Contact Graphs of Circular Arcs

M. J. Alam, **D. Eppstein**, M. Kaufmann, S. G. Kobourov,
S. Pupyrev, A. Schulz, and T. Ueckerdt

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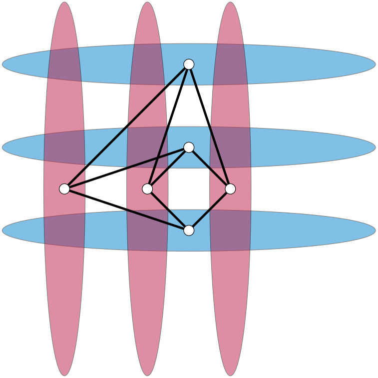
Intersection graphs vs contact graphs

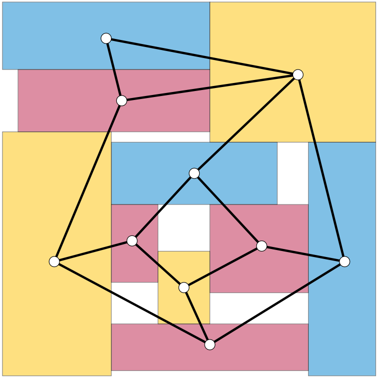
Intersection graph:

- Vertices: geometric objects
- Edges: overlapping pairs

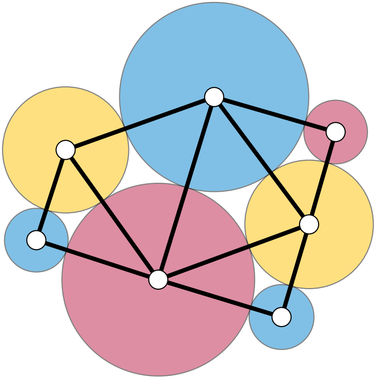


Contact graph:

- Objects cannot overlap
- Edges: touching pairs



Examples of contact graphs: disks



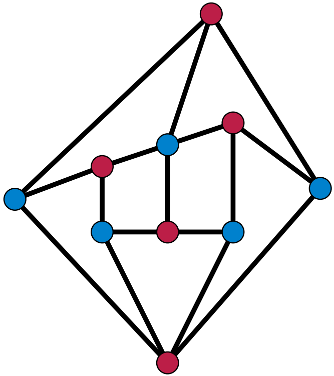
Koebe–Andreev–Thurston circle packing theorem:

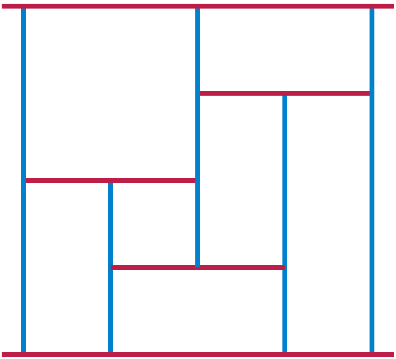
The contact graphs of disks are exactly the planar graphs

Many applications in graph theory, graph drawing, mesh generation, neuroanatomy, etc.

Another example: Axis-aligned segments

Each contact has one endpoint and one interior point



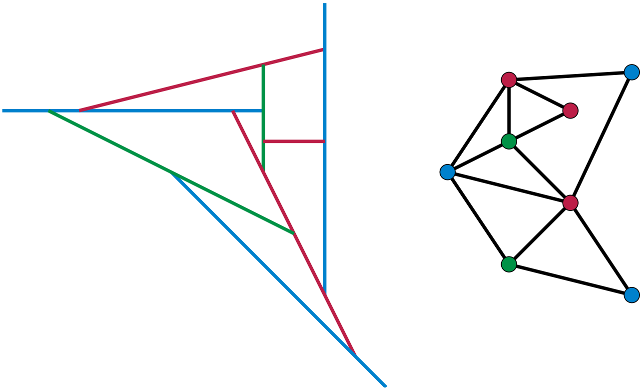


Realizable graphs are exactly the planar bipartite graphs

Hartman, Newman, and Ziv, "On grid intersection graphs", Disc. Math. 1991

Another example: Non-aligned segments

Each subset of k segments has ≥ 3 non-contact endpoints at the vertices of its convex hull, $2k - 3$ remaining potential contacts

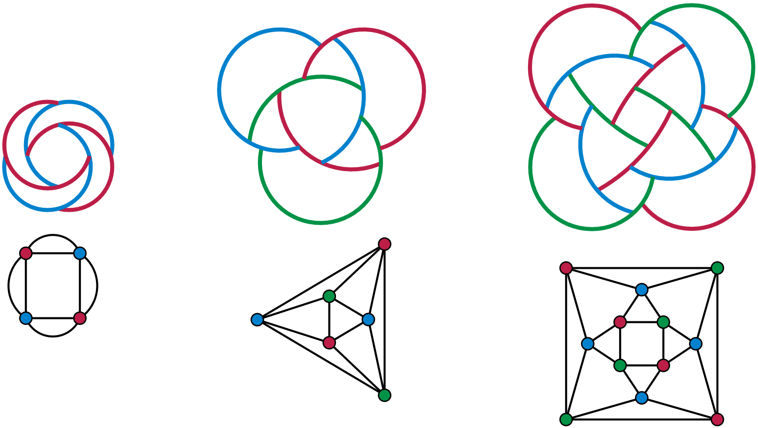


Realizable graphs are exactly the planar graphs in which every k vertices induce a subgraph with at most $2k - 3$ edges

Alam et al., "Proportional contact representations of planar graphs", JGAA 2012

Our question: What about circular arcs?

