Matrix Decomposition and Latent Semantic Indexing (LSI)

Introduction to Information Retrieval
CS 221
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Outline

- Introduction
- Linear Algebra Refresher

Star Cluster NGC 290 - ESA & NASA

Star Cluster NGC 290 - ESA & NASA

- A picture of the sky is two dimensional
- The stars are not in two dimensions
- When we take a photo of stars we are projecting them into 2-D
  - projecting can be defined mathematically
- When we see two stars that are close..
  - They may not be close in space
- When we see two stars that appear far...
  - They may not be far in 3-D space

Star Cluster NGC 290 - ESA & NASA

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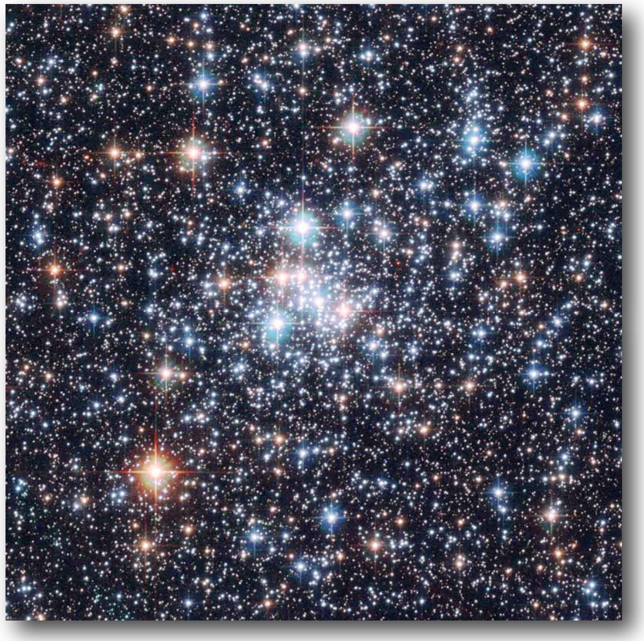

Star Cluster NGC 290 - ESA & NASA

- When we see two stars that are close in a photo
  - They really are close for some applications
  - For example pointing a big telescope at them
  - Large shared telescopes order their views according to how "close" they are.

Overhead projector example

Overhead projector example

- Depending on where we put the light (and the wall) we can make things in three dimensions appear close or far away in two dimensions.
- Even though the “real” position of the 3-d objects never moved.

Mathematically speaking

- This is taking a 3-D point and projecting it into 2-D

(x, y, z) → (x, y)

(10, 10, 10) → (10, 10)

$$\begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} \qquad \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

- The arrow in this picture acts like the overhead projector

Mathematically speaking

- We can project from any number of dimensions into any other number of dimensions.
- Increasing dimensions adds redundant information
  - But sometimes useful
  - Support Vector Machines (kernel methods) do this effectively
- Latent Semantic Indexing always reduces the number of dimensions

Mathematically speaking

- Latent Semantic Indexing always reduces the number of dimensions

(x,y) → (x)

(10, 10) → (10)

$$\begin{bmatrix} 10 \\ 10 \end{bmatrix} \rightarrow \begin{bmatrix} 10 \end{bmatrix}$$

Mathematically speaking

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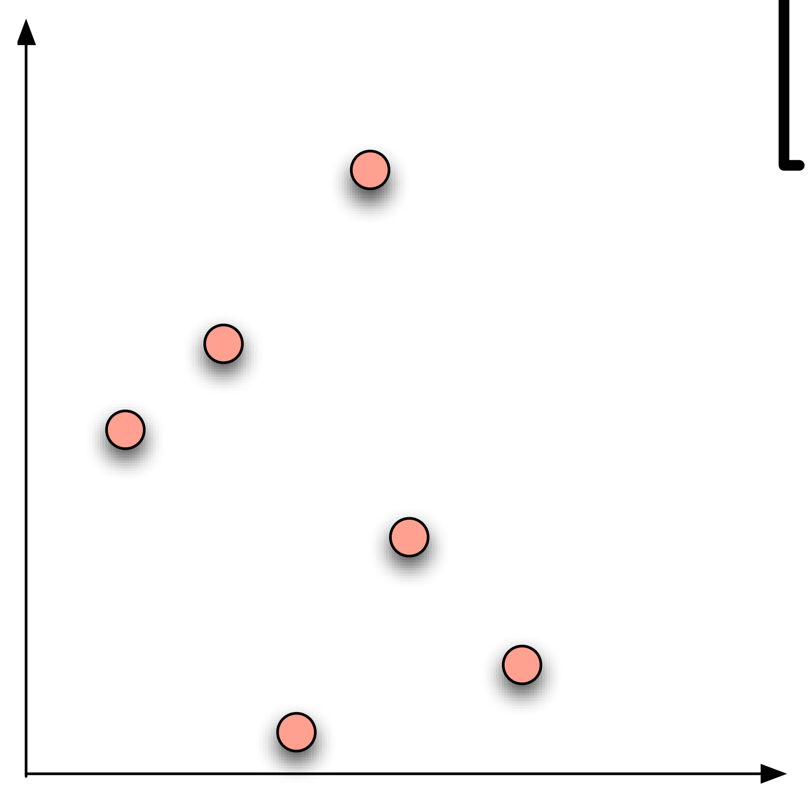

