
is st-planar (acyclic)

(same argument as before:
a cycle would have incoming
paths of the other color,
nowhere for them to go)

Boundary line segment =
face in single-color subgraph

Topologically order the boundary
line segments of the partition

Use position in topological order
as Cartesian coordinate

Consequences of rectangular regular labelings

Simple criterion for existence of a rectangular partition
(no separating triangles in adjacency graph)

Linear time construction of a partition from its dual

Separation of combinatorics (left-to-right and top-to-bottom ordering of boundary segments) from shape (coordinates of segments)

Later in this talk:

Language to describe constraints on orientations in a layout and ability to find orientation-constrained layouts

Orthogonal polyhedra

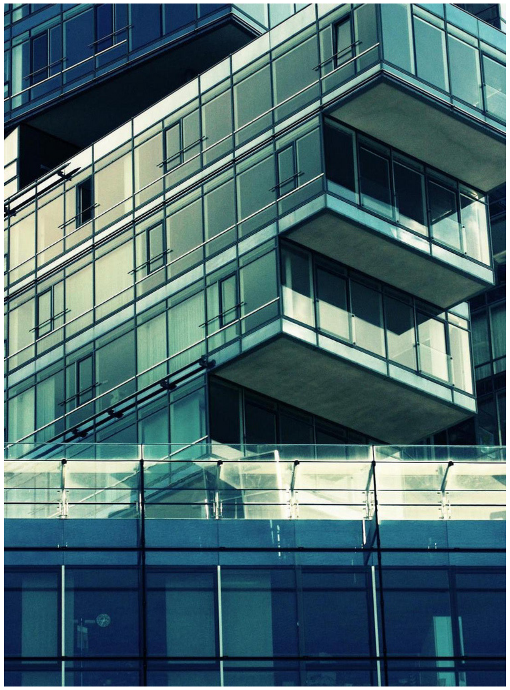
Used frequently in architecture

Fallingwater
Frank Lloyd Wright

Sxenko, http://commons.wikimedia.org/wiki/File:Wrightfallingwater.jpg

Habitat 67
Moshe Safdie, Montreal

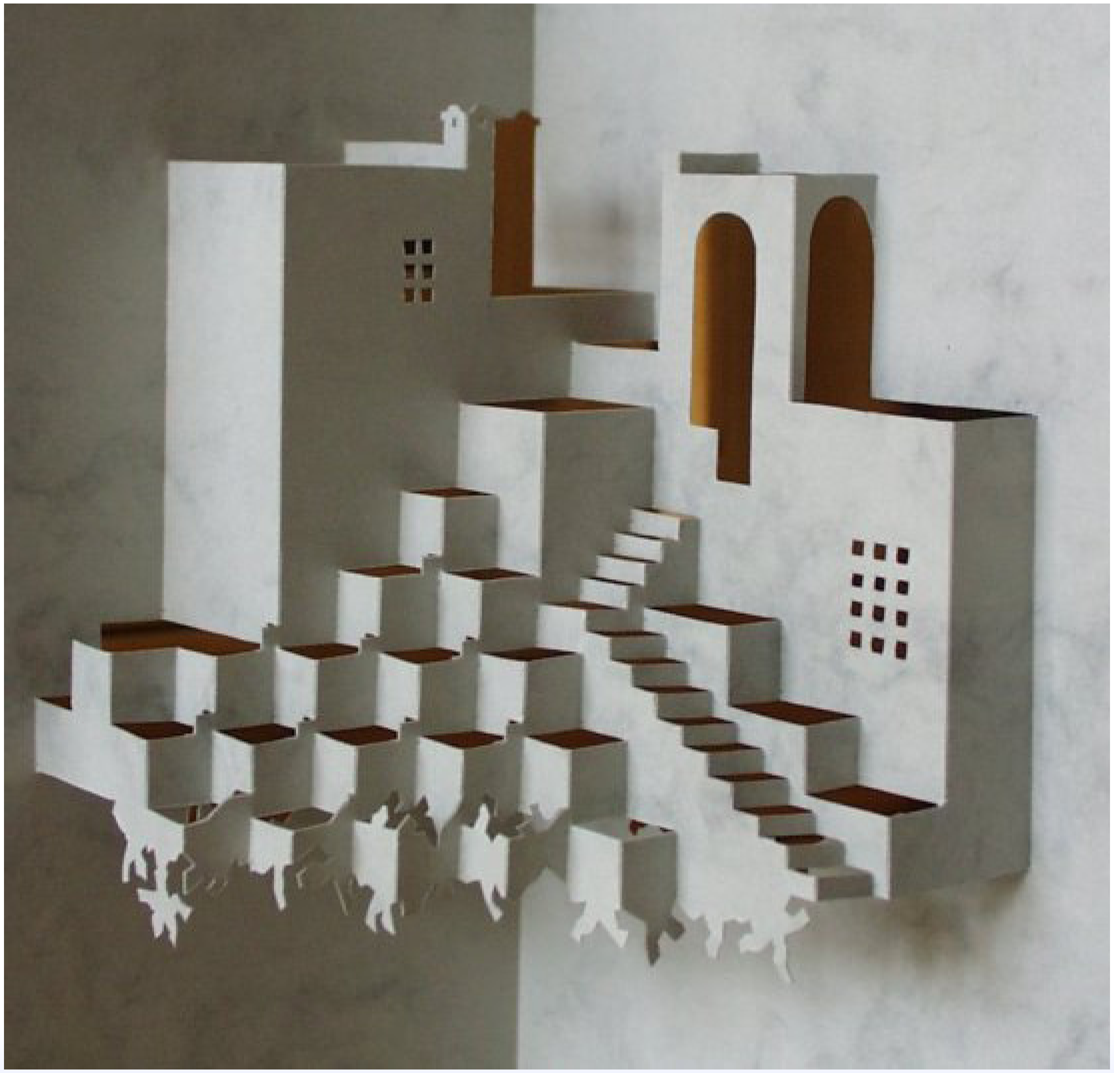
nnova on Flickr, http://www.flickr.com/photos/nnova/2919165183/



