ICS 6A
Solution to Homework Assignment 8
Winter 2004

Instructor: Rina Dechter

Answer the following questions (explain your answers).

1. Rosen, page 270, problem 2c.
   $f(1) = 1, f(2) = -3, f(3) = 13, f(4) = 141, f(5) = 19,597$

2. Rosen, page 270, problem 2d.
   $f(1) = f(2) = f(3) = f(4) = f(5) = 3$

3. Rosen, page 270, problem 3. $f(0) = -1,\ f(1) = 2$
   a) $f(n+1) = f(n) + 3 \cdot f(n-1)$
   $f(2) = f(1) + 3 \cdot f(0) = 2 + 3 \cdot (-1) = -1$ for n = 1
   $f(3) = f(2) + 3 \cdot f(1) = -1 + 3 \cdot 2 = 5$ for n = 2
   $f(4) = f(3) + 3 \cdot f(2) = 5 + 3 \cdot -1 = 2$ for n = 3
   $f(5) = f(4) + 3 \cdot f(3) = 2 + 3 \cdot 5 = 17$ for n = 4

   b) $f(n+1) = f(n)^2 \cdot f(n-1)$
   $f(2) = f(1)^2 \cdot f(0) = 2^2 \cdot (-1) = -4$ for n = 1
   $f(3) = f(2)^2 \cdot f(1) = (-4)^2 \cdot 2 = 32$ for n = 2
   $f(4) = f(3)^2 \cdot f(2) = 32^2 \cdot (-4) = -4096$ for n = 3
   $f(5) = f(4)^2 \cdot f(3) = (-4096)^2 \cdot 32 = 536870912$ for n = 4

   c) $f(n+1) = 3 \cdot f(n)^2 - 4 \cdot f(n-1)^2$
   $f(2) = 3 \cdot f(1)^2 - 4 \cdot f(0)^2 = 3 \cdot 2^2 - 4 \cdot (-1)^2 = 8$ for n = 1
   $f(3) = 3 \cdot f(2)^2 - 4 \cdot f(1)^2 = 3 \cdot 8^2 - 4 \cdot 2^2 = 176$ for n = 2
   $f(4) = 3 \cdot f(3)^2 - 4 \cdot f(2)^2 = 3 \cdot 176^2 - 4 \cdot 8^2 = 92672$ for n = 3
   $f(5) = 3 \cdot f(4)^2 - 4 \cdot f(3)^2 = 3 \cdot 92672^2 - 4 \cdot 176^2 = 25764174848$ for n = 4

   d) $f(n+1) = f(n-1)/f(n)$
   $f(2) = f(0)/f(1) = -\frac{1}{2}$ for n = 1
   $f(3) = f(1)/f(2) = \frac{2}{-\frac{1}{2}} = -4$ for n = 2
   $f(4) = f(2)/f(3) = \frac{-\frac{1}{2}}{-4} = \frac{1}{8}$ for n = 3
   $f(5) = f(3)/f(4) = \frac{-4}{\frac{1}{8}} = -32$ for n = 4

4. Rosen, page 271, problem 8.
   There are many possible answers.

   a) $a_n = 4n - 2$: $a_{n+1} = a_n + 4$, for $n \geq 1$ and $a_1 = 2$
   b) $a_n = 1 + (-1)^n$: $a_{n+1} = 2 - a_n$, for $n \geq 1$ and $a_1 = 0$
   c) $a_n = n(n+1)$: $a_{n+1} = a_n + 2(n+1)$, for $n \geq 1$ and $a_1 = 2$
   d) $a_n = n^2$: $a_{n+1} = a_n + 2n + 1$, for $n \geq 1$ and $a_1 = 1$

5. Rosen, page 271, problem 13.
   Please see *"Solutions to Odd-Numbered Exercises"* of Rosen.(Page S-28)

6. Rosen, page 272, problem 22.
   Proof: Let A be the set of all positive integers. To prove that A = S, we must show that $A \subseteq S$ and $S \subseteq A$.