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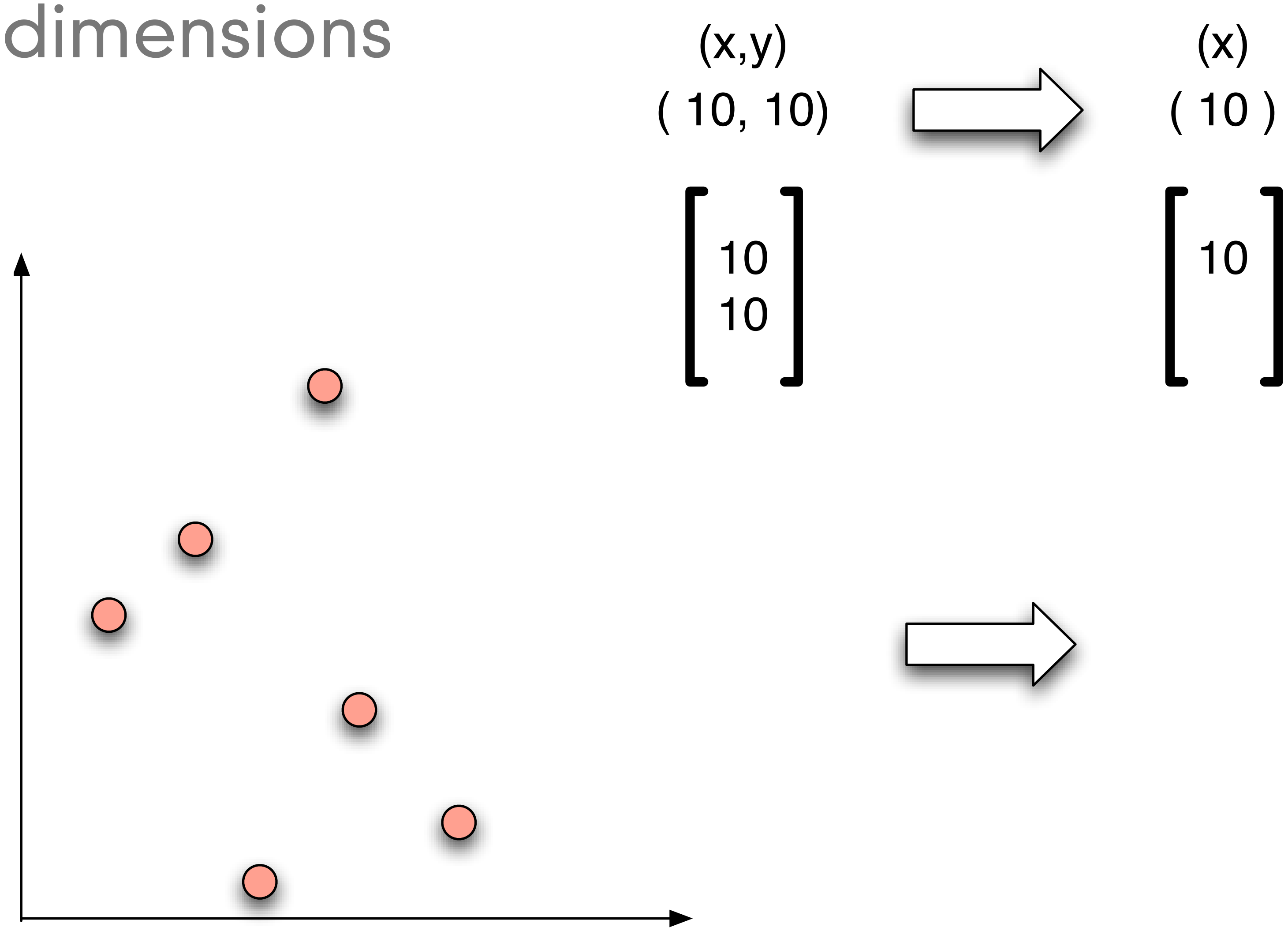