Offices in
Hannover, Germany

Jasmic on Flickr, http://www.flickr.com/photos/jasmic/2318463768/

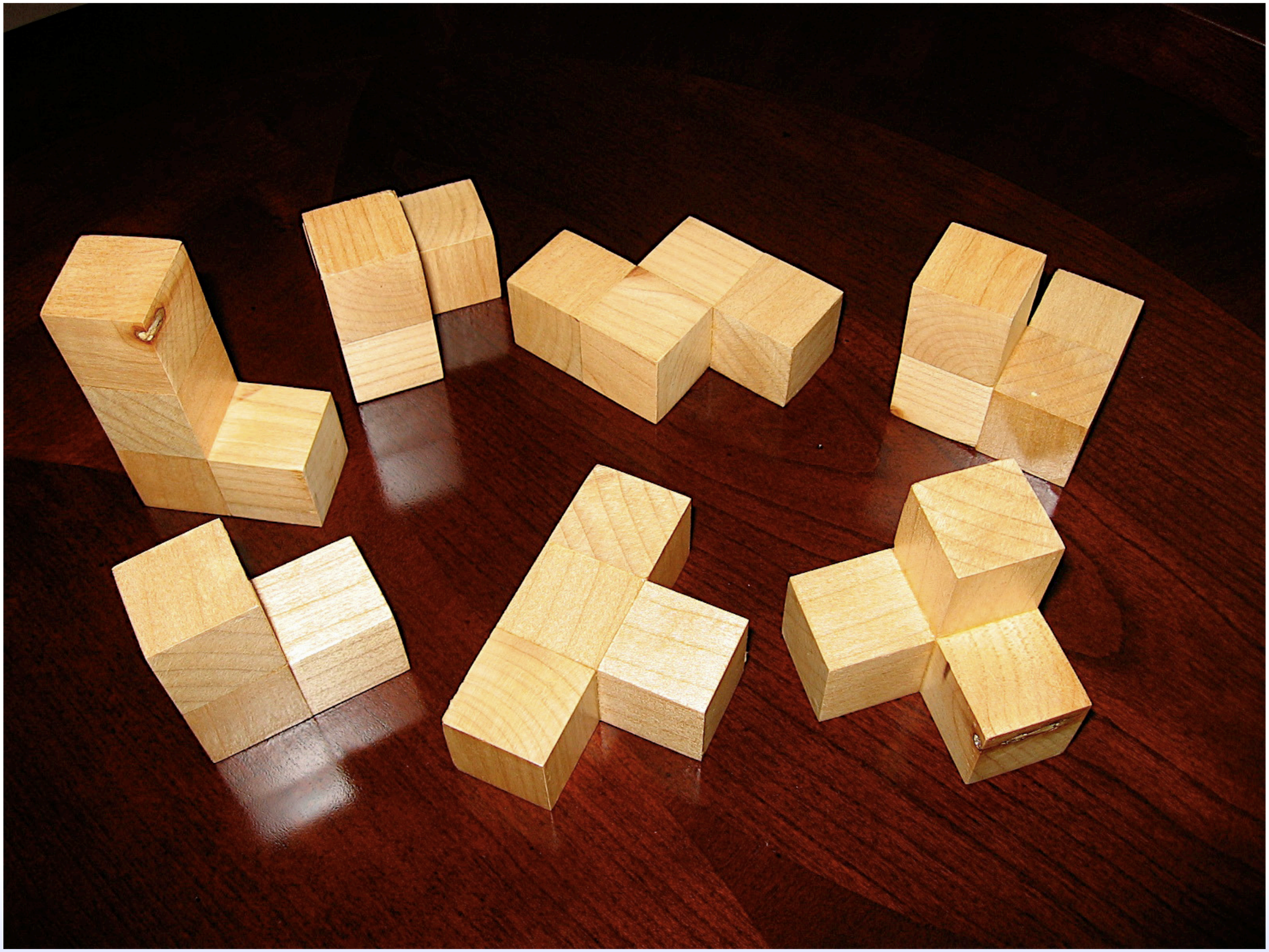
Orthogonal polyhedra in papercraft



Ingrid Siliakus

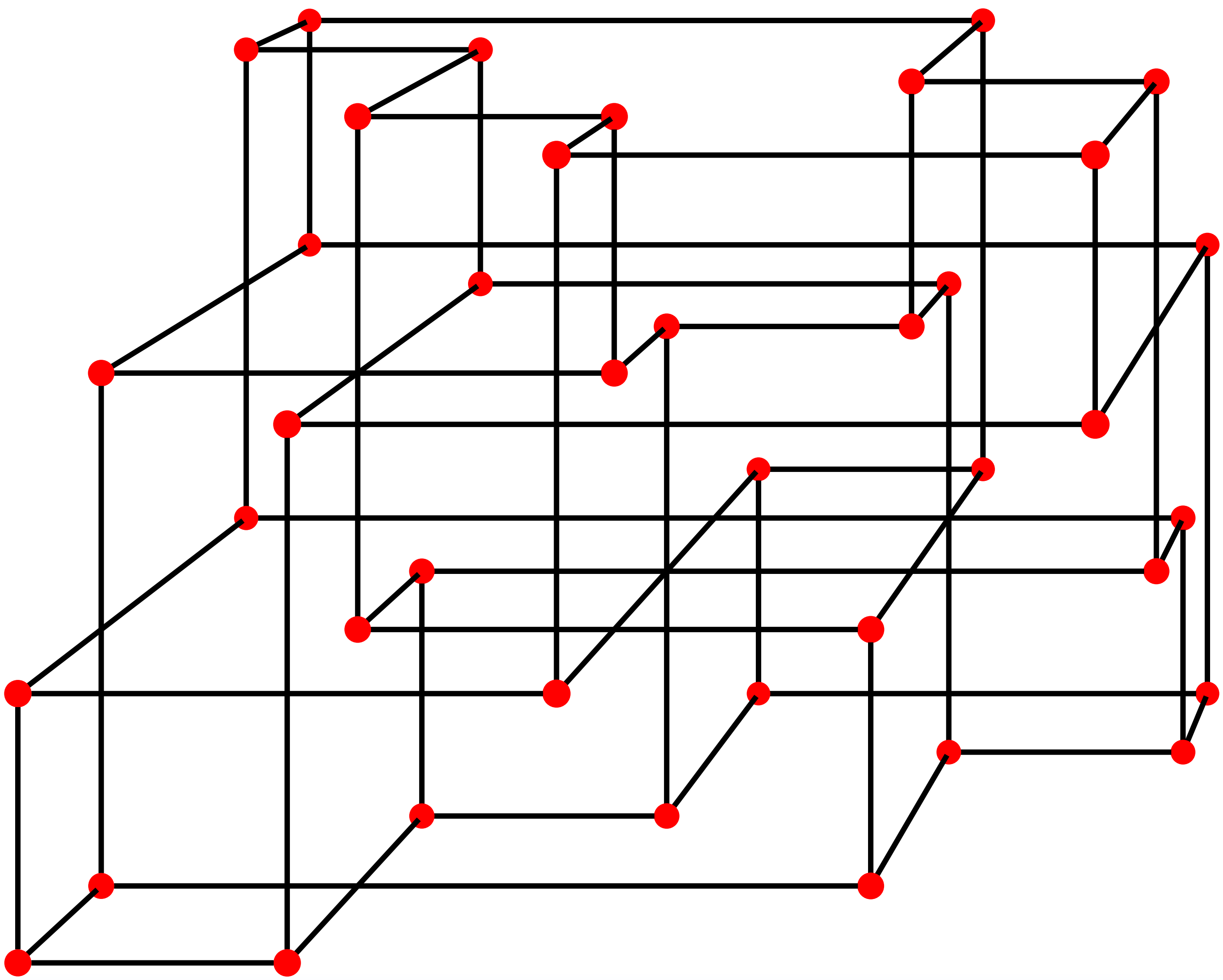
Orthogonal polyhedra

Geometric puzzles such as the Soma Cube



Mr. Hobbie on Picasa, http://picasaweb.google.com/lh/photo/bx8Cj3M8hSXTdIr6xSA1_w