- Case 1: $A \subseteq S$.
  To prove $A \subseteq S$, we must show that every positive integer is in S. We will use mathematical induction to prove this.
  Let P(n) be the statement that "For every positive integer n, $n \in S$"
  - Basis step: $1 \in S \Rightarrow$ P(1) is true, since it is the first part of the recursive definition of S.
  - Inductive step: Assume P(n) is true, i.e. "$n \in S$". Since $n \in S$ and $1 \in S$, it follows from the second part of the recursive definition of S that $(n+1) \in S$. This means P(n+1) is true. So $A \subseteq S$.
- Case 2: $S \subseteq A$.
  To prove $S \subseteq A$, we must show that every element of S is a positive integer. We will use the recursive definition of S.
  - First the basis step of the definition specifies that $1 \in S$. Since 1 is a positive integer, all elements specified in S (only one element "1" in this step) are in A.
  - To finish the proof, we must show that all integers in S generated using the second part of the recursive definition are in A. This consists of showing that $(s+t) \in A$ whenever $s \in A$ and $t \in A$, it follows that both s and t are positive integers. obviously (s+t) is also a positive integer, so we completed the proof.

7. Rosen, page 272, problem 31.
   Please see *"Solutions to Odd-Numbered Exercises"* of Rosen.(Page S-28)

8. Rosen, page 409, problem 3 a,c,e,h. $a_n = 8a_{n-1} - 16a_{n-2}$
   a) $a_n = 0$: YES.
   $a_{n-1} = 0$, $a_{n-2} = 0$,
   $8a_{n-1} - 16a_{n-2} = 8 \times 0 - 16 \times 0 = 0 = a_n$

   c) $a_n = 2^n$: NO.
   $a_{n-1} = 2^{n-1}$, $a_{n-2} = 2^{n-2}$,
   $8a_{n-1} - 16a_{n-2} = 8 \cdot 2^{n-1} - 16 \cdot 2^{n-2} = 0 \neq 2^n = a^n$

   e) $a_n = n4^n$: YES.
   $a_{n-1} = (n-1)4^{n-1}$, $a_{n-2} = (n-2)4^{n-2}$,
   $8a_{n-1} - 16a_{n-2} = 8 \cdot (n-1)4^{n-1} - 16 \cdot (n-2)4^{n-2} = 2(n-1)4^n - (n-2)4^n = n4^n = a^n$

   h) $a_n = n^2 4^n$: NO.
   $a_{n-1} = (n-1)^2 4^{n-1}$, $a_{n-2} = (n-2)^2 4^{n-2}$,
   $8a_{n-1} - 16a_{n-2} = 8 \cdot (n-1)^2 4^{n-1} - 16 \cdot (n-2)^2 4^{n-2} = 2(n-1)^2 4^n - (n-2)^2 4^n = (n^2-2)4^n \neq n^2 4^n = a^n$

9. Rosen, page 409, problem 9 a,c,e,g.
   a) $a_n = 3a_{n-1}$, $a_0 = 2$
   Answer: $a_n = 2 \times 3^n$

$$
\begin{aligned}
a_n &= 3a_{n-1} \\
&= 3 \cdot (3a_{n-2}) \ = 3^2 \cdot a_{n-2} \\
&= 3^2 \cdot (3a_{n-3}) = 3^3 \cdot a_{n-3} \\
&= 3^3 \cdot (3a_{n-4}) = 3^4 \cdot a_{n-4} \\
&\vdots \\
&= 3^n \cdot a_{n-n} \\
&= 3^n \cdot a_0 \\
&= 2 \cdot 3^n
\end{aligned}
$$