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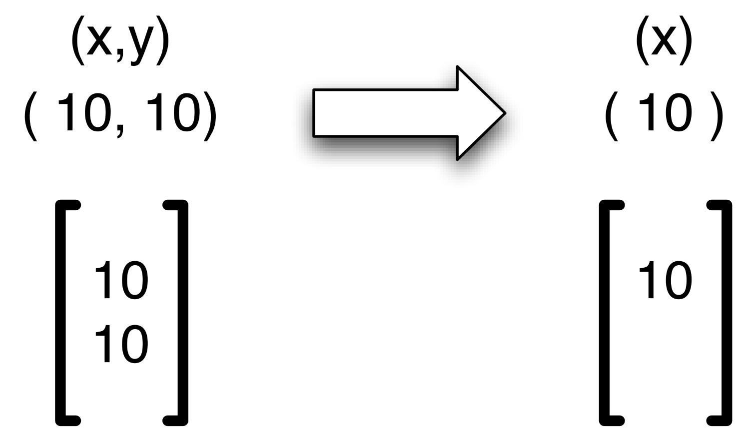
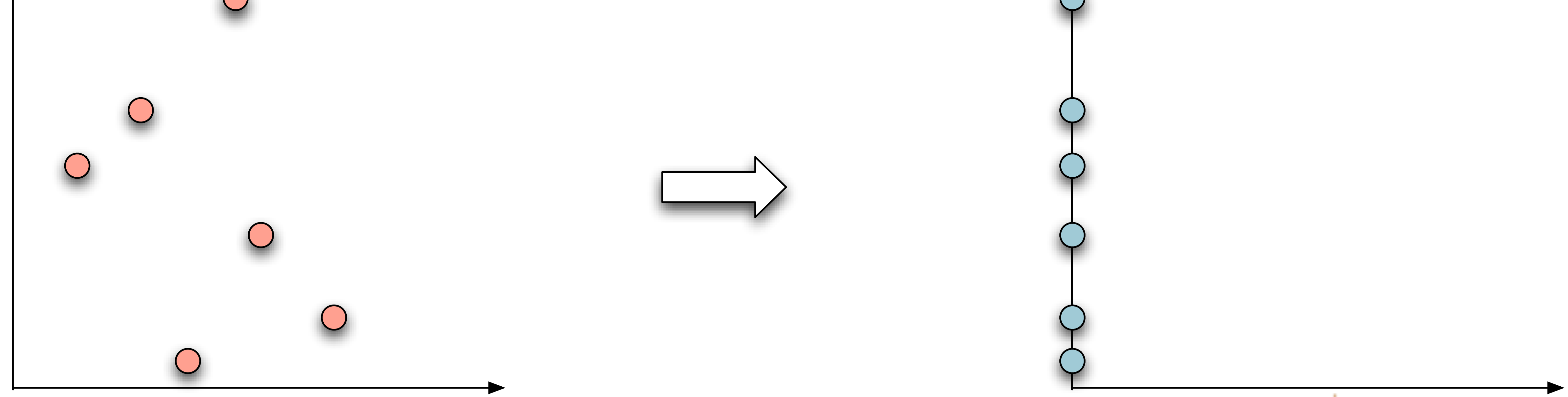
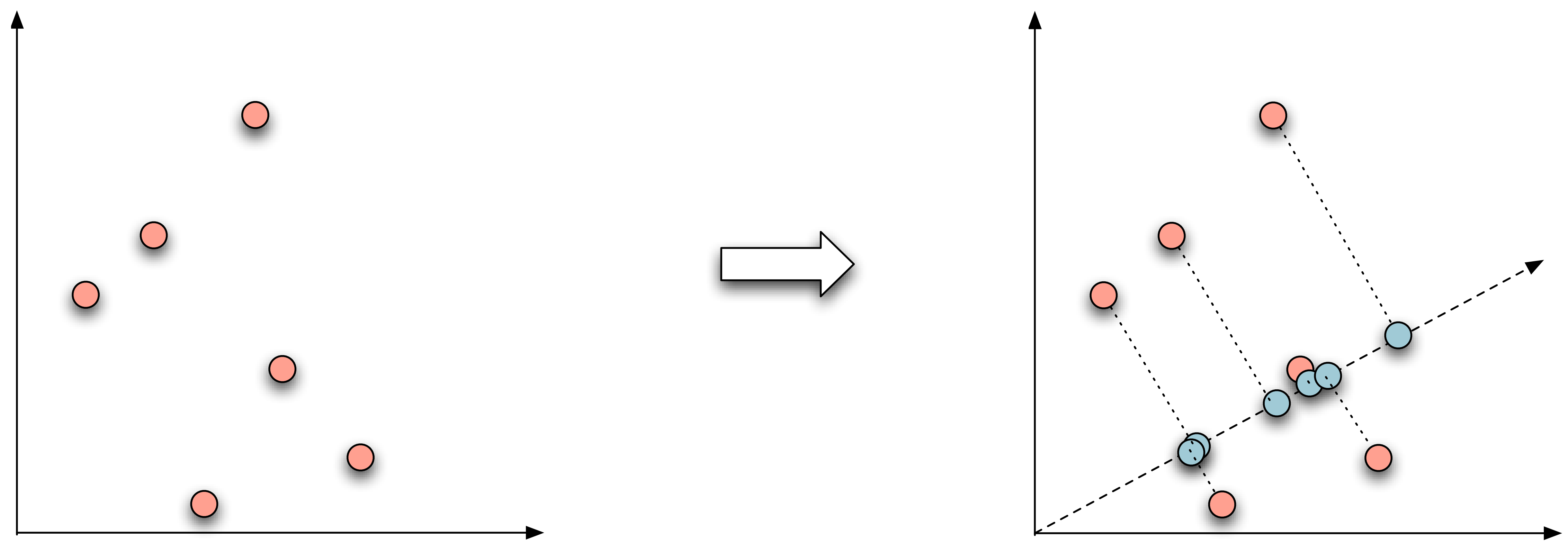




