# Exploring UFO's

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## CONTENTS



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## GLOSSARY

**$\tau$-underflowed function**

($f_\tau$) A function such that output values less than $\tau$ are replaced with 0.0.

$$f_{\alpha_\tau}(\boldsymbol{a} \in D_\alpha) = \begin{cases} f_\alpha(\boldsymbol{a}), & f_\alpha(\boldsymbol{a}) \geq \tau \\ 0, & \text{otherwise.} \end{cases}$$

4

**graphical model**

($\mathcal{M} = \langle \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F} \rangle$). Mathematical tool for modeling complex systems composed of a set of variables $\boldsymbol{X}$, a set of domains $\boldsymbol{D} = \{D_X | X \in \boldsymbol{X}\}$ for each variable $X$, and a set of functions $\boldsymbol{F}$ with each function defined over a subset of the model's variables $\alpha \subseteq \boldsymbol{X}$. 4

**marginal maximum a-posteriori**

(MMAP). The marginal likelihood associated with the configuration of a target subset of variables $\boldsymbol{Q}$ that maximizes their marginal likelihood.
In the context of discrete graphical models without evidence, with $\boldsymbol{Q} \subset \boldsymbol{X}$, $\boldsymbol{S} = \boldsymbol{X} \setminus \boldsymbol{Q}$ be the variables to sum over, $\boldsymbol{F_Q} = \{f_\alpha \mid \alpha \subseteq \boldsymbol{Q}\}$ be the set of functions defined only over $\alpha \in \boldsymbol{Q}$, and $\boldsymbol{F_S} = \boldsymbol{F} \setminus \boldsymbol{F_Q}$ be functions that include some $X \in \boldsymbol{S}$ in their scope,

$$MMAP = \max_{\boldsymbol{Q}} \sum_{\boldsymbol{S}} \prod_{f_\alpha \in \boldsymbol{F}} f_\alpha(\boldsymbol{q} \cup \boldsymbol{s}) \tag{1}$$

$$= \max_{\boldsymbol{Q}} \prod_{f'_\alpha \in \boldsymbol{F_S}} f'_\alpha(\boldsymbol{q}) \sum_{\boldsymbol{S}} \prod_{f''_\alpha \in \boldsymbol{F_S}} f''_\alpha(\boldsymbol{q} \cup \boldsymbol{s}) \tag{2}$$

3, 4, 7

**maximum a-posteriori**

(MAP). With respect to a graphical model, the likelihood value associated with the **most probable explanation** or **MPE**.
In the context of discrete graphical models without evidence, $MAP = \max_{\boldsymbol{Q}=\boldsymbol{X}} \prod_{f_\alpha \in \boldsymbol{F}} f_\alpha(\boldsymbol{q})$. 3, 4

**most probable explanation**

(MPE). Assignment to variables the variables of a graphical model that maximizes the conditional probability of the observed evidence.
In the context of discrete graphical models without any evidence, $MPE = \text{argmax}_{\boldsymbol{Q}=\boldsymbol{X}} \prod_{f_\alpha \in \boldsymbol{F}} f_\alpha(\boldsymbol{q})$. 2, 3

**partial configuration**

A joint assignment to a subset of the variables of a graphical model. 4

**partition function**

(Z). A mathematical quantity that characterizes the distribution among a system's possible states and serves as a normalizing constant for calculating probabilistic measures associated with these states.
In the context of discrete graphical models, $Z = \sum_{\boldsymbol{X}} \prod_{f_\alpha \in \boldsymbol{F}} f_\alpha(\boldsymbol{x})$. 3

## ABBREVIATIONS

**MAP**
: **Maximum a-posteriori**. 7

**MMAP**
: **Marginal maximum a-posteriori**. 7

**MPE**
: **Most probable explanation**. 2

**Z**
: **Partition function**. 7

## NOTATION

**[capital letters] (ex. $X$)**
: Represents a variable of a **graphical model**. 7

**[lower-case letters] (ex. $x$)**
: Represent assignments to variable corresponding to their capitalized form. For example, $x$ represents a particular assignment to the variable represented by $X$, or $X \leftarrow x$. 7

**[bold-faced capital letters] (ex. $\boldsymbol{X}$)**
: A set of variables of a **graphical model**. ($\boldsymbol{X}$, in particular, often refers to the set of all variables of a graphical model). 7

$\boldsymbol{X}$
: The set of all variables $X$ of a **graphical model**. 4

$\boldsymbol{Q}$
: In the context of the **maximum a-posteriori** or **marginal maximum a-posteriori** task, the [sub]set of the variables $\boldsymbol{Q}$ that are to be maximized over (know as "query" or "MAP" variables). 7

$\boldsymbol{S}$
: In the context of the **marginal maximum a-posteriori** task, the subset of variables to be summed over. $\boldsymbol{S} = \boldsymbol{X} \setminus \boldsymbol{Q}\}$ 7

$\boldsymbol{x}$
: A full configuration, ie. assignment to all variables $\boldsymbol{X}$ of a **graphical model**. $\boldsymbol{X} \leftarrow \boldsymbol{x} \in D_{\boldsymbol{X}}$. 7

$D_{\boldsymbol{Y}}$
: The set of all possible **partial configurations** to the variables of the subset $\boldsymbol{Y} \subset \boldsymbol{X}$. $D_{\boldsymbol{Y}}$ is the Cartesian product of the domains of the variables in $\boldsymbol{Y}$. $D_{\boldsymbol{Y}} = \bigotimes_{\{D_Y | Y \in \boldsymbol{Y}\}} D_Y$, where $\bigotimes$ is the Cartesian product operator. 7

$\boldsymbol{F}_\tau$
: $\boldsymbol{F}_\tau = \{f_\tau \mid f \in \boldsymbol{F}\}$ 7

$\boldsymbol{F}_{\boldsymbol{Q}}$
: In the context of the **maximum a-posteriori** or **marginal maximum a-posteriori** task, the subset of functions defined only on a subset of the variables $\boldsymbol{Q}$ that are to be maximized over (know as "query" or "MAP" variables). $\boldsymbol{F}_{\boldsymbol{Q}} = \{f_\alpha \mid \alpha \subseteq \boldsymbol{Q}\}$ 7