   c) $a_n = a_{n-1} + n$, $a_0 = 1$
   Answer: $a_n = 1 + \frac{n^2+n}{2}$

$$\begin{aligned}
a_n &= a_{n-1} + n \\
&= (a_{n-2} + n - 1) + n & &= a_{n-2} + 2n - 1 \\
&= (a_{n-3} + n - 2) + 2n - 1 & &= a_{n-3} + 3n - (1+2) \\
&= (a_{n-4} + n - 3) + 3n - 3 & &= a_{n-4} + 4n - (1+2+3) \\
&\vdots \\
&= a_{n-n} + n \cdot n - (1 + 2 + 3 + ... + (n-1)) \\
&= a_0 + n^2 - \tfrac{(n-1)(1+n-1)}{2} \\
&= 1 + n^2 - \tfrac{n^2 - n}{2} \\
&= 1 + \tfrac{n^2 + n}{2}
\end{aligned}$$

e) $a_n = 2a_{n-1} - 1$, $a_0 = 1$
Answer: $a_n = 1$

$$\begin{aligned}
a_n &= 2a_{n-1} - 1 \\
&= 2(2a_{n-2} - 1) - 1 & &= 2^2 a_{n-2} - (1+2) \\
&= 2^2(2a_{n-3} - 1) - (1+2) & &= 2^3 a_{n-3} - (1+2+2^2) \\
&= 2^3(2a_{n-4} - 1) - (1+2+2^2) & &= 2^4 a_{n-4} - (1+2+2^2+2^3) \\
&\vdots \\
&= 2^n a_{n-n} - (1 + 2 + 2^2 + 2^3 + ... + 2^{n-1}) \\
&= 2^n a_0 - \tfrac{2^n - 1}{2-1} \\
&= 2^n - (2^n - 1) \\
&= 1
\end{aligned}$$

g) $a_n = na_{n-1}$, $a_0 = 5$
Answer: $a_n = 5 \cdot n!$

$$\begin{aligned}
a_n &= na_{n-1} \\
&= n((n-1)a_{n-2}) & &= (n(n-1)) \cdot a_{n-2} \\
&= (n(n-1))((n-2)a_{n-3}) & &= (n(n-1)(n-2)) \cdot a_{n-3} \\
&\vdots \\
&= (n(n-1)(n-2)\cdots 1) \cdot a_{n-n} \\
&= n! \cdot a_0 \\
&= 5 \cdot n!
\end{aligned}$$

10. Rosen, page 409, problem 11.
a) **Answer:** $a_n = 3a_{n-1}$, where a_n is the number of bacteria after n hours, and a_0 is the number of bacteria used to begin a new colony.
b) **Answer:** 5,904,900
Repeat the same steps as that in the problem 5, part a),

$$\begin{aligned}
a_n &= 3a_{n-1} \\
&= 3 \cdot (3a_{n-2}) = 3^2 \cdot a_{n-2} \\
&= 3^2 \cdot (3a_{n-3}) = 3^3 \cdot a_{n-3} \\
&= 3^3 \cdot (3a_{n-4}) = 3^4 \cdot a_{n-4} \\
&\vdots \\
&= 3^n \cdot a_{n-n} \\
&= 3^n \cdot a_0
\end{aligned}$$

Where $a_0 = 100$, so $a_{10} = 3^{10} \cdot 100 = 5,904,900$

11. Rosen, page 409, problem 12.
a) **Answer:** $a_n = a_{n-1} + 1.3\% \cdot a_{n-1} = 1.013 \cdot a_{n-1}$, where a_n is the population of the world n years after 2002, and a_0 is the population of the world in 2002, which is 6.2 billion.

b) Similar with problem 11 on page 409 of Rosen,

$$
\begin{aligned}
a_n &= 1.013 a_{n-1} \\
&= 1.013 \cdot (1.013 a_{n-2}) \;\; = 1.013^2 \cdot a_{n-2} \\
&= 1.013^2 \cdot (1.013 a_{n-3}) = 1.013^3 \cdot a_{n-3} \\
&= 1.013^3 \cdot (1.013 a_{n-4}) = 1.013^4 \cdot a_{n-4} \\
&\vdots \\
&= 1.013^n \cdot a_{n-n} \\
&= 1.013^n \cdot a_0
\end{aligned}
$$

c) In part b), $a_0 = 6$ billion, so $a_{20} = 1.013^{20} \cdot 6.2 = 8.0270$ billion, where a_{20} is the population of