Orthogonal polyhedra



Special case of a more general problem:

Embedding graphs in a 3d grid without bends

Characterizing orthogonal polyhedra

[E. & Mumford, SoCG 2010]

Simplifying assumptions:

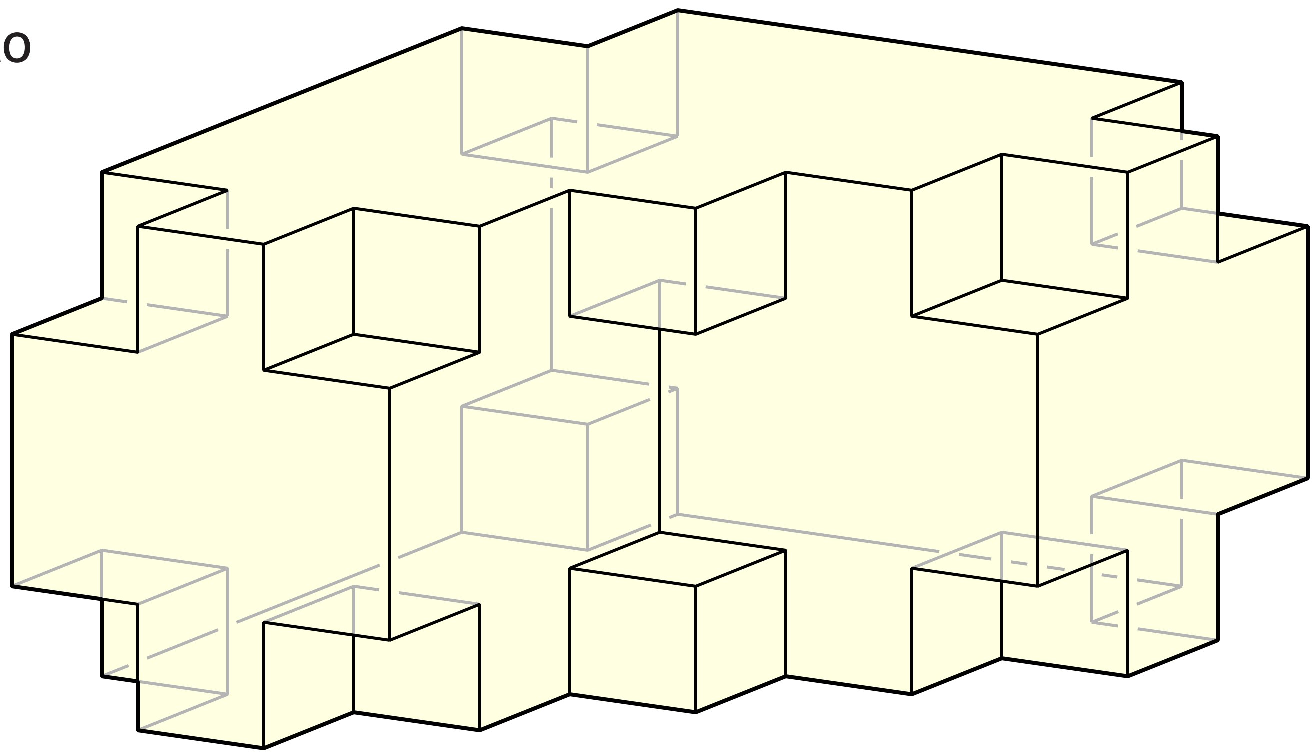
Three perpendicular edges at each vertex

Each face is parallel to two coordinate axes

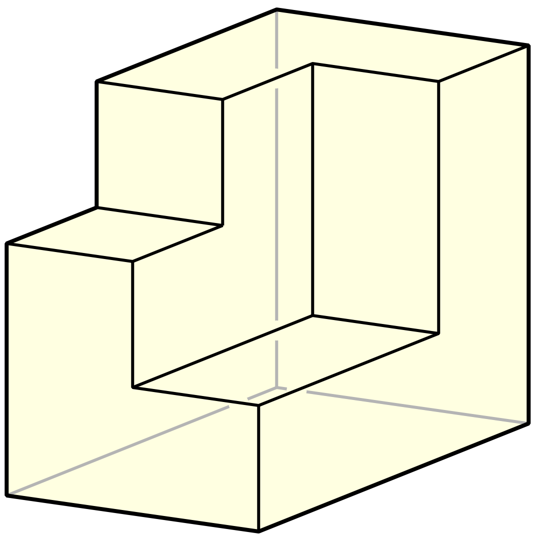
Topology of a sphere

As a result:

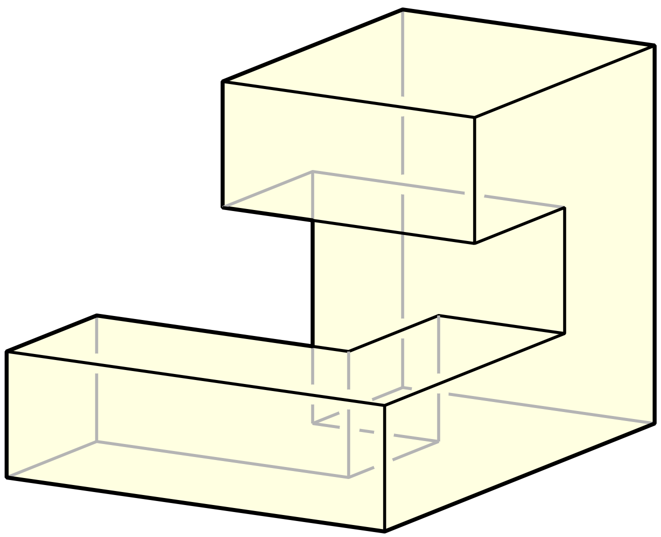
Underlying graph is planar, bipartite, and 3-regular



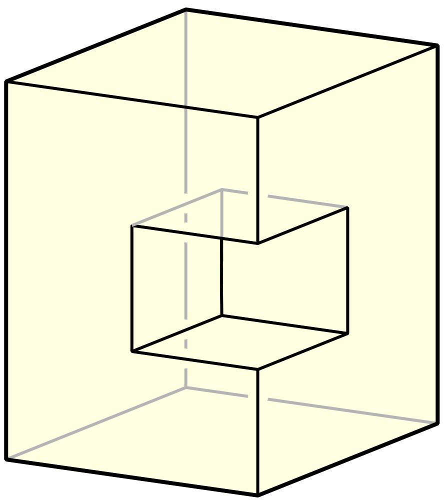
Restricted classes of orthogonal polyhedra



Corner polyhedron:
all but three faces
oriented positively



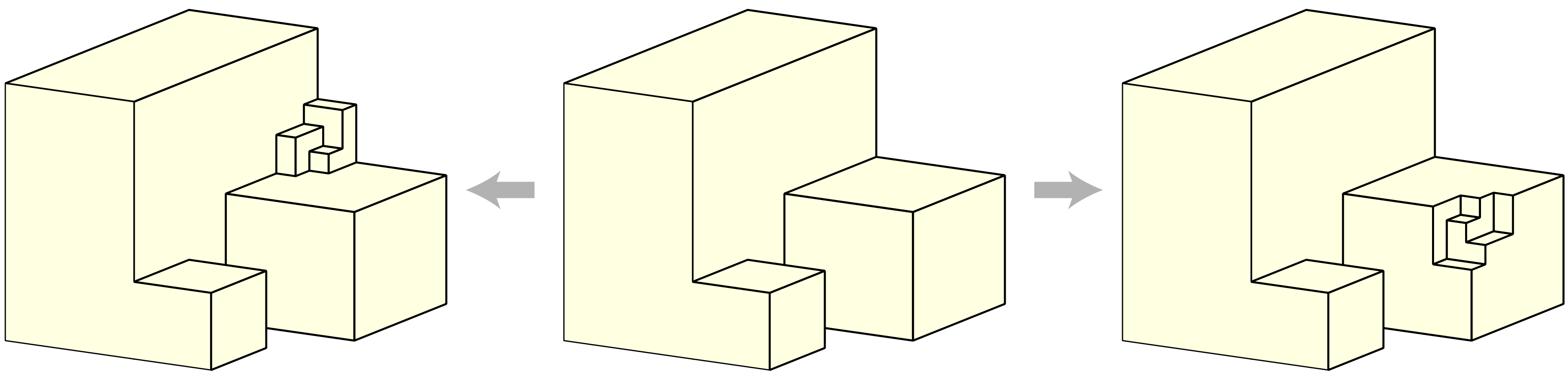
xyz polyhedron:
Each axis-parallel line
has ≤ 2 vertices



Unconstrained

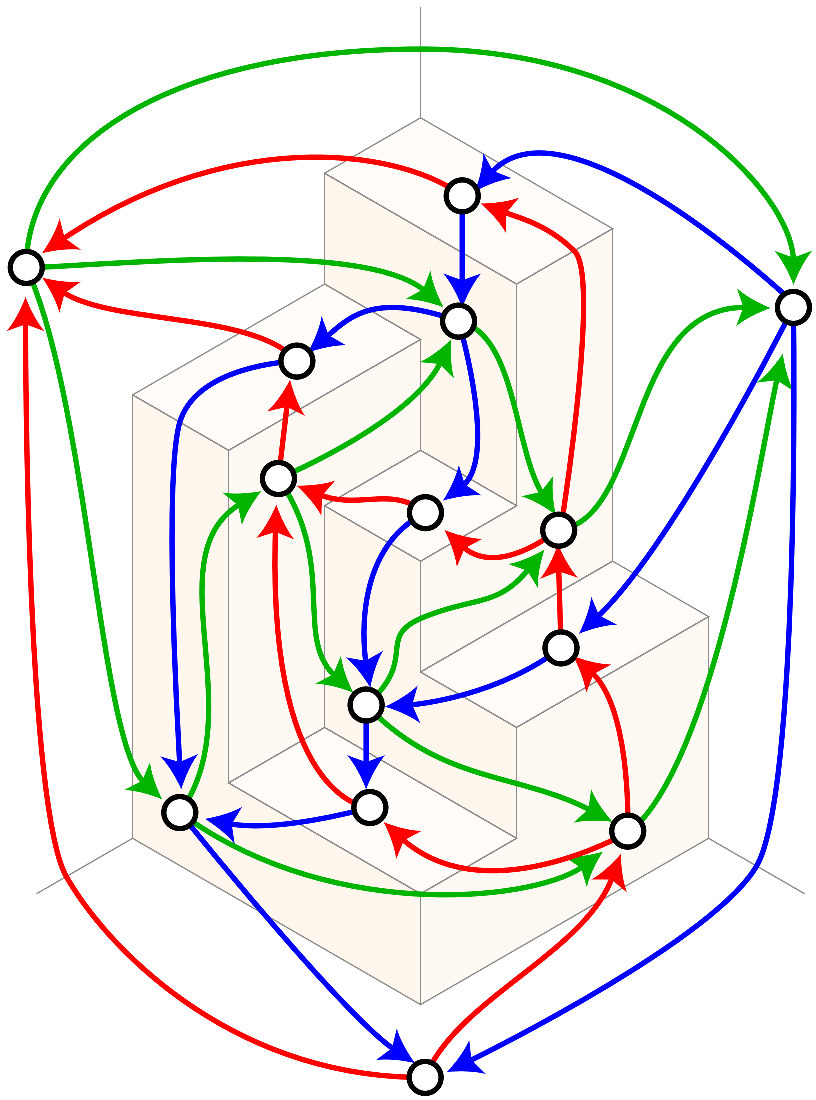
Corner polyhedra as base case

Unconstrained polyhedra can be formed by gluing together corner polyhedra



Each corner-polyhedron component has no separating triple of faces
(dual graph is 4-connected)

Regular labelings from corner polyhedra

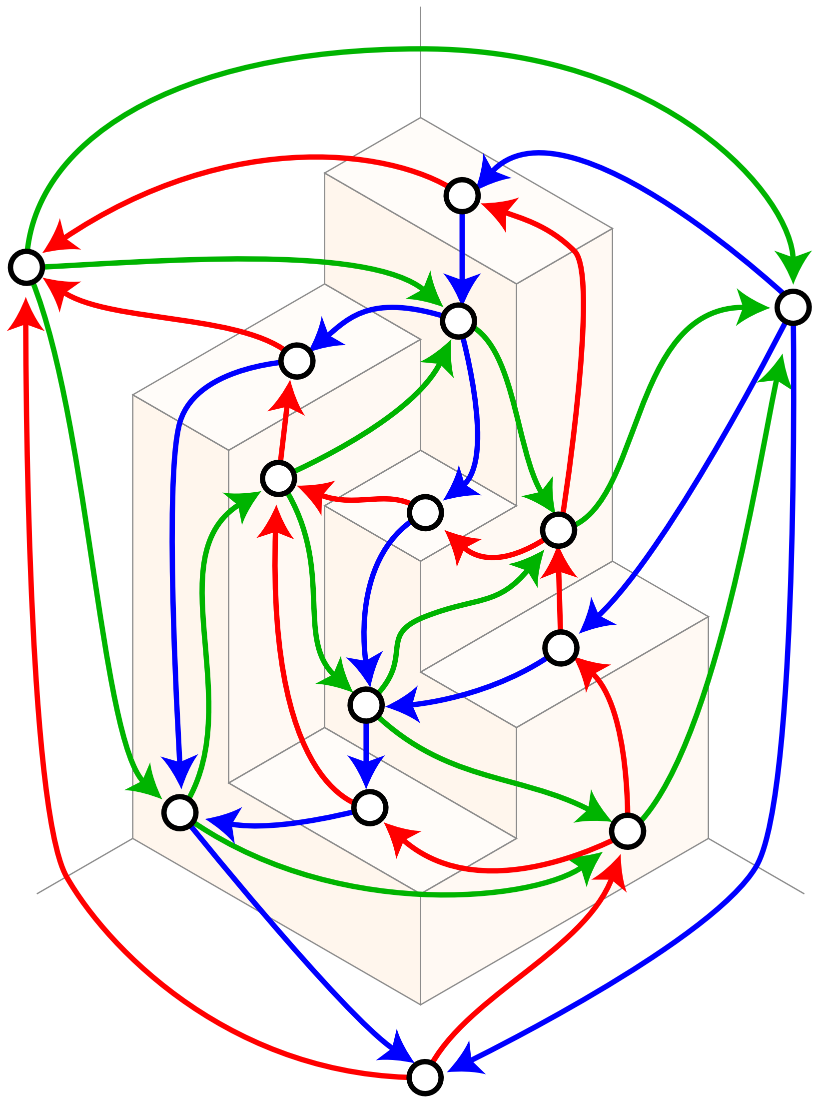


Draw polyhedron isometrically (120 degree angles)

Color edges of dual graph red, green, and blue according to slopes of the polyhedron edge they cross

Direct edges of each color from one side of the drawing to the other

Defining regular labelings for corner polyhedra



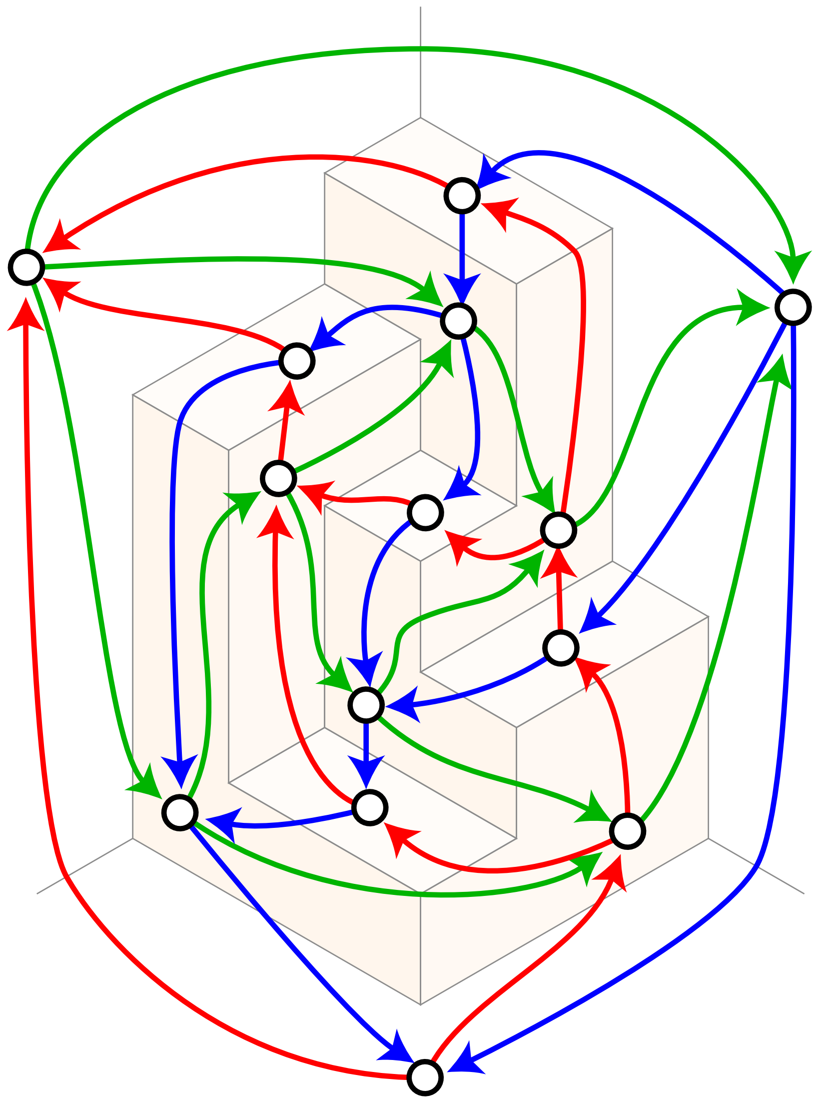
Each triangle has edges of all three colors

Each vertex has edges alternating between two colors

At the three outside vertices, orientations strictly alternate

At the inside vertices, there are exactly two breaks in alternation of orientations (one with two incoming edges, one with two outgoing edges)

Existence and construction of labelings



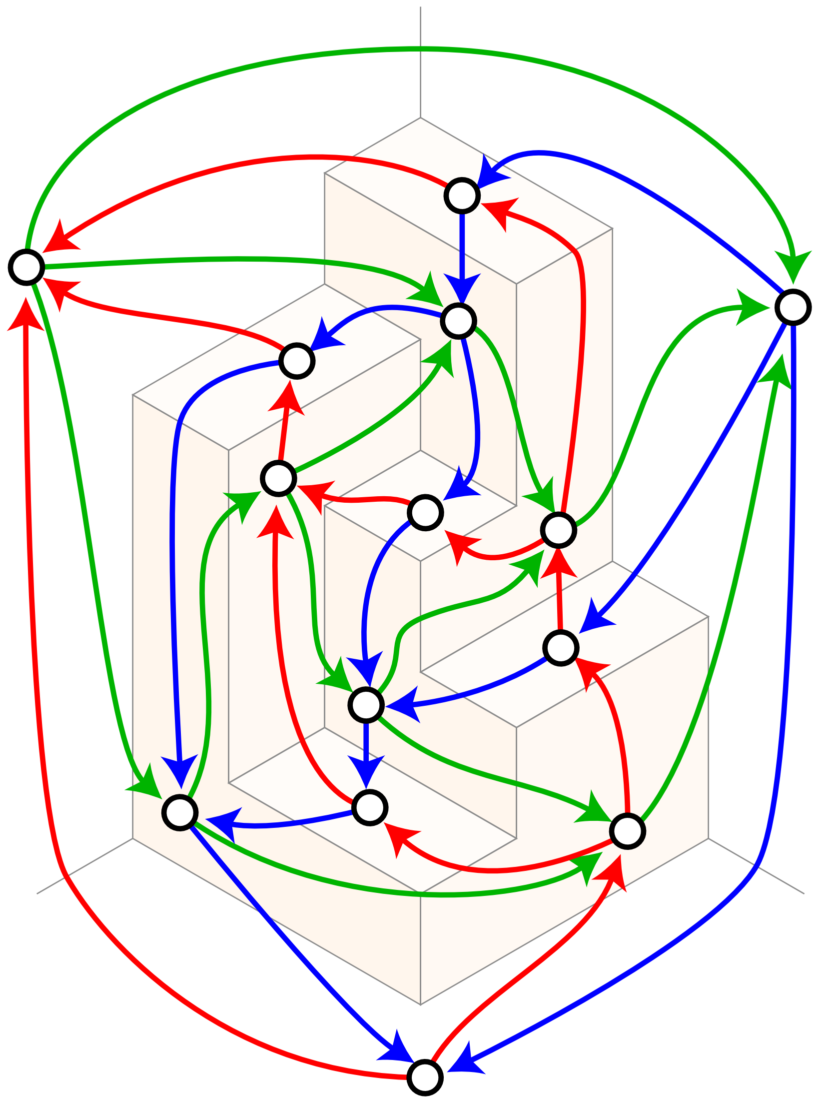
Sufficient condition:

- Dual graph is 4-connected, Eulerian, maximal planar

Proof and near-linear-time construction algorithm:

- Contract two edges or split on a 4-cycle
- Recurse on smaller graph(s)
- Undo simplification and locally relabel

Corner polyhedra from labelings



Each two-color subgraph
with one color reversed
is st-planar (acyclic)

(same argument as before)

Topologically order these
st-planar graphs

Face coordinate =
position in ordering for graph
with same colors as dual vertex

Vertex coordinate =
coordinates of 3 adjacent faces

Consequences of polyhedral regular labelings

Underlying graphs of corner polyhedra are exactly
planar bipartite 3-connected 3-regular graphs
s.t. every dual separating triangle has same parity

Underlying graphs of xyz polyhedra are exactly
planar bipartite 3-connected 3-regular graphs

Underlying graphs of simple orthogonal polyhedra are exactly
planar bipartite 2-connected 3-regular graphs
such that removing any two vertices leaves $\leq$ 2 components

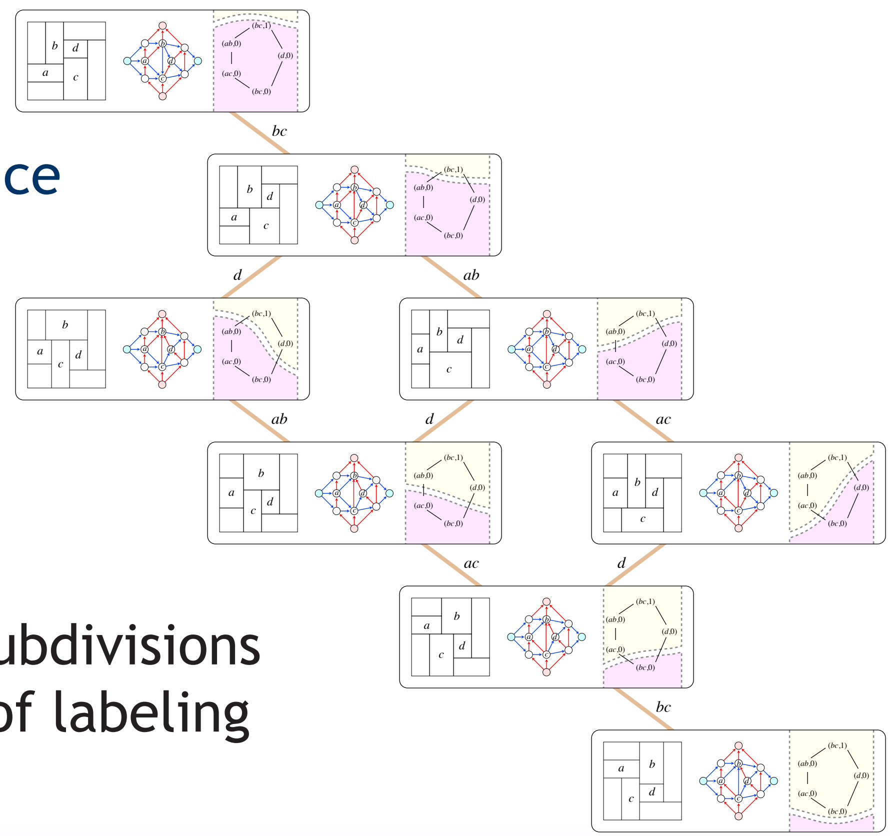
All can be recognized (and a polyhedral representation constructed)
in linear randomized expected time or near-linear deterministic time

Structure in the set of all regular labelings

Fix a (near-)maximal planar graph and its outer face

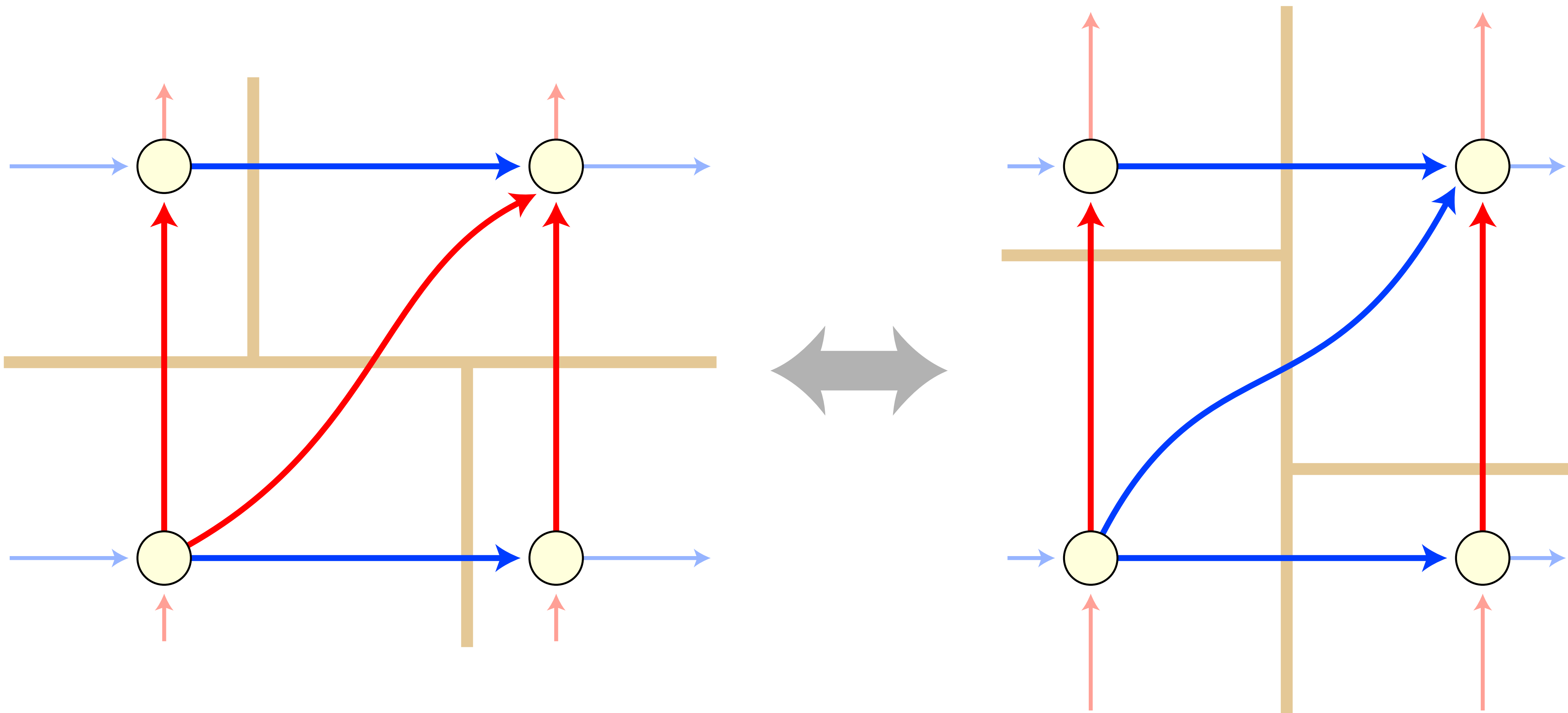
Look at all possible regular labelings on the graph