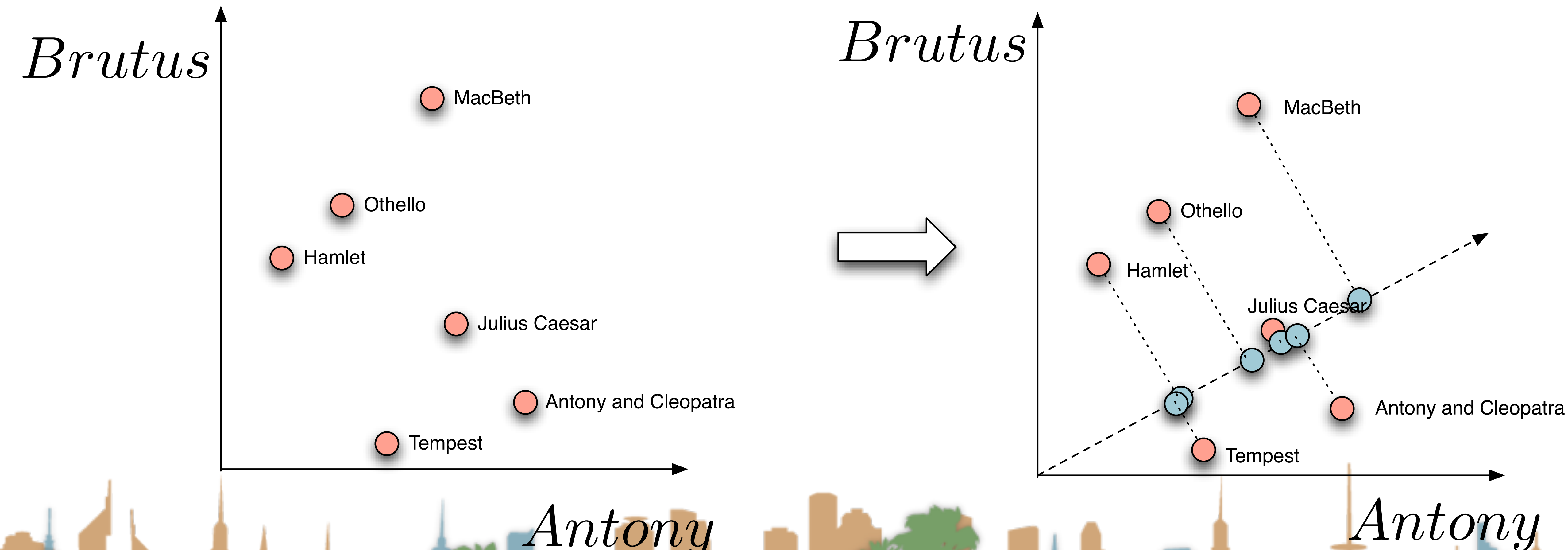
Mathematically speaking

- Latent Semantic Indexing can project on an arbitrary axis, not just a principal axis