$\boldsymbol{F}_{\boldsymbol{S}}$
: In the context of the **marginal maximum a-posteriori** task, $\boldsymbol{F}_{\boldsymbol{S}} = \boldsymbol{F} \setminus \boldsymbol{F}_{\boldsymbol{Q}} = \{f_\alpha \mid \exists X \in \alpha s.t. X \notin \boldsymbol{Q}\}$ 7

$\mathcal{M}$
: **Graphical model** $\mathcal{M} = \langle \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F} \rangle$ with non-negative functions. 7

$\mathcal{M}_\tau$
: An altered **Graphical model** with non-negative functions such that each original function is replaced by its corresponding $\tau$**-underflowed function**. $\mathcal{M}_\tau = \langle \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F}_\tau \rangle$. 7

$MMAP(\mathcal{M}, \boldsymbol{Q})$
: The **marginal maximum a-posteriori** of graphical model $\mathcal{M}$ maximizing over the subset of variables $\boldsymbol{Q}$. 7

# 1 BACKGROUND

Probabilistic graphical models are powerful tools for modeling complex systems with local structure. A discrete graphical model can be defined as a 3-tuple $\mathcal{M} = \langle \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F} \rangle$, where: $\mathbf{X}$ is a set of variables for which the model is defined; $\mathbf{D} = \{D_X : X \in \mathbf{X}\}$ is a set of finite domains, each defining the possible assignments for a variable; each $f_\alpha \in \mathbf{F}$ is a real-valued function defined over a subset of the model's variables $\alpha \subseteq \mathbf{X}$ known as the function's **scope**. More concretely, if we let $D_\alpha$ denote the Cartesian product of the domains of the variables in $\alpha$, then $f_\alpha : D_\alpha \to \mathbb{R}_{\geq 0}$. We let capital letters ($X$) represent variables and small letters ($x$) represent their assignment. Boldfaced capital letters ($\mathbf{X}$) denote a collection of variables, $|\mathbf{X}|$ its cardinality, $D_{\boldsymbol{X}}$ its joint domain, and $\boldsymbol{x}$ a particular realization in that joint domain called a **configuration**. Operations denoted $\bigoplus_{\boldsymbol{X}}$ (ex. $\sum_{\boldsymbol{X}}$) imply $\bigoplus_{\boldsymbol{X}} \implies \bigoplus_{\boldsymbol{x} \in D_{\boldsymbol{X}}}$.

Some common queries to a graphical model include asking for the most likely assignment to the variables, marginal probabilities of some variables, or more complex tasks such as determination of the **marginal maximum a-posteriori (or MMAP)** (Definition 1.0.0.1, [Dechter, 2019]).

**Definition 1.0.0.1** (MMAP)
*Given a graphical model $\mathcal{M} = \langle \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F} \rangle$, the marginal maximum a-posteriori of $\mathcal{M}$ is:*

$$MMAP(\mathcal{M}, \boldsymbol{Q} \subset \boldsymbol{X}) = \max_{\boldsymbol{Q}} \sum_{\boldsymbol{S} = \boldsymbol{X} \setminus \boldsymbol{Q}} \prod_{\boldsymbol{F}} f(\boldsymbol{q}, \boldsymbol{s}) \tag{3}$$

Answering these queries involve computational tasks that are NP-Hard thus for large models obtaining reasonable approximations is often the goal. To this end, many approximate algorithms have been developed for a wide variety of these tasks. In this work, we add a new tool to this arsenal in the form of a new algorithm called UFO.

# 2 UFO

Utilizing constraint propagation (CP) as a tool for pruning inconsistent search paths has been shown to be able to greatly speed up search Dechter [2019], Mateescu and Dechter [2008], Darwiche [2009]. Similar ideas have been explored in mixed integer programming [Danna et al., 2005]. More recently in the scope of protein design Pezeshki et al. [2022] demonstrated that introducing artificially generated determinism by underflowing function values under a provided threshold (Definition 2.0.0.1) can further leverage CP and enhance the speed of solving K* optimization problems. Thus, being able to identify a good threshold and use it to speed up search could be invaluable to solving certain problems.

**Definition 2.0.0.1** ($\tau$-underflow of $f$, $f_\tau$)
*Let $f$ be a non-negative function and $\tau \in \mathbb{R}^+$. The $\tau$-underflow of $f$ is $f_\tau(x) = f(x)$ if $f(x) \geq \tau$ and $0$, otherwise.*

**Definition 2.0.0.2** ($\tau$-underflow of $\mathcal{M}$, $\mathcal{M}_\tau$)
*For $\mathcal{M} = \langle \mathbf{X}, \mathbf{D}, \mathbf{F} \rangle$, the $\tau$-underflow of $\mathcal{M}$ is $\mathcal{M}_\tau = \langle \mathbf{X}, \mathbf{D}, \mathbf{F}_\tau \rangle$, where $\mathbf{F}_\tau = \{f_\tau \mid f \in \boldsymbol{F}\}$.*

**Definition 2.0.0.3** (Inconsistent Model)
*A model is said to be inconsistent if $\forall \boldsymbol{x} \in D_{\boldsymbol{X}}$, $\prod_{\boldsymbol{F}} f(x) = 0$.*

Choosing a larger underflow-threshold leads to more underflows and consequently more infused determinism thus resulting in more aggressive pruning via CP. However, if the threshold is set too high, the resulting model becomes inaccurate and may even become inconsistent altogether, resulting in pruning of all configurations. Therefore it is useful to find a threshold that is as high as possible yet still results in a consistent model.

Algorithm 1: UFO (underflow-threshold optimization) describes a general methodology for choosing such an underflow-threshold. To achieve this UFO employs binary search to find the largest threshold that still results in a satisfiable model (lines 6-12). Then UFO decreases the threshold using a hyper-parameter $\delta$ (line 14) to enable a wider array of solutions.