Set of labelings forms a distributive lattice connected by local twist operations allowing efficient listing of all labelings, construction of constrained labelings



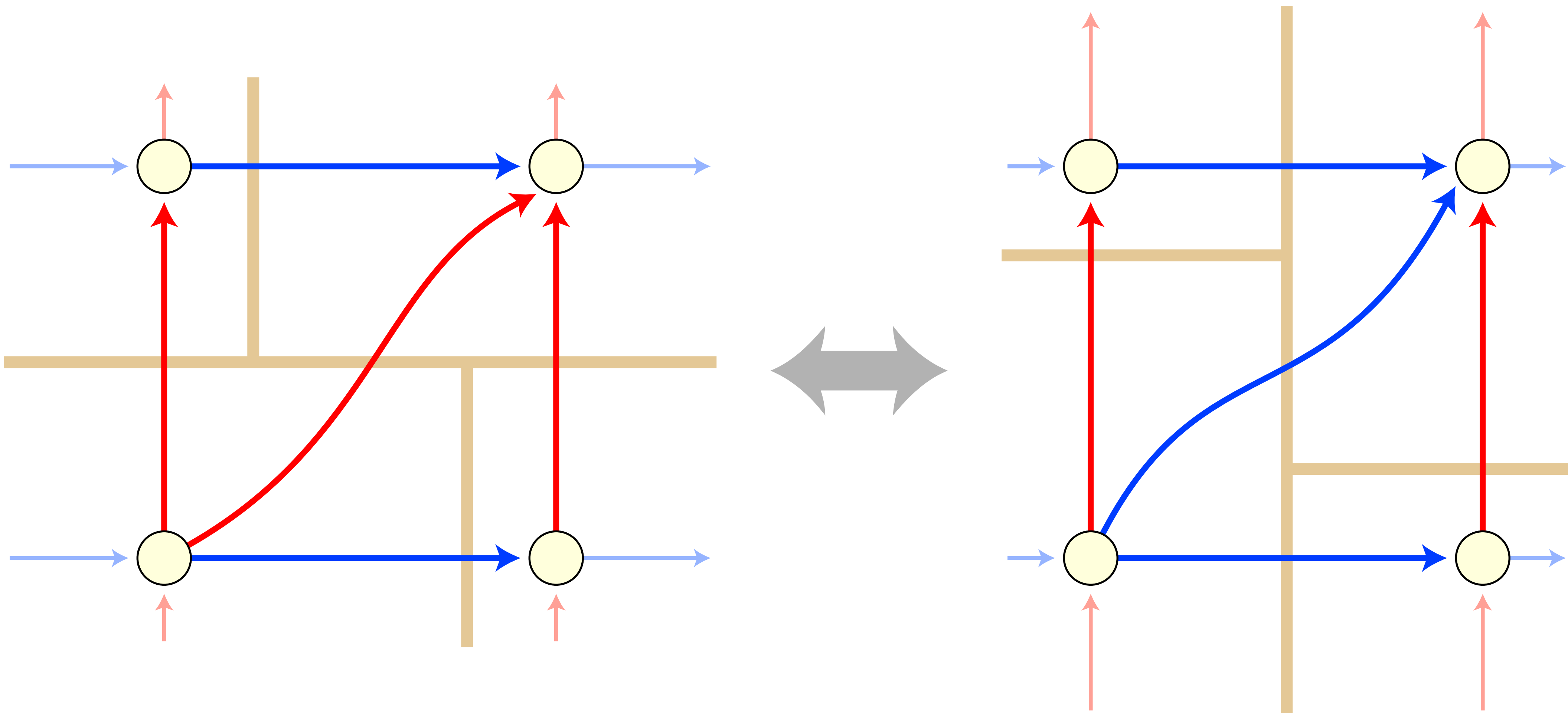
Theory is best-developed for rectangular subdivisions but applies equally well to all three kinds of labeling

Local moves in rectangular subdivisions



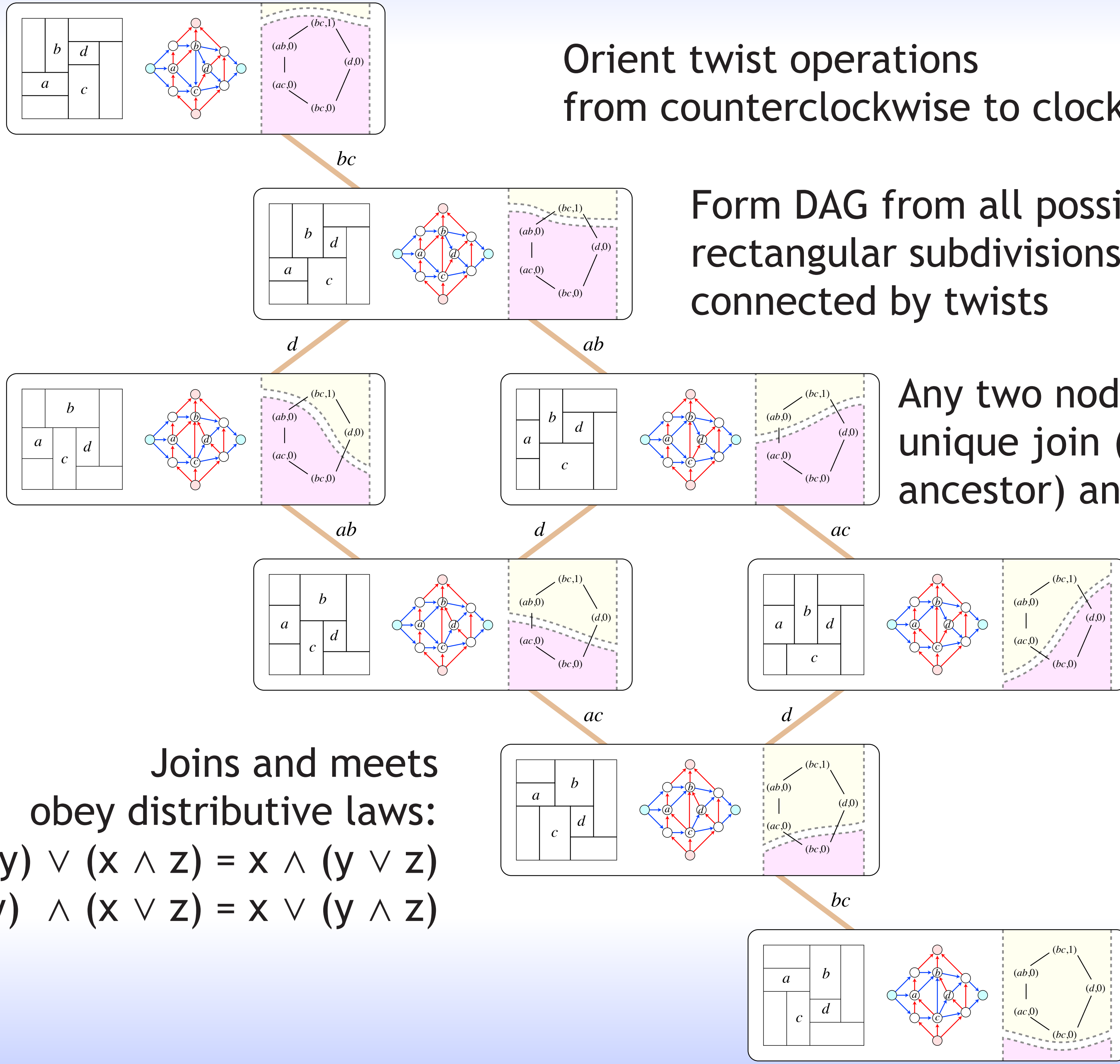
Find a 4-cycle with alternating colors
change the color of everything inside it, adjust orientations as necessary