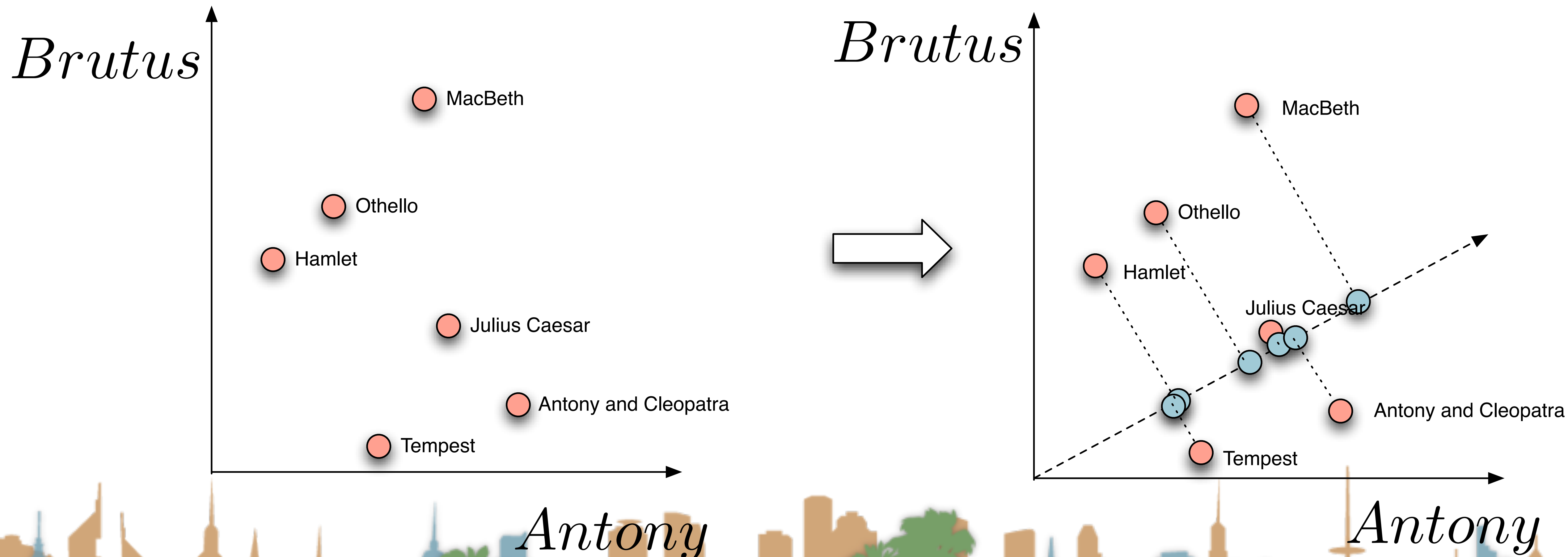
Mathematically speaking

- Our documents were just points in an N-dimensional term space
- We can project them also



Mathematically speaking

- Latent Semantic Indexing makes the claim that these new axes represent semantics - deeper meaning than just a term



Mathematically speaking

- A term vector that is projected on new vectors may uncover deeper meanings
  - For example
    - Transforming the 3 axes of a term matrix from “ball” “bat” and “cave” to
      - An axis that merges “ball” and “bat”
      - An axis that merges “bat” and “cave”
    - Should be able to separate differences in meaning of the term “bat”
    - Bonus: less dimensions is faster

Linear Algebra Refresher

- Let C be an M by N matrix with real-valued entries
    - for example our term document matrix
- A matrix with the same number of rows and columns is called a square matrix
- An M by M matrix with elements only on the diagonal is called a diagonal matrix
- The identity matrix is a diagonal matrix with ones on the main diagonal

Linear Algebra Refresher

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$N=5$, $M=3$

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N=5

$$M=3 \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

C

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

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N=5

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C

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- The identity matrix is a diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Decomposition

- Singular Value Decomposition
  - Splits a matrix into three matrices
  - Such that
    - If
    - then
    - and
    - and
    - also Sigma is almost a diagonal matrix

Matrix Decomposition

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  - Splits a matrix into three matrices $U \quad \Sigma \quad V^T$
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$$U \quad \Sigma \quad V^T$$

$$C = U\Sigma V^T$$

$$C \; is \; (M \; by \; N)$$

Matrix Decomposition

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    - then $U\ is\ (M\ by\ M)$
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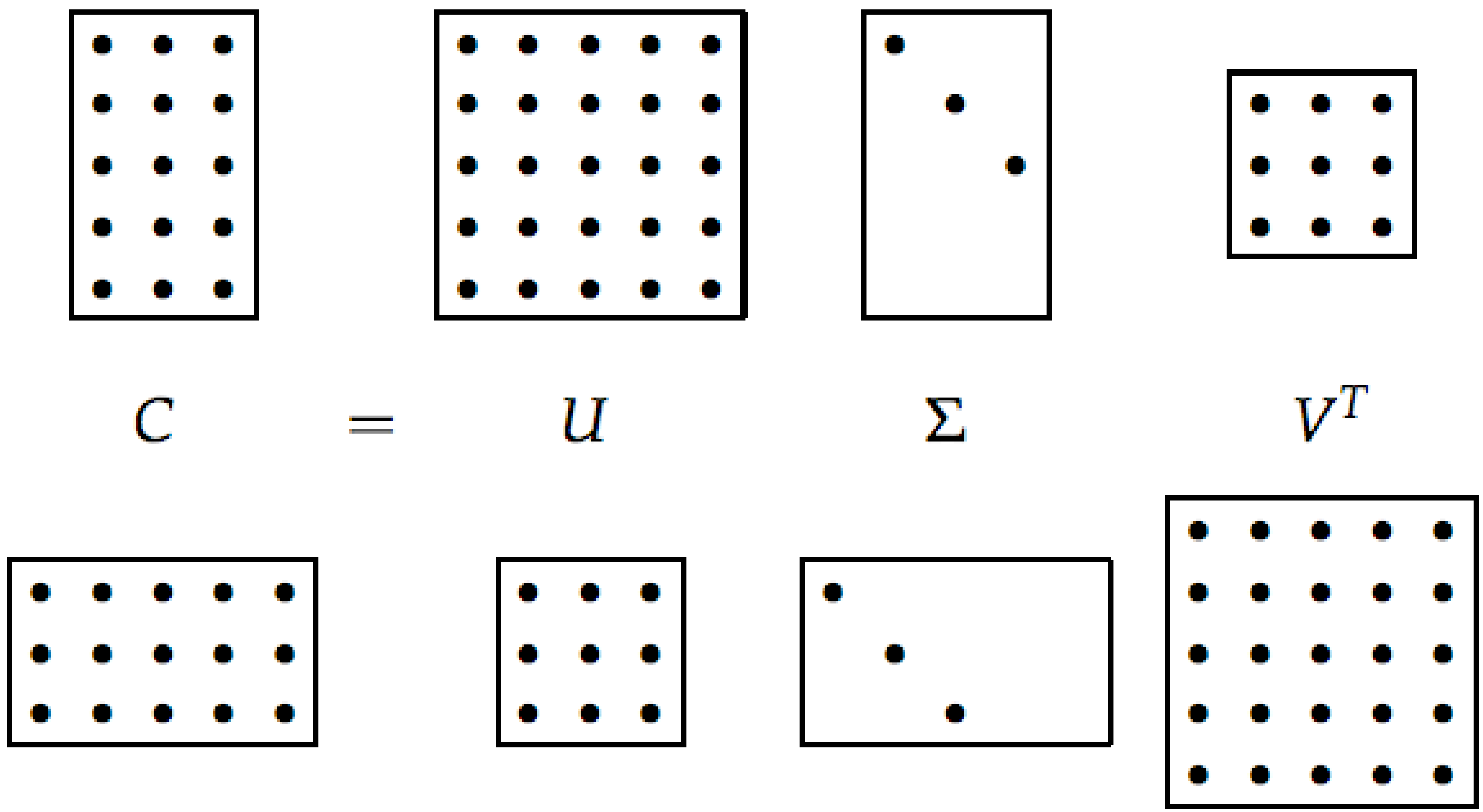
Matrix Decomposition