Note that UFO operates under the assumption that satisfiability of a model can be determined quickly. This is not true in general, nevertheless we have found that the satisfiability sub-task underlying many optimization problems tends to be easy. In other cases, satisfiability can be approximated by constraint propagation schemes [Dechter, 2003].

**Algorithm 1:** UFO

**input** : Graphical model $\mathcal{M} = \langle \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{F} \rangle$; SAT solving algorithm, $SAT(.)$ ; time limit for binary search; a deflation factor $0 < \delta \leq 1$

**output** : A proposed threshold $\tau$ to use

**begin**
    **if** $SAT(\mathcal{M}) = False$ **then**
        return $FAILURE$
    $\tau_{min} = 0; \;\; \tau_{max} = \max_{\boldsymbol{F},\boldsymbol{X}} f(\boldsymbol{x})$
    $\tau = \frac{\tau_{max} + \tau_{min}}{2}$
    **while** *time remains for $\tau$ binary search* **do**
        **if** $SAT(\mathcal{M}_\tau) = False$ **then**
            $\tau_{max} = \tau$
        **else**
            $\tau_{min} = \tau$
        $\tau = \frac{\tau_{max} + \tau_{min}}{2}$
    **end**
    $\tau = \tau_{min} \cdot \delta$
    **return** $\tau$
**end**

## 3 PROPERTIES

**Theorem 3.0.0.1** (Lower-bounds from $\tau$-underflows)
$\forall task \in \{Z, MAP, MMAP\}, \forall \mathcal{M}, task(\mathcal{M}_\tau) \leq task(\mathcal{M})$

*Proof*
The evaluated value of summation, maximization, or mixed max-sum operations over products of non-negative functions can only decrease if function values decrease (but remain non-negative). By definition, $\tau$-underflows either do not affect function values or decrease them to 0.0. Thus, the evaluated value of summation, maximization, or mixed max-sum operations can only decrease, and never increase, due to a $\tau$-underflows. □

**Corollary 3.0.0.2** (Monotonicity from $\tau$-underflows)
$\forall task \in \{Z, MAP, MMAP\}, \forall \mathcal{M}, \forall \tau' < \tau'', task(\mathcal{M}_{\tau''}) \leq task(\mathcal{M}_{\tau'})$.

*Proof*
$\forall \mathcal{M}, \forall \tau' < \tau'', \mathcal{M}_{\tau''} = \mathcal{M}_{\tau'_{\tau''}}$. Thus, the result follows directly from Theorem 3.0.0.1 □

**Corollary 3.0.0.3** (Solution persistence)
*$\forall task \in \{Z, MAP, MMAP\}$, if a $\tau$-underflow does not affect the function values involved in the computation of the solution cost, then $task(\mathcal{M}_\tau) = task(\mathcal{M})$.*

*More formally, $\forall \mathcal{M}$, if $\forall f \in \boldsymbol{F}, f_\tau = f$, then $Z(\mathcal{M}_\tau) = Z(\mathcal{M})$ and $\forall task \in \{MAP, MMAP\}, \forall \mathcal{M}$, if $\exists \boldsymbol{Q} \leftarrow \boldsymbol{q}^* \mid task(\mathcal{M}|\boldsymbol{q}^*) = task(\mathcal{M}), \forall f \in \boldsymbol{F}, f_\tau(\boldsymbol{q}^*) = f(\boldsymbol{q}^*)$, then $task(\mathcal{M}_\tau) = task(\mathcal{M})$.*

*Proof*
Case 1 (Z):

If $\forall f \in \boldsymbol{F}, f_\tau = f$, then $\mathcal{M}_\tau = \mathcal{M}$ and $task(\mathcal{M}_\tau) = task(\mathcal{M})$ follows trivially.

Case 2 (MAP and MMAP):

If $\exists \boldsymbol{Q} \leftarrow \boldsymbol{q}^* \mid task(\mathcal{M}|\boldsymbol{q}^*) = task(\mathcal{M}), \forall f \in \boldsymbol{F}, f_\tau(\boldsymbol{q}^*) = f(\boldsymbol{q}^*)$, then $task(\mathcal{M}_\tau|\boldsymbol{q}^*) = task(\mathcal{M}|\boldsymbol{q}^*) = task(\mathcal{M})$. Since $\forall \boldsymbol{q} \neq \boldsymbol{q}^*, task(\mathcal{M}|\boldsymbol{q}) \leq task(\mathcal{M}|\boldsymbol{q}^*)$ and by $\tau$-underflows can only decrease costs for these tasks, the optimal solution in the $\tau$-underflowed problem will still correspond to $\boldsymbol{q}^*$.

□

**Theorem 3.0.0.4** (Tractable upper-bound on error $MMAP(\mathcal{M}, \boldsymbol{Q}) - MMAP(\mathcal{M}_{\boldsymbol{S}_\tau}, \boldsymbol{Q})$)
*Given a graphical model $\mathcal{M}$ for computing the **marginal maximum a-posteriori** maximizing over query variables $\boldsymbol{Q} \subset \boldsymbol{X}$ and a $\tau$-underflow of the model, $\mathcal{M}_{\boldsymbol{S}_\tau}$, such that underflows are only applied to functions $\boldsymbol{F_S}$ that include summation variables in their scope ($\boldsymbol{F_S} = \{f_\alpha \mid f_\alpha \in F$ and $\exists X \in \alpha$ s.t. $X \in \boldsymbol{S} = \boldsymbol{X} \setminus \boldsymbol{Q}\}$), the error $\epsilon = MMAP(\mathcal{M}, \boldsymbol{Q}) - MMAP(\mathcal{M}_{\boldsymbol{S}_\tau}, \boldsymbol{Q})$ is upper-bounded by*

$$\epsilon \leq (v^*_{\boldsymbol{F_Q}})^{|\boldsymbol{F_Q}|} \cdot |D_{X_{\boldsymbol{S}}}| \cdot (v^*_{\boldsymbol{F_S}})^{|\boldsymbol{F_S}|-1} \cdot (\tau)$$

