ICS 6A
Solution to Homework Assignment 9
Winter 2004

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Answer the following questions (explain all your answers).

1. Rosen, page 423, problem 3 (b,d,f.)
   b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$
   Answer: Obvious result: $a_n = 2$
   OR this is a linear homogeneous recurrence relation with degree 1. We can also do it like this:
   $c_1 = 1$, $r - 1 = 0 \Rightarrow r = 1$, with the initial condition: $a_0 = 2 \Rightarrow a_n = 2$

   d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
   Answer: $a_n = 6 \cdot 2^n - 2n \cdot 2^n$
   $c_1 = 4$, $c_2 = -4$,
   $r^2 - 4r + 4 = 0 \Rightarrow r_1 = r_2 = 2$
   $\Rightarrow a_n = b_1 \cdot 2^n + b_2 \cdot n \cdot 2^n$, with the initial conditions: $a_0 = 6$, $a_1 = 8$
   $\Rightarrow 6 = b_1 \cdot 2^0 + b_2 \cdot 0 \cdot 2^0 = b_1$ and
   $8 = b_1 \cdot 2^1 + b_2 \cdot 1 \cdot 2^1 = 2b_1 + 2b_2$
   $\Rightarrow b_1 = 6, b_2 = -2$
   $\Rightarrow a_n = 6 \cdot 2^n - 2n \cdot 2^n$

   f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$
   Answer: $a_n = 2^n - (-2)^n$
   $c_1 = 0$, $c_2 = 4$,
   $r^2 - 4 = 0 \Rightarrow r_1 = 2$, $r_2 = -2$
   $\Rightarrow a_n = b_1 \cdot 2^n + b_2 \cdot (-2)^n$, with the initial conditions: $a_0 = 0$, $a_1 = 4$
   $\Rightarrow 0 = b_1 \cdot 2^0 + b_2 \cdot (-2)^0 = b_1 + b_2$ and
   $4 = b_1 \cdot 2^1 + b_2 \cdot (-2)^1 = 2b_1 - 2b_2$
   $\Rightarrow b_1 = 1$, $b_2 = -1$
   $\Rightarrow a_n = 2^n - (-2)^n$

2. Rosen, page 423, problem 4 (a,c,e.)
   a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
   Answer: $a_n = \frac{12}{5} \cdot 3^n + \frac{3}{5}(-2)^n$
   $c_1 = 1$, $c_2 = 6$,
   $r^2 - r - 6 = 0 \Rightarrow r_1 = 3$, $r_2 = -2$
   $\Rightarrow a_n = b_1 \cdot 3^n + b_2 \cdot (-2)^n$, with the initial conditions: $a_0 = 3$, $a_1 = 6$
   $\Rightarrow 3 = b_1 \cdot 3^0 + b_2 \cdot (-2)^0 = b_1 + b_2$ and
   $6 = b_1 \cdot 3^1 + b_2 \cdot (-2)^1 = 3b_1 - 2b_2$
   $\Rightarrow b_1 = \frac{12}{5}$, $b_2 = \frac{3}{5}$
   $\Rightarrow a_n = \frac{12}{5} \cdot 3^n + \frac{3}{5}(-2)^n$

   c) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$
   Answer: $an = 3 \cdot 2^n + 4^n$
   $c_1 = 6$, $c_2 = -8$,
   $r^2 - 6r + 8 = 0$
   $\Rightarrow r_1 = 2$, $r_2 = 4$
   $\Rightarrow a_n = b_1 \cdot 2^n + b_2 \cdot 4^n$, with the initial conditions: $a_0 = 4$, $a_1 = 10$
   $\Rightarrow 4 = b_1 \cdot 2^0 + b_2 \cdot 4^0 = b_1 + b_2$ and
   $10 = b_1 \cdot 2^1 + b_2 \cdot 4^1 = 2b_1 + 4b_2$
   $\Rightarrow b_1 = 3$, $b_2 = 1$

$\Rightarrow a_n = 3 \cdot 2^n + 4^n$

e) $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$
Answer: $a_n = 2 + 3 \cdot (-1)^n$
$c_1 = 0$, $c_2 = 1$,
$r^2 - 1 = 0$
$\Rightarrow r_1 = 1$, $r_2 = -1$
$\Rightarrow a_n = b_1 \cdot 1^n + b_2 \cdot (-1)^n = b_1 + b_2 \cdot (-1)n$, with the initial conditions: $a_0 = 5$, $a_1 = -1$
$\Rightarrow 5 = b_1 + b_2 \cdot (-1)^0 = b_1 + b_2$ and
$-1 = b_1 + b_2 \cdot (-1)^1 = b_1 - b_2$
$\Rightarrow b_1 = 2$, $b_2 = 3$
$\Rightarrow a_n = 2 + 3 \cdot (-1)^n$

3. Rosen, page 423, problem 7. (see solutions in book)

4. Rosen, page 423, problem 13. $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$
Answer: $a_n = 8 \cdot (-1)^n + 4 \cdot 3^n - 3 \cdot (-2)^n$
$c_1 = 0$, $c_2 = 7$, $c_3 = 6$
$r^3 - 7r - 6 = 0$
$\Rightarrow r^3 - r - 6r - 6 = r(r^2 - 1) - 6(r + 1) = r(r + 1)(r - 1) - 6(r + 1) = (r + 1)(r^2 - r - 6) = (r + 1)(r - 3)(r + 2) = 0$
$\Rightarrow r_1 = -1$, $r_2 = 3$, $r_2 = -2$
$\Rightarrow a_n = b_1 \cdot (-1)^n + b_2 \cdot 3^n + b_3 \cdot (-2)^n$, with the initial conditions: $a_0 = 9$, $a_1 = 10$, $a_2 = 32$
$\Rightarrow 9 = b_1 \cdot (-1)^0 + b_2 \cdot 3^0 + b_3 \cdot (-2)^0 = b_1 + b_2 + b_3$,
$10 = b_1 \cdot (-1)^1 + b_2 \cdot 3^1 + b_3 \cdot (-2)^1 = -b_1 + 3b_2 - 2b_3$ and
$32 = b_1 \cdot (-1)^2 + b_2 \cdot 3^2 + b_3 \cdot (-2)^2 = b_1 + 9b_2 + 4b_3$
$\Rightarrow b_1 = 8, b_2 = 4, b_2 = -3$
$\Rightarrow a_n = 8 \cdot (-1)^n + 4 \cdot 3^n - 3 \cdot (-2)^n$

5. Rosen, page 423, problem 18. Solve the recurrence $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5$ $a_1 = 4$ and$a_2 = 15$. The characteristic equations is: $r^3 - 6r^2 + 12r - 8 = 0$. There is a single root for the equation $r = 2$ with multiplicity 3. Therefore, by Theorem 4 (page 418) the solution is of the form: $a_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3 n^2 2^n$. The initial condition can be used to find $\alpha_1, \alpha_2, \alpha_3$.

6. Rosen, page 423, problem 23. (see solutions in book)

7. Rosen, page 424, problem 28. a) The solution fo $a_n = 2a_{n-1} + 2n^2$ is a sum of the solution of the homogeneous recursions and a single solution of the given recurrence relation. The solution for $a_n = 2a_{n-1}$ is $a_n = \alpha 2^n$. A specific solution for the relation is of the form $a_n = cn^2 + bn + a$. pluging this in the relation will dictate a solution fo c b and a. b) Once we found the general form of the solution the initial condition $a_1 = 4$ will determine the constants.