Minimum Forcing Sets for Miura Folding Patterns

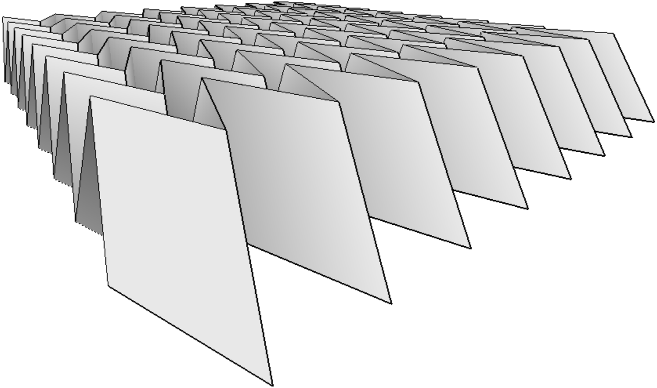
Brad Ballinger, Mirela Damian, **David Eppstein**, Robin Flatland, Jessica Ginepro, and Thomas Hull

SODA 2014

The Miura fold

Fold the plane into congruent parallelograms
(with a careful choice of mountain and valley folds)

Has a continuous folding motion from its unfolded state to a compact flat-folded shape



Applications of the Miura fold

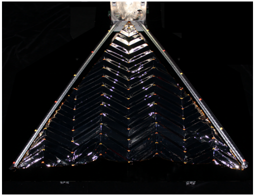
Paper maps

http://theopencompany.net/products/san-francisco-map

High-density batteries

http://www.extremetech.com/extreme/168288-folded-paper-lithium-ion-battery-increases-energy-density-by-14-times

Satellite solar panels

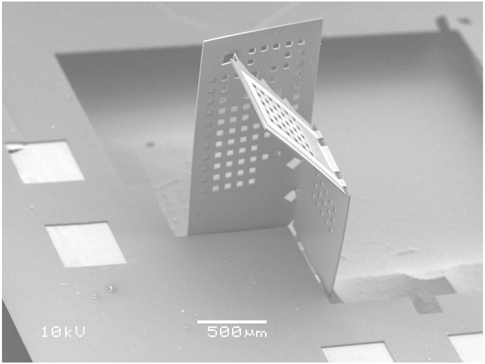


http://sat.aero.cst.nihon-u.ac.jp/sprout-e/1-Mission-e.html

Acoustic architecture

Persimmon Hall, Meguro Community Campus

Self-folding devices



http://newsoffice.mit.edu/2009/nano-origami-0224

Motorize some hinges

Leave others free to fold as either mountain or valley

Our main question:

How many motorized hinges do we need?

GPL image SwarmRobot_org.jpg by Sergey Kornienko from Wikimedia commons

Optimal solution = *minimum forcing set*

We solve this for the Miura fold
and for all other folds with same pattern

Non-standard Miura folds

To find minimum forcing sets for the Miura fold we need to understand the other folds that we want to prevent

http://www.umass.edu/researchnext/feature/new-materials-origami-style

E.g. easiest way to fold the Miura: accordion-fold a strip, zig-zag fold the strip, then reverse some of the folds

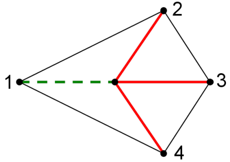
Locally flat-foldable: creases in same position as Miura, and a neighborhood of each vertex can be folded flat

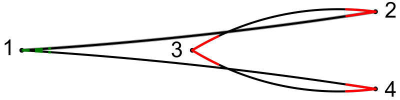
Bird's foot theorem

Describe folds by assignment of mountain fold or valley fold to each segment of the crease pattern

Locally flat-foldable Miura ⇔ at each vertex,

- The three toes of the bird foot are not all folded the same
- The leg is folded the same as the majority of the toes

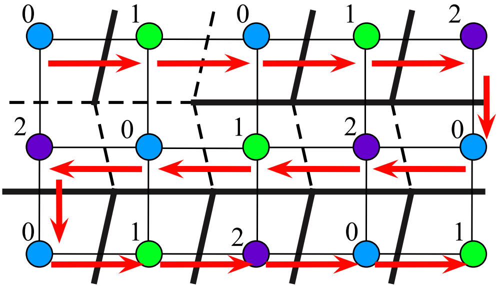




Follows from Maekawa's theorem ($|\#\text{Mountain} - \#\text{Valley}| = 2$) together with the observation that the fold with three toes one way and the leg the other way doesn't work

Locally flat-foldable Miura ≡ grid 3-coloring

[Hull & Ginepro, *J. Integer Seq.* 2014]