*Proof*
Let $\boldsymbol{S} = \boldsymbol{X} \setminus \boldsymbol{Q}$ be the set of variables to sum over, $\boldsymbol{F_Q} = \{f_\alpha \mid \alpha \subseteq \boldsymbol{Q}\}$ be the set of functions defined only over $\alpha \in \boldsymbol{Q}$, and $\boldsymbol{F_S} = \boldsymbol{F} \setminus \boldsymbol{F_Q}$ be functions that include some $X \in \boldsymbol{S}$ in their scope, the MMAP can be expressed as:

$$MMAP(\mathcal{M}, \boldsymbol{Q}) = \max_{\boldsymbol{Q}} \prod_{f' \in \boldsymbol{F_Q}} f'(\boldsymbol{q}) \sum_{\boldsymbol{S}} \prod_{f'' \in \boldsymbol{F_S}} f''(\boldsymbol{q} \cup \boldsymbol{s}) \tag{4}$$

for which the summation can be more explicitly be expressed via the notation

$$MMAP(\mathcal{M}, \boldsymbol{Q}) = \max_{\boldsymbol{Q}} \prod_{f' \in \boldsymbol{F_Q}} f'(\boldsymbol{q}) \sum_{\boldsymbol{s} \in D_{\boldsymbol{S}}} \prod_{f'' \in \boldsymbol{F_S}} f''(\boldsymbol{q} \cup \boldsymbol{s}) \tag{5}$$

Assume the MMAP solution is due to some unique assignment $\boldsymbol{Q} \leftarrow \boldsymbol{q}^*$. (Although a MMAP solution can be due to a set of possible $\boldsymbol{q}^*$'s, it will be easy to see that the proof pertaining to a unique MMAP $\boldsymbol{q}^*$ assignment is easily extendable to cases with multiple assignments). The MMAP value can now be expressed as

$$MMAP(\mathcal{M}, \boldsymbol{Q}) = Z(\mathcal{M}, \boldsymbol{q}^*) = \prod_{f' \in \boldsymbol{F_Q}} f'(\boldsymbol{q}^*) \sum_{\boldsymbol{s} \in D_{\boldsymbol{S}}} \prod_{f'' \in \boldsymbol{F_S}} f''(\boldsymbol{q}^* \cup \boldsymbol{s}) \tag{6}$$

Now partition $D_{\boldsymbol{S}}$ into subsets $D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{\geq\tau}}$ and $D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{<\tau}}$ where

$$D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{\geq\tau}} = \{\boldsymbol{s} \mid \forall f \in \boldsymbol{F_S}, f(\boldsymbol{q}^* \cup \boldsymbol{s}) \geq \tau\} \tag{7}$$

$$D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{<\tau}} = \{\boldsymbol{s} \mid \exists f \in \boldsymbol{F_S}, f(\boldsymbol{q}^* \cup \boldsymbol{s}) < \tau\} \tag{8}$$

$D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{\geq\tau}}$ corresponds to terms in the summation that are not affected by a $\tau$-underflow of the problem whereas $D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{<\tau}}$ corresponds to terms of the summation that end up being 0.0 after a $\tau$-underflow of the model is performed. As such, the cost of $\boldsymbol{q}^*$ in the $\tau$-underflowed model is

$$\begin{aligned}
Z(\mathcal{M}_{\boldsymbol{S}_\tau}, \boldsymbol{q}^*) &= \prod_{f' \in \boldsymbol{F_Q}} f'(\boldsymbol{q}^*) \sum_{\boldsymbol{s} \in D_{\boldsymbol{S}}} \prod_{f''_\tau \in \boldsymbol{F}_{\boldsymbol{S}_\tau}} f''_\tau(\boldsymbol{q}^* \cup \boldsymbol{s}) & (9)\\
&= \prod_{f' \in \boldsymbol{F_Q}} f'(\boldsymbol{q}^*) \cdot \Big( \sum_{\boldsymbol{s} \in D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{\geq\tau}}} \prod_{f''_\tau \in \boldsymbol{F}_{\boldsymbol{S}_\tau}} f''_\tau(\boldsymbol{q}^* \cup \boldsymbol{s}) \; + \sum_{\boldsymbol{s} \in D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{<\tau}}} \prod_{f''_\tau \in \boldsymbol{F}_{\boldsymbol{S}_\tau}} f''_\tau(\boldsymbol{q}^* \cup \boldsymbol{s}) \Big) & (10)\\
&= \prod_{f' \in \boldsymbol{F_Q}} f'(\boldsymbol{q}^*) \sum_{\boldsymbol{s} \in D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{\geq\tau}}} \prod_{f''_\tau \in \boldsymbol{F}_{\boldsymbol{S}_\tau}} f''_\tau(\boldsymbol{q}^* \cup \boldsymbol{s}) \; + \\
&\qquad \prod_{f' \in \boldsymbol{F_Q}} f'(\boldsymbol{q}^*) \sum_{\boldsymbol{s} \in D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{<\tau}}} \prod_{f''_\tau \in \boldsymbol{F}_{\boldsymbol{S}_\tau}} f''_\tau(\boldsymbol{q}^* \cup \boldsymbol{s}) & (11)\\
&= \prod_{f' \in \boldsymbol{F_Q}} f'(\boldsymbol{q}^*) \sum_{\boldsymbol{s} \in D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{\geq\tau}}} \prod_{f''_\tau \in \boldsymbol{F}_{\boldsymbol{S}_\tau}} f''_\tau(\boldsymbol{q}^* \cup \boldsymbol{s}) \; + \; 0.0 & (12)\\
&= \prod_{f' \in \boldsymbol{F_Q}} f'(\boldsymbol{q}^*) \sum_{\boldsymbol{s} \in D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{\geq\tau}}} \prod_{f'' \in \boldsymbol{F_S}} f''(\boldsymbol{q}^* \cup \boldsymbol{s}) & (13)
\end{aligned}$$

Thus, the error $\epsilon = MMAP(\mathcal{M}, \boldsymbol{Q}) - MMAP(\mathcal{M}_{\boldsymbol{S}_\tau}, \boldsymbol{Q})$ can be expressed as

$$\epsilon = \prod_{f' \in \boldsymbol{F_Q}} f'(\boldsymbol{q}^*) \sum_{\boldsymbol{s} \in D^{(\boldsymbol{q}^*)}_{\boldsymbol{S}_{\geq\tau}}} \prod_{f''_\tau \in \boldsymbol{F}_{\boldsymbol{S}_\tau}} f''_\tau(\boldsymbol{q}^* \cup \boldsymbol{s}) \tag{14}$$

and, since for each $\boldsymbol{s} \in D_{\boldsymbol{S}_{\geq\tau}}$ at least one $f''_\tau \in \boldsymbol{F}_{\boldsymbol{S}_\tau}$ has to evaluate to be zero by definition of $D_{\boldsymbol{S}_{\geq\tau}}$, $\epsilon$ can be at most

$$\epsilon \leq (v^*_{\boldsymbol{F_Q}})^{|\boldsymbol{F_Q}|} \cdot |D_{X_{\boldsymbol{S}}}| \cdot (v^*_{\boldsymbol{F_S}})^{|\boldsymbol{F_S}|-1} \cdot (\tau) \tag{15}$$

□

*Complexity*

Testing the condition in Corollary 3.0.0.4 is can be done in linear time.

## 4 PRELIMINARY TESTS OF AOBB-UFO ON UAI 2022 COMPETITION MMAP

Tables 1- 2 shows preliminary test results of AOBB [Marinescu et al., 2018] empowered with UFO. For these preliminary results, the UFO binary search was done over log space for two seconds and no deflation was applied to the resulting threshold. Per the competition restrictions, the algorithm was constrained to using 8G of memory.

**Table Key**

For each problem, provided are:

- $|X|$ ; $|F|$: the number of variables and functions, respectively
- $w^*$: the induced width (see Extended Supplemental) due to the variable ordering used by AOBB-UFO
- $k$: the maximum domain size
- $Anytime$: the time it took AOBB-UFO to originally find its final solution
- $Time$: the time it took AOBB-UFO to terminate
- $Solution$: AOBB-UFO's best found solution
- the best found solution for each of the competing solvers

| | PROBLEM STATISTICS | | | AOBB-UFO | | | UAI 2022 COMPETITION COMPETING SOLVERS | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | toulbar2 | | | |
| Problem | \|X\| ; \|F\| | w* | k | Anytime | Time | Solution | braobb | daoopt-lh | daoopt | lbp | ipr | vacint | vns | uai14 |
| 75-17-5.Q0.5.I4 | 289 ; 289 | 110 | 2 | 18.1 | 18.06 | -7.7 | -7.5 | -7.7 | -7.7 | | -7.7 | -7.7 | -7.7 | -14.1 |
| 75-19-5.Q0.5.I2 | 361 ; 361 | 133 | 2 | 15.7 | 15.66 | -10.1 | -9.7 | -9.6 | -9.6 | | -9.6 | -9.6 | -9.6 | -15.9 |
| 75-22-5.Q0.5.I2 | 484 ; 484 | 110 | 2 | 37.6 | 37.59 | -12.8 | | -11.8 | -11.7 | | -11.7 | -11.7 | -11.7 | |
| 75-23-5.Q0.5.I3 | 529 ; 529 | 177 | 2 | 17.9 | 17.95 | -13.7 | -13.8 | -12.5 | -12.5 | | -12.5 | -12.5 | -12.5 | |
| 75-26-5.Q0.5.I4 | 676 ; 676 | 208 | 2 | 31.5 | 31.46 | -19 | -18.3 | -18.1 | -18.1 | | -18.1 | -18.1 | -18.1 | |
| 90-22-5.Q0.5.I4 | 484 ; 484 | 173 | 2 | 39.4 | 39.39 | -6.4 | -5.6 | -5.6 | -5.6 | | -5.6 | -5.6 | -5.6 | |
| 90-24-5.Q0.5.I2 | 576 ; 576 | 131 | 2 | 27.4 | 27.37 | -5.8 | -5.6 | -5.7 | -5.7 | | -5.7 | -5.7 | -5.7 | |
| 90-25-5.Q0.5.I2 | 625 ; 625 | 174 | 2 | 21.4 | 21.44 | -7.9 | | -7.8 | -7.7 | | -7.7 | -7.7 | -7.7 | |
| 90-26-5.Q0.5.I1 | 676 ; 676 | 110 | 2 | 27 | 26.96 | -8.8 | | -9.1 | -8.7 | | -8.7 | -8.7 | -8.7 | |
| 90-30-5.Q0.5.I1 | 900 ; 900 | 246 | 2 | 36.1 | 36.14 | -11 | -11.5 | -10.9 | -10.9 | | -10.9 | -10.9 | -10.9 | |
| 90-34-5.Q0.5.I2 | 1156 ; 1156 | 352 | 2 | 29.6 | 29.63 | -12.7 | | -14.2 | -12.2 | | -12.2 | -12.2 | -12.2 | |
| 90-38-5.Q0.5.I4 | 1444 ; 1444 | 371 | 2 | 44.9 | 44.86 | -17.1 | | | -16.8 | | -16.8 | -16.8 | -16.8 | |
| 90-42-5.Q0.5.I4 | 1764 ; 1764 | 300 | 2 | 40.6 | 40.59 | -17.3 | | | -17 | | -17 | -17 | -17 | |
| 90-46-5.Q0.5.I4 | 2116 ; 2116 | 356 | 2 | 46 | 45.95 | -26 | | | -24.8 | | -24.9 | -24.9 | -24.5 | |
| 90-50-5.Q0.5.I3 | 2500 ; 2500 | 788 | 2 | 201.2 | 201.21 | -26.3 | -36.9 | | -26.8 | | -25.7 | -25.7 | -25.7 | |
| bw_p24_16 | 937 ; 937 | 136 | 3 | 82.8 | 3682.8 | | | | | | | | | |
| bw_p24_20 | 1169 ; 1169 | 171 | 3 | 105.2 | 3705.2 | | | | | | | | | |
| bw_p34_15 | 2191 ; 2191 | 294 | 3 | 59.2 | 3659.2 | | | | | | | | | |
| bw_p34_20 | 2916 ; 2916 | 389 | 3 | 108.5 | 3708.5 | | | | | | | | | |
| bw_p44_15 | 4075 ; 4075 | 638 | 3 | 514.1 | 4114.1 | | | | | | | | | |
| bw_p44_19 | 5155 ; 5155 | 722 | 3 | 828.9 | 4428.9 | | | | | | | | | |
| bw_p54_10 | 4366 ; 4366 | 777 | 3 | 911.2 | 911.25 | -1 | | | | | | | | |
| bw_p54_16 | 6964 ; 6964 | 1134 | 3 | | | | | | | | | | | |
| comm_p01_16 | 4477 ; 4477 | 831 | 2 | 266.9 | 3866.9 | | | | | | | | | |
| comm_p01_20 | 5585 ; 5585 | 1039 | 2 | 453.8 | 4053.8 | | | | | | | | | |
| Grids_20 | 6400 ; 19040 | 122 | 2 | 63.8 | 3663.8 | | 4518.7 | 4833.9 | 4839 | 4804.9 | 4837.5 | 4816.7 | 4836.5 | |
| Grids_21 | 1600 ; 4800 | 108 | 2 | 50.9 | 3650.9 | | 8002.4 | 8429.3 | 8499.7 | 8322.9 | 8499.7 | 8357.8 | 8438.1 | 8222.4 |
| Grids_22 | 1600 ; 4800 | 97 | 2 | 53.1 | 3653.1 | | 2687.2 | 2833.8 | 2835.2 | 2763.4 | 2835.2 | 2797.1 | 2823.4 | 2790.8 |
| Grids_23 | 1600 ; 4720 | 73 | 2 | 40 | 3640 | | 2787.3 | 2793.4 | 2793 | 2739.2 | 2793 | 2778.7 | 2780.3 | 2765 |
| Grids_24 | 1600 ; 4720 | 61 | 2 | 40.9 | 3640.9 | | 8204.5 | 8234.4 | 8237.3 | 8106.5 | 8237.3 | 8160.9 | 8175.2 | 8023.4 |
| Grids_25 | 1600 ; 4720 | 63 | 2 | 71.2 | 3671.2 | | 1209.9 | 1208.9 | 1209.3 | 1204.6 | 1209.3 | 1199.5 | 1209.1 | 1209 |
| Grids_26 | 400 ; 1200 | 59 | 2 | 58.7 | 3658.7 | | 1322.2 | 1326.3 | 1326.3 | 1300.4 | 1326.3 | 1321.4 | 1322.2 | 1280.9 |
| Grids_27 | 1600 ; 4720 | 69 | 2 | 38.5 | 3638.5 | | 5507 | 5506.8 | 5508.9 | 5383.1 | 5508.9 | 5422.5 | 5489.7 | 5378.4 |
| Grids_28 | 400 ; 1200 | 53 | 2 | 90 | 3690 | | 1969.1 | 1982.5 | 1982.5 | 1924.7 | 1982.5 | 1978.2 | 1972.6 | 1948.2 |
| Grids_29 | 400 ; 1200 | 54 | 2 | 83.6 | 3683.6 | | 669.2 | 673 | 673 | 662.6 | 673 | 672.3 | 670.4 | 666.3 |