Matrix Decomposition

- Singular Value Decomposition
  - Is a technique that splits a matrix into three components with these properties.
  - They also have some other properties which are relevant to latent semantic indexing

Matrix Decomposition

- Singular Value Decomposition
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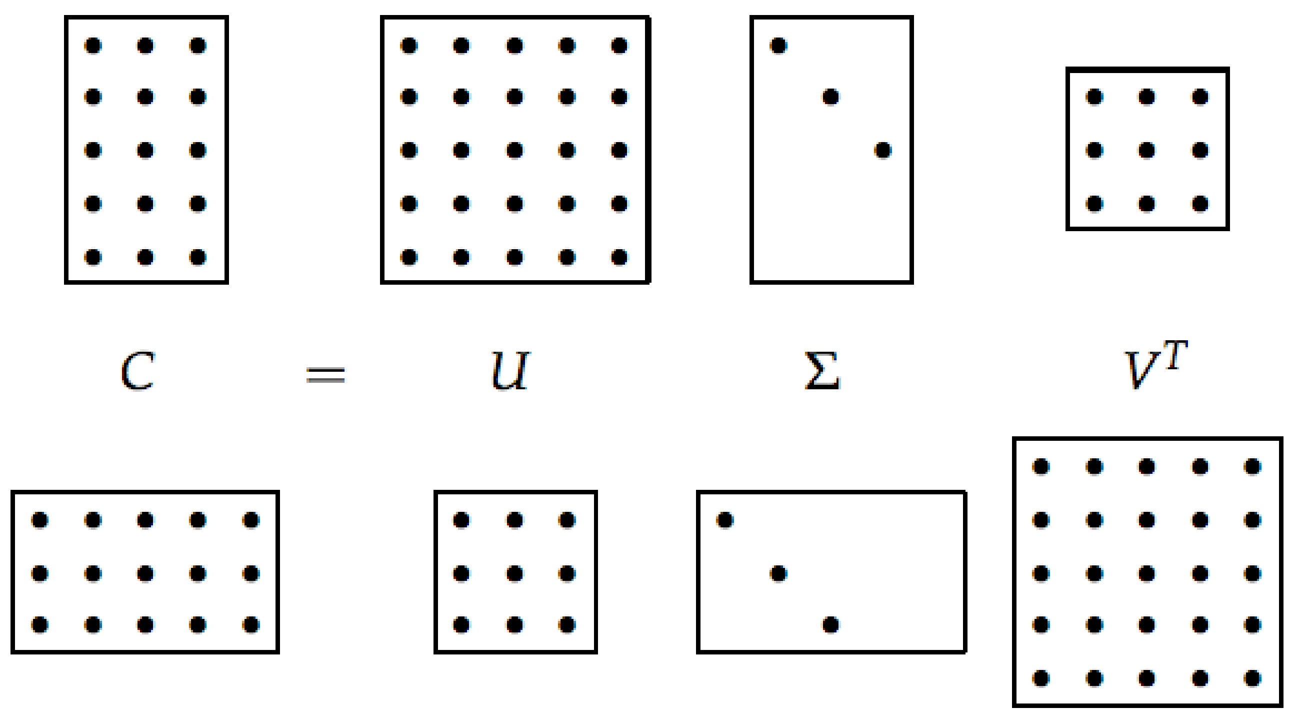
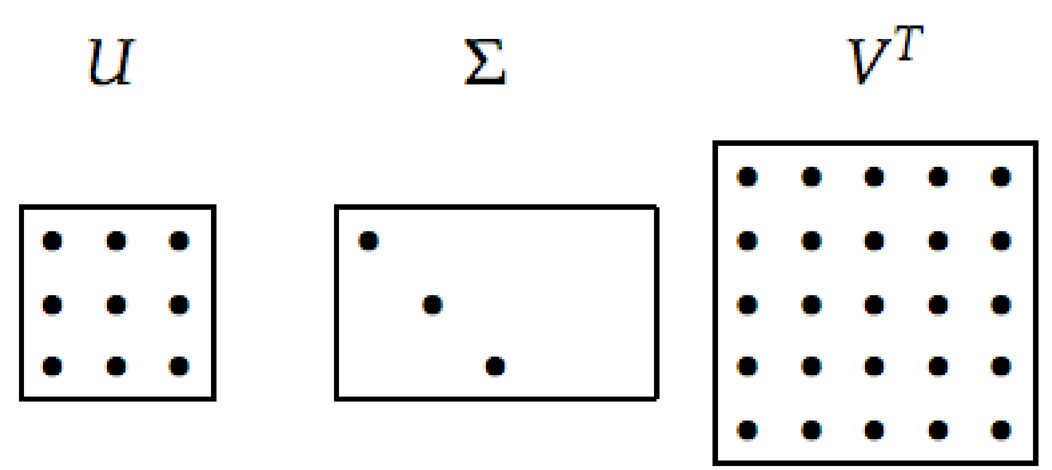