Local moves in rectangular subdivisions



Find a 4-cycle with alternating colors
twist boundaries between edge color/orientation groups at its four vertices

Distributive lattice of rectangular subdivisions [Fusy, GD 2005]



Orient twist operations from counterclockwise to clockwise

Form DAG from all possible rectangular subdivisions, connected by twists

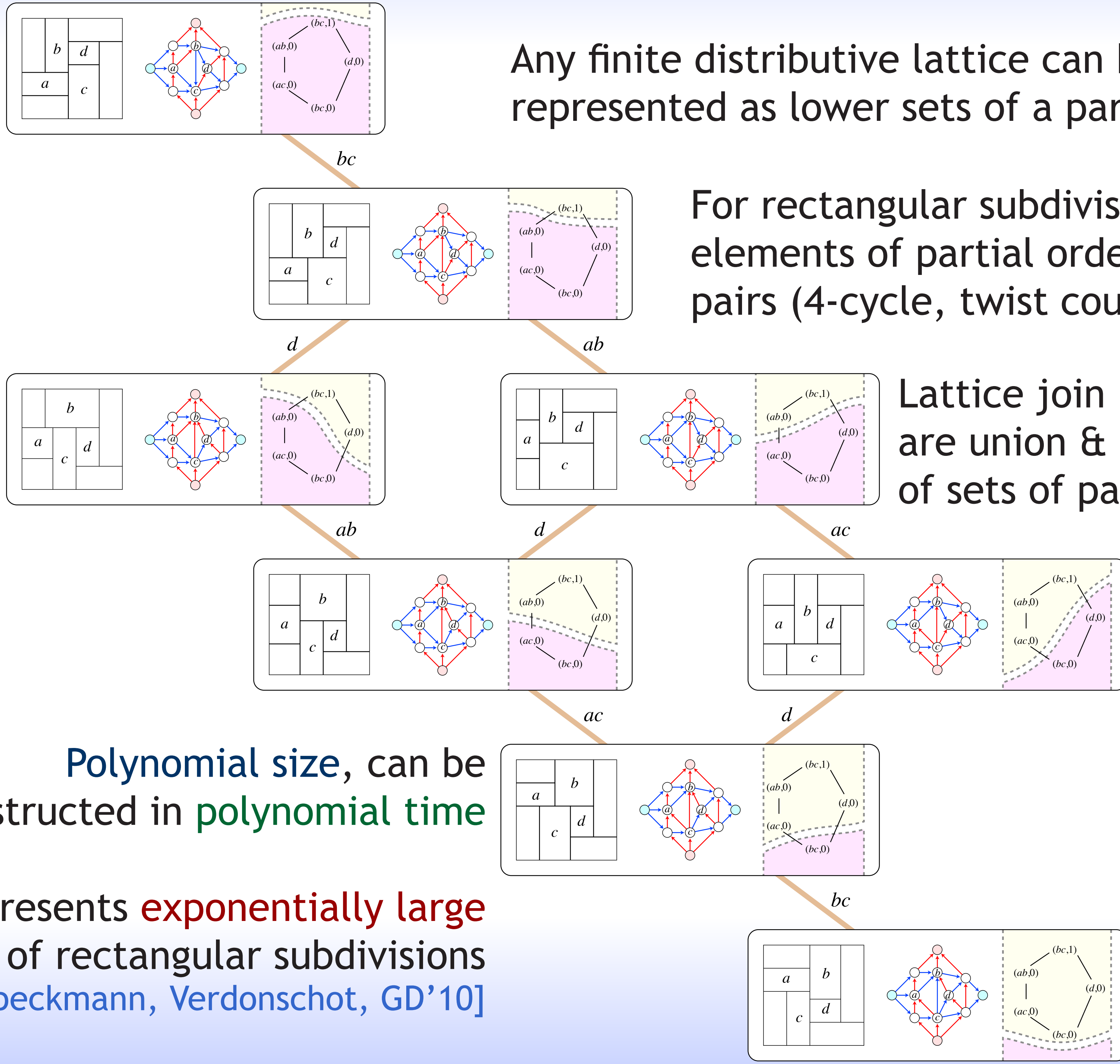
Any two nodes have unique join (minimal ancestor) and meet

Joins and meets obey distributive laws:

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$$

$$(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$$

Birkhoff's representation theorem for distributive lattices



Any finite distributive lattice can be represented as lower sets of a partial order

For rectangular subdivisions, elements of partial order are pairs (4-cycle, twist count)

Lattice join & meet are union & intersect of sets of pairs

Polynomial size, can be constructed in polynomial time

Represents exponentially large set of rectangular subdivisions
[Buchin, Speckmann, Verdonschot, GD'10]

Consequences of distributive lattice structure

List all regular labelings in time poly(n) + O(k)
Build partial order and use algorithms for listing
all lower sets of a partial order [Habib et al., Disc. Appl. Math. 2001]

Fixed-parameter-tractable algorithms for
finding area-universal rectangular cartograms
(can be morphed to match any assignment of areas)
param is # separating 4-cycles [E., Mumford, Speckmann, Verbeek, SoCG 2009]

Find layout with constraints on orientations of
adjacent rectangles in polynomial time
(translate constraints into edge contractions of underlying partial order)
[E., Mumford, WADS 2009]

Summary of common features of regular labelings for grid embeddings, rectangular partitions, polyhedra

Base graph is (near-)maximal planar
with specified outer face

Each edge is directed and
colored from a finite set of colors

Local constraints on the cyclic order
of labels at each vertex

Single-colored and two-colored
subgraphs are automatically st-planar

Every geometric structure
gives a labeling and every labeling
gives a geometric structure

Characterize labelable graphs
in terms of their connectivity

Inductive proof of characterization
leads to recursive labeling algorithm

Local twist operations
generate a distributive lattice
on all possible labelings

Partial order dual to the lattice
represents all labelings
in polynomial space

Some questions

Why do these three different geometric objects
lead to such similar combinatorial structures?

Distributive lattice structure can be explained in terms of
bounded-outdegree orientations of planar graphs [Propp 1993]
but this doesn't help explain the other similarities

What other analogies do these three types of object have?

Are there other geometric structures that can
also be explained by similar kinds of regular labeling?

Washington D.C., 1922

http://commons.wikimedia.org/wiki/
File:Unclothed_woman_behind_question_mark_sign.jpg