**Table 1:** AOBB-UFO on UAI 2022 Competition Final Problems (3600s)

| | PROBLEM STATISTICS | | | AOBB-UFO | | | UAI 2022 COMPETITION COMPETING SOLVERS | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | toulbar2 | | | |
| Problem | \|X\| ; \|F\| | w* | k | Anytime | Time | Solution | braobb | daoopt-lh | daoopt | lbp | ipr | vacint | vns | uai14 |
| ImageAlignment_11 | 350 ; 3563 | 33 | 77 | 25.5 | 108.67 | -1005.8 | -824.2 | -824.2 | -824.2 | -824.2 | -824.2 | -824.2 | -824.2 | -824.2 |
| ImageAlignment_12 | 30 ; 465 | 29 | 58 | 5.3 | 5.28 | -436.7 | -436.7 | -436.7 | -436.7 | -436.7 | -436.7 | -436.7 | -436.7 | -436.7 |
| ImageAlignment_13 | 400 ; 3334 | 21 | 83 | 78 | 128.24 | -3081.4 | -2998.9 | -2999.8 | -2999.8 | -2998.9 | -2999.8 | -2999.8 | -2999.8 | -2998.9 |
| ImageAlignment_14 | 200 ; 2128 | 23 | 69 | 9.1 | 12.89 | -1632 | -1557.5 | -1557.5 | -1557.5 | -1557.5 | -1557.5 | -1557.5 | -1557.5 | -1557.5 |
| ImageAlignment_15 | 300 ; 2732 | 23 | 68 | 59.7 | 415.05 | -1339.5 | -1177.5 | -1177.5 | -1177.5 | -1177.5 | -1177.5 | -1177.5 | -1177.5 | |
| ObjectDetection_13 | 60 ; 1830 | 59 | 21 | 6.2 | 3606.1 | -528.6 | 6684.3 | 9967.7 | 9854.2 | 8826.5 | 9970.6 | 9970.6 | 9970.6 | 8883.1 |
| ObjectDetection_14 | 60 ; 1830 | 59 | 11 | 31.6 | 3631.1 | 236.6 | 6712.5 | 9020.8 | 9020.8 | 8341.9 | 9093.7 | 9093.7 | 9093.7 | 8562.9 |
| ObjectDetection_15 | 60 ; 1830 | 59 | 16 | 2155.8 | 3637 | -404 | 10628 | 11703.8 | 12479 | 11544 | 12633 | 12633 | 12633 | 11976 |
| ObjectDetection_16 | 60 ; 1830 | 59 | 21 | 1753.5 | 3606.2 | -471 | 12023 | 14154.2 | 13536 | 13548 | 14347 | 14347 | 14347 | 14022 |
| ObjectDetection_17 | 60 ; 1830 | 59 | 11 | 30.6 | 3630.5 | -335.9 | 2454.9 | 4716.4 | 4816 | 3794.9 | 4887.4 | 4887.4 | 4887.4 | 4518.2 |
| or_chain_11.fg.Q0.5.I3 | 900 ; 915 | 191 | 2 | 30.8 | 30.85 | -22.9 | | | | | | | | |
| or_chain_16.fg.Q0.5.I3 | 1675 ; 1700 | 318 | 2 | 98.5 | 98.52 | -38.1 | -53.8 | -25.2 | -23.4 | | -23.4 | -23.4 | -23.4 | |
| or_chain_22.fg.Q0.5.I3 | 1044 ; 1054 | 196 | 2 | 28.9 | 28.92 | -15.3 | | | | | | | | |
| or_chain_24.fg.Q0.5.I3 | 1155 ; 1171 | 247 | 2 | 32.3 | 32.31 | -24.4 | | | | | | | | |
| or_chain_25.fg.Q0.5.I3 | 1075 ; 1086 | 88 | 2 | 30.4 | 30.37 | -16.8 | | | | | | | | |
| or_chain_32.fg.Q0.5.I3 | 1466 ; 1478 | 108 | 2 | 32 | 3632 | | | | | | | | | |
| or_chain_36.fg.Q0.5.I3 | 933 ; 943 | 91 | 2 | 30.5 | 30.46 | -15.3 | | | | | | | | |
| or_chain_39.fg.Q0.5.I3 | 1751 ; 1766 | 430 | 2 | 96.3 | 96.32 | -22.9 | | | | | | | | |
| or_chain_40.fg.Q0.5.I3 | 988 ; 998 | 96 | 2 | 16.3 | 16.26 | -15.1 | | | | | | | | |
| or_chain_41.fg.Q0.5.I3 | 1847 ; 1863 | 203 | 2 | 118.5 | 3645 | -27 | | | | | | | | |
| or_chain_43.fg.Q0.5.I3 | 1692 ; 1712 | 216 | 2 | 44.5 | 44.5 | -30.5 | | | | | | | | |
| or_chain_6.fg.Q0.5.I3 | 1849 ; 1876 | 386 | 2 | 2351.2 | 3710.9 | -41.2 | | | | | | | | |
| or_chain_60.fg.Q0.5.I3 | 1997 ; 2023 | 552 | 2 | 3691.2 | 3736.6 | -42.8 | | | | | | | | |
| or_chain_63.fg.Q0.5.I3 | 731 ; 744 | 97 | 2 | 26.3 | 26.35 | -10.8 | | | | | | | | |
| or_chain_8.fg.Q0.5.I3 | 1195 ; 1203 | 63 | 2 | 23.6 | 23.58 | -12.2 | | | | | | | | |
| pedigree1.Q0.5.I3 | 298 ; 334 | 105 | 4 | 3319.4 | 3641.4 | -35.2 | | -35.9 | -37.3 | | -35.7 | -35.7 | -36.1 | -37.8 |
| pedigree13.Q0.5.I1 | 888 ; 1077 | 138 | 3 | 2941.6 | 3623.5 | -63.1 | | -62.8 | -62.4 | | -62.8 | -62.7 | -62.5 | |
| pedigree18.Q0.5.I1 | 931 ; 1184 | 181 | 5 | 38.8 | 3638.8 | | | -112.5 | -111.9 | | -112.5 | -112.9 | -113.5 | -116.4 |
| pedigree19.Q0.5.I4 | 693 ; 793 | 173 | 5 | 3280 | 3625.2 | -97.5 | -95.3 | -89.5 | -87.5 | | -87.1 | -89.5 | -89.6 | |
| pedigree20.Q0.5.I2 | 387 ; 437 | 76 | 5 | 2455.1 | 3614.7 | | | -46.8 | -46.7 | | -47.9 | -46.1 | -46.1 | -72.8 |
| pedigree25.Q0.5.I2 | 993 ; 1289 | 178 | 5 | 3236.3 | 3702.4 | -148.7 | -154.6 | -148.5 | -147.8 | | -147.6 | -147.3 | -147.8 | -170 |
| pedigree30.Q0.5.I2 | 1015 ; 1289 | 109 | 5 | 30.9 | 3630.9 | | | -123.6 | -124.7 | | -123.6 | -123.1 | -124.3 | -140.8 |
| pedigree31.Q0.5.I2 | 1006 ; 1183 | 153 | 5 | 16 | 3616 | | -118.4 | -116.2 | -116 | | -117 | -117.6 | -115.6 | -161.1 |
| pedigree33.Q0.5.I2 | 581 ; 798 | 118 | 4 | 28.7 | 3628.7 | | -69.4 | -70.3 | -70.9 | | -71 | -70.4 | -71.1 | |