Matrix Decomposition

- Singular Value Decomposition
  - SVD enables lossy compression of your term-document matrix
    - reduces the dimensionality or the rank
    - you can arbitrarily reduce the dimensionality by putting zeros in the bottom right of sigma
    - this is a mathematically optimal way of reducing dimensions



Matrix Decomposition

- Singular Value Decomposition
  - If the old dimensions were based on terms
    - after reducing the rank of the matrix the dimensionality is based on concepts or semantics
    - a concept is a linear combination of terms

$$SVD_{dimension_1} = a * td_{dim_1} + b * td_{dim_2} + c * td_{dim_3} + d * td_{dim_4}$$

$$SVD_{dimension_2} = a' * td_{dim_1} + b' * td_{dim_2} + c' * td_{dim_3} + d' * td_{dim_4}$$

$$SVD_{dimension_3} = a'' * td_{dim_1} + b'' * td_{dim_2} + c'' * td_{dim_3} + d'' * td_{dim_4}$$

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- 4 dimensions to 3 dimensions

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{vmatrix}$$

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- 4 dimensions to 3 dimensions

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{vmatrix} * \begin{vmatrix} td_{dim_1} \\ td_{dim_2} \\ td_{dim_3} \\ td_{dim_4} \end{vmatrix}$$

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$$\left| \begin{array}{c} SVD_{dim_1} \\ SVD_{dim_2} \\ SVD_{dim_3} \end{array} \right| = \left| \begin{array}{cccc} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{array} \right| * \left| \begin{array}{c} td_{dim_1} \\ td_{dim_2} \\ td_{dim_3} \\ td_{dim_4} \end{array} \right|$$

Matrix Decomposition

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$$SVD_{ConceptSpace} = M * query_{TermSpace}$$

Matrix Decomposition

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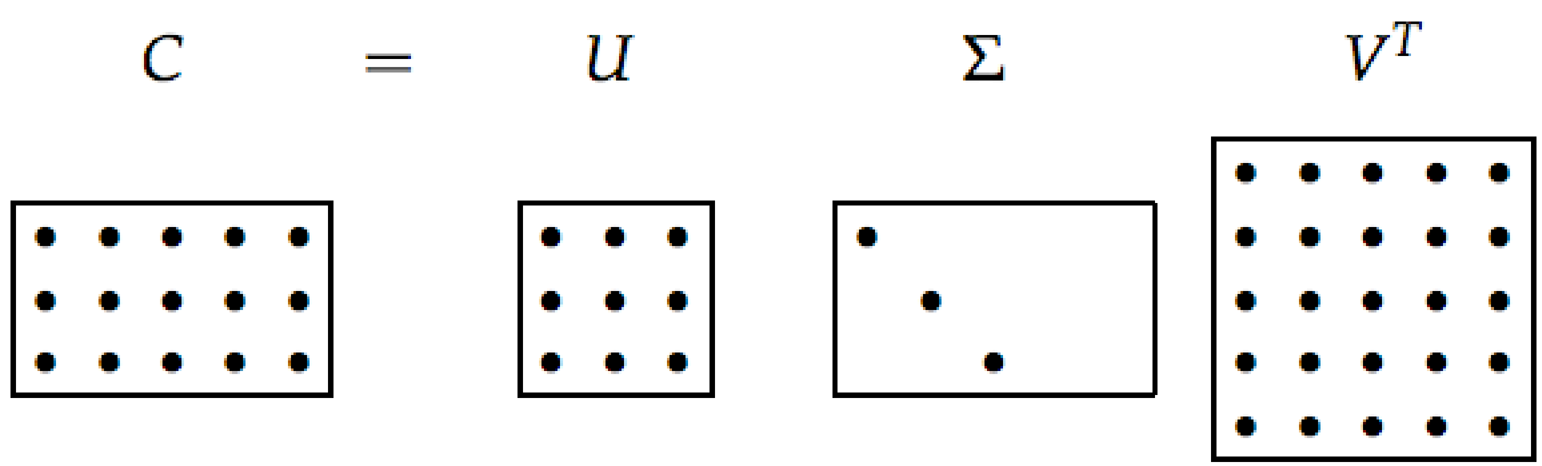

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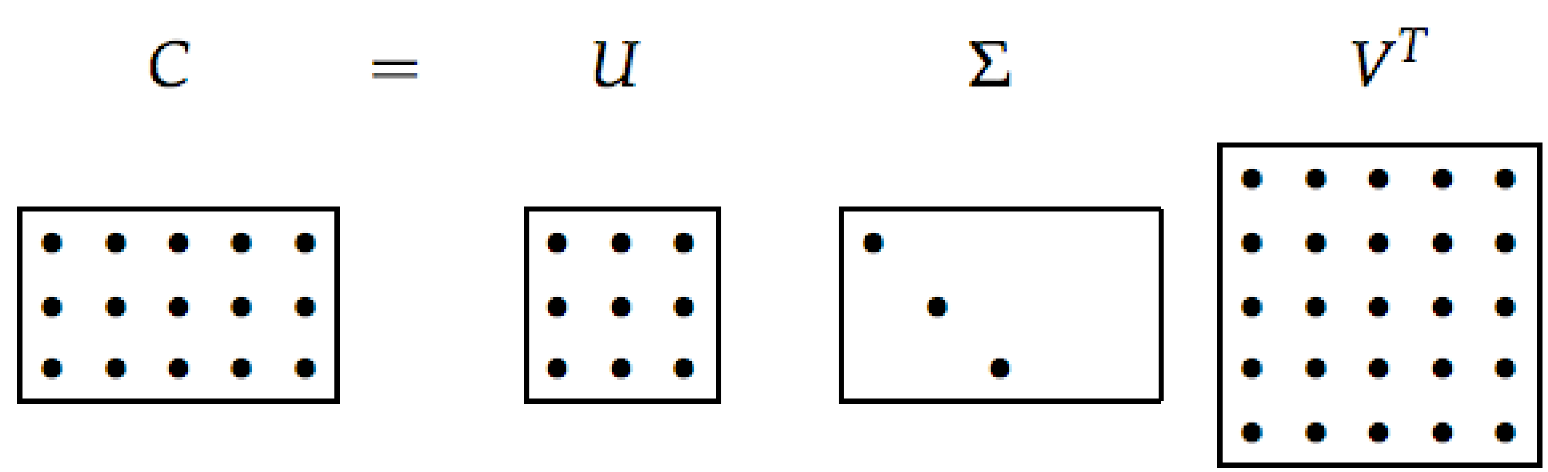

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$$SVD_{ConceptSpace} = M * query_{TermSpace}$$

$$M = \Sigma_k^{-1} U_k^T$$

$$query_{ConceptSpace} = \Sigma_k^{-1} U_k^T query_{TermSpace}$$

Matrix Decomposition

- Singular Value Decomposition
  - SVD is an algorithm that gives us $\Sigma \; U \; V^T$
  - With these quantities we can reduce dimensionality
  - With reduced dimensionality
    - synonyms are mapped onto the same location
      - “bat” “chiroptera”
    - polysemies are mapped onto different locations
      - “bat” (baseball) vs. “bat” (small furry mammal)

Latent Semantic Indexing - Linear Algebra Refresher

- Computing SVD takes a significant amount of CPU
- It is possible to add documents to a corpus without recalculating SVD
  - The result becomes an approximation
  - To get mathematical guarantees the whole SVD needs to be computed from scratch
- LSI doesn't support negation queries
- LSI doesn't support boolean queries

Matrix Decomposition

- "I am not crazy"
  - Netflix

Latent Semantic Indexing - Linear Algebra Refresher

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Matrix Decomposition

- “I am not crazy”
  - Netflix
  - Machine translations
    - Just like “bat” and “chiroptera” map the same
    - “bat” and “murciélago” can map to the same thing

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That one mathematically is hard, but is beautiful and at long range next...

LUCI