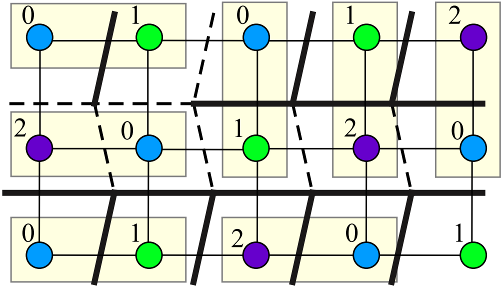
Follow boustrophedon (alternating left-to-right then right-to-left) path through the pattern, coloring quads with numbers mod 3

Path crosses mountain fold ⇒ next color is +1 mod 3
Path crosses valley fold ⇒ next color is −1 mod 3

Obeys bird's foot theorem ⇔ proper 3-coloring

Forcing sets from domino tilings

Forced crease $\equiv$ fixed difference (mod 3) between colors in the two squares of a domino (rectangle covering two adjacent Miura quads)

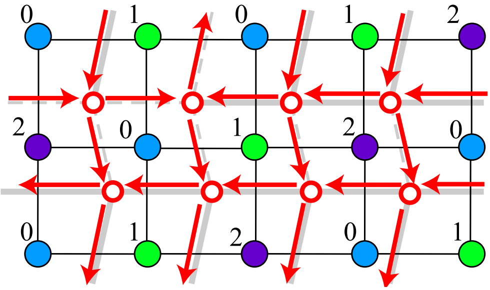


For standard Miura, two-domino tiling of a 2×2 square fixes the color differences for the other two dominoes in the square

All edges belong to some domino tiling, and all domino tilings are connected by 2×2 flips $\Rightarrow$ every domino tiling is a forcing set

But how good is it?

Grid 3-coloring $\equiv$ Eulerian orientation



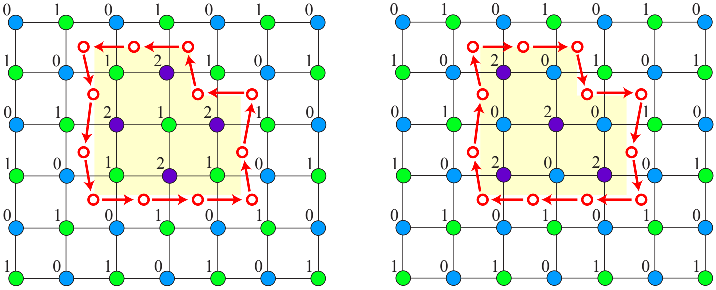
For each grid edge from color i to color $i+1$ (mod 3)
orient the crease segment that crosses it 90° clockwise
(so when viewed from the cell with color i, it goes left-to-right)

Form a directed graph with a vertex at each crease vertex
(+one more vertex, attached to all creases that reach paper edge)

Then at all vertices, indegree = outdegree

Recolorings and cycle reorientations

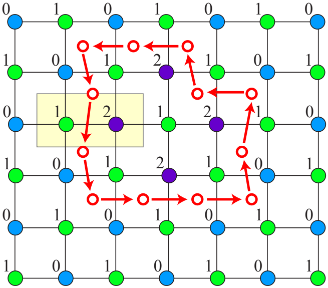
If the directed graph has a cycle, we can add $\pm 1 \pmod 3$ to all colors inside it (reversing the orientation of the cycle edges)

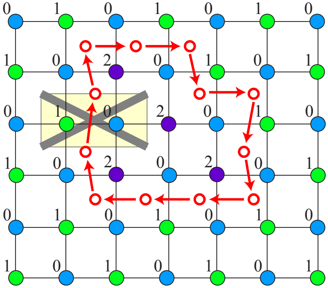


For every two different grid colorings, the difference between them can be broken down into recoloring steps of this type

Forcing set ≡ feedback arc set

Fixing the color difference of one grid edge prevents any recoloring step whose cycle crosses it





To prevent all recolorings, we must find a set of directed edges that intersect every cycle, force the crease type on those segments, and fix the color differences for the grid edges that cross them

What does this tell us about Miura forcing sets?

Planar minimum feedback arc set solvable in polynomial time

⇒ We can compute minimum forcing set for any non-standard Miura

Standard Miura ≡ checkerboard coloring ≡ orientation in which all quads are cyclically oriented

⇒ for every quad, at least one crease must be forced

⇒ domino tiling is optimal

Basil Rathbone as Sherlock Holmes

Conclusions and open problems

CC-BY-SA image "elf" (origamijoel/1450169902) by Joel Cooper on Flickr

For the standard Miura with $m \times n$ quadrilaterals, optimal forcing set size $= \left\lceil \frac{mn}{2} \right\rceil$

Every non-standard Miura fold has smaller forcing sets (sometimes $O(\sqrt{mn})$) that can be constructed in polynomial time

What about optimal forcing sets for other folding patterns?