| pedigree38.Q0.5.I2 | 581 ; 724 | 164 | 5 | 115.5 | 3621.9 | -94.2 | -91.4 | -78.2 | -77.6 | | -77.5 | -77.6 | -77.8 | -90.5 |
| pedigree41.Q0.5.I2 | 885 ; 1062 | 205 | 5 | 27.5 | 3620.7 | -117 | | -105.6 | -107.3 | | -106.8 | -107.7 | -108.2 | -160.7 |
| pedigree44.Q0.5.I4 | 644 ; 811 | 186 | 4 | 3177.3 | 3628.3 | -89.3 | -98.2 | -89.1 | -88.5 | | -89.1 | -89.4 | -89.7 | |
| pedigree50.Q0.5.I1 | 478 ; 514 | 124 | 6 | 214.5 | 3677.4 | -51.5 | -55.7 | -52.4 | -52.8 | | -53.1 | -53.1 | -52 | -69.6 |
| pedigree7.Q0.5.I2 | 867 ; 1068 | 89 | 4 | 3579.1 | 3628.3 | -103.8 | | -97.5 | -97.6 | | -98.4 | -98 | -98.3 | -136.4 |
| pedigree9.Q0.5.I3 | 935 ; 1118 | 235 | 7 | 913.3 | 3626.6 | -123.9 | | -113.8 | -113.7 | | -113.4 | -112.7 | -114 | -159 |
| pomdp10-12_7_3_8_4 | 2673 ; 2701 | 2599 | 32 | | | | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 | 1.2 |
| pomdp6-12_6_2_6_3 | 250 ; 265 | 198 | 18 | 63.9 | 3663.9 | 0.9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| pomdp7-20_10_2_10_3 | 3166 ; 3193 | | | | | | | | | | | | | |
| pomdp8-14_9_3_12_4 | 2145 ; 2189 | 2057 | 48 | | | | 1.4 | 1.4 | 1.4 | 1.4 | 1.4 | 1.4 | 1.4 | 1.4 |
| pomdp9-14_8_3_10_4 | 5277 ; 5313 | | | | | | | | | | | | | |
| ProteinFolding_11 | 400 ; 1160 | 28 | 2 | 22.4 | 22.44 | 1926.6 | 1962.3 | 1962.3 | 1962.3 | 1839.4 | 1962.3 | 1959.7 | 1944.8 | 1528.7 |
| ProteinFolding_12 | 250 ; 2098 | 20 | 60 | 10 | 13.34 | -1586.4 | -1547 | -1547 | -1547 | -1548 | -1547 | -1547 | -1547 | |
| ProteinFolding_13 | 100 ; 1055 | 21 | 92 | 24.5 | 24.56 | -143.3 | -143.3 | -143.3 | -143.3 | -143.3 | -143.3 | -143.3 | -143.3 | -143.3 |
| ProteinFolding_14 | 80 ; 847 | 20 | 49 | 12.6 | 12.61 | -331.7 | -331.7 | -331.7 | -331.7 | -331.7 | -331.7 | -331.7 | -331.7 | -331.7 |
| ProteinFolding_15 | 50 ; 536 | 22 | 47 | 10 | 9.97 | -51.6 | -51.6 | -51.6 | -51.6 | -51.6 | -51.6 | -51.6 | -51.6 | -51.6 |
| Segmentation_11 | 228 ; 852 | 19 | 21 | 6.4 | 135.02 | -152.1 | -132.1 | -132.3 | -132.3 | -133.3 | -132.3 | -132.3 | -132.3 | -133.4 |
| Segmentation_12 | 231 ; 856 | 20 | 2 | 2.9 | 2.88 | -36.7 | -21.9 | -21.9 | -21.9 | -30.1 | -21.9 | -21.9 | -21.9 | -21.9 |
| Segmentation_13 | 225 ; 832 | 18 | 2 | 3.8 | 3.76 | -28.9 | -21.4 | -21.4 | -21.4 | -21.4 | -21.4 | -21.4 | -21.4 | -21.4 |
| Segmentation_14 | 231 ; 863 | 18 | 2 | 3 | 2.96 | -40.2 | -39.3 | -39.3 | -39.3 | -40.2 | -39.3 | -39.3 | -39.3 | -39.8 |
| Segmentation_15 | 229 ; 851 | 18 | 21 | 6.9 | 336.2 | -182.9 | -169.6 | -163.8 | -163.8 | -168.5 | -163.8 | -163.8 | -163.8 | -168.8 |
| Segmentation_16 | 228 ; 838 | 18 | 2 | 2.9 | 2.94 | -42.5 | -40.4 | -40.4 | -40.4 | -40.4 | -40.4 | -40.4 | -40.4 | -40.8 |
| Segmentation_17 | 225 ; 837 | 20 | 21 | 10.2 | 23.49 | -184.1 | -176.2 | -174.3 | -174.3 | -175.3 | -174.3 | -174.3 | -174.3 | -175.7 |
| Segmentation_18 | 235 ; 882 | 20 | 2 | 3 | 3.04 | -44.8 | -34 | -34 | -34 | -35.9 | -34 | -34 | -34 | -34 |
| Segmentation_19 | 228 ; 852 | 20 | 2 | 3.1 | 3.06 | -32.6 | -24.1 | -24.1 | -24.1 | -25.4 | -24.1 | -24.1 | -24.1 | -24.1 |
| Segmentation_20 | 232 ; 867 | 21 | 21 | 8.2 | 3199.8 | -138.1 | -112 | -112 | -112 | -112.4 | -112 | -112 | -112 | -113.5 |
| wcsp_14 | 301 ; 19161 | 48 | 8 | 68.1 | 75.36 | 1.5 | -29.2 | -9.6 | 1.1 | -402.4 | 1.1 | 1.1 | 1.1 | -20.9 |
| wcsp_15 | 125 ; 736 | 66 | 4 | 16.3 | 3610.4 | -162 | -163.6 | -80 | -74.6 | -1503.1 | -75.4 | -78.6 | -83.4 | -193.8 |
| wcsp_16 | 200 ; 1970 | 57 | 44 | 8.9 | 3608.9 | | -5.3 | 4.8 | 23.1 | -251.5 | 22.9 | 22.8 | 23.2 | -8.5 |
| wcsp_17 | 340 ; 3417 | 95 | 44 | 12.1 | 3612.1 | | -101.8 | 8.3 | 33.6 | -549.1 | 34.9 | 35 | 34.8 | |
| wcsp_18 | 239 ; 18016 | 38 | 24 | 80.4 | 80.47 | 1.1 | -540 | -24.1 | 0.2 | -16 | 0.2 | 0.2 | 0.2 | -16.9 |

**Table 2:** AOBB-UFO on UAI 2022 Competition Final Problems (3600s)

**Summary Statistics**

- Total number of problems = 100
- Number of problems for which AOBB-UFO equalized or did better than all of the competing solvers = 34
- Average number of competing solvers AOBB-UFO equalized or did better than = 3.62
- Number of problems for which AOBB-UFO did strictly better than all of the competing solvers = 18
- Average number of competing solvers AOBB-UFO did strictly better than = 2.24
- Number of problems for which AOBB-UFO terminated before 120 seconds = 42

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