Only allow endpoint-interior contacts (else same as circle packings)



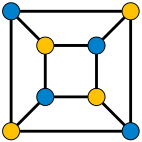
May not have any endpoint non-contacts on convex hull

Pairs of arcs may have multiple contacts $\Rightarrow$ multigraphs

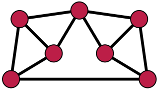
Sparse and tight graphs

(a, b)-sparse: each k-vertex subgraph has $\leq ak - b$ edges

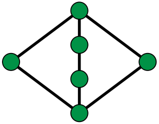
(a, b)-tight: (a, b)-sparse and whole graph has exactly $an - b$ edges



(2, 4)-tight



(2, 3)-tight



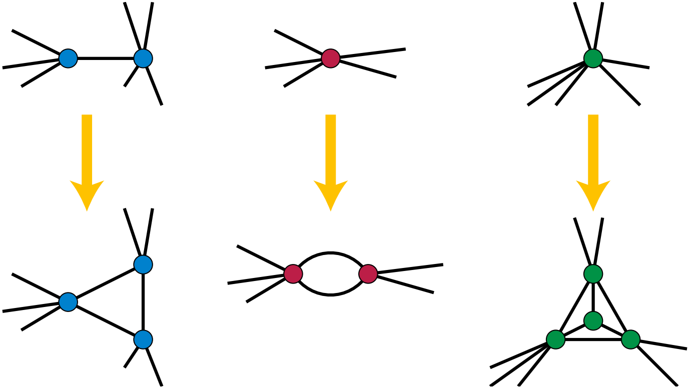
(2, 4)-sparse

For planar graphs:

- $(2, 3)$-tight = Laman (minimally rigid)
- $(2, 3)$-sparse = contact graph of line segments
- $(2, 4)$-tight = maximal bipartite
- $(2, 4)$-sparse = triangle-free
- For $a \in \{2, 3, 4\}$, dual of $(2, a)$-tight is always $(2, 4 - a)$-tight

Henneberg moves

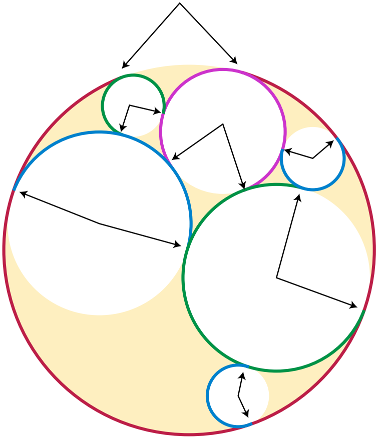
All $(2,2)$-tight and dual-$(2,3)$-tight graphs can be constructed by sequences of three moves, starting from simple base cases:



Each move can be performed in any circular arc representation
(Proof: messy case analysis)

Corollary: All such graphs can be represented by circular arcs

Arc representations from circle packings



Break circles into arcs turning tangencies into arc contacts

Extra property: each arc has empty convex hull

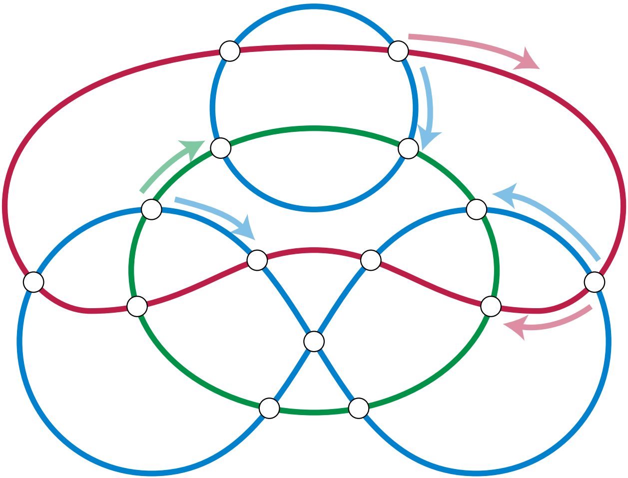
This method works $\iff$ graph has an edge orientation with

- Outdegree ≤ 2
- When outdegree $= 2$, both out-edges are adjacent

Which graphs have such orientations?

4-regular graphs have good orientations

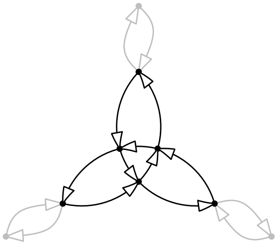
Group opposite pairs of edges at each vertex into curves, orient each curve consistently



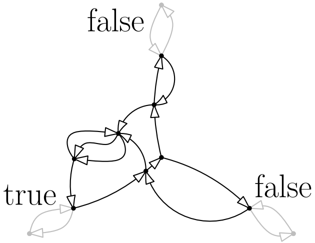
Orienting $(2,0)$-tight graphs is NP-hard

Reduction: positive planar 1-in-3 SAT $\rightarrow$ multigraph orientation $\rightarrow$ simple graph orientation

Wire gadget



Splitter gadget



Clause gadget

Conclusions and open problems

Simple necessary condition for arc representation: $(2,0)$-sparse

Simple sufficient conditions: dual $(2,a)$-tight, $a \in \{2,3,4\}$

Related hardness results possibly indicating
the actual story may be more complicated...

Do all planar $(2,0)$-sparse graphs have arc representations?
Not true for multigraphs with fixed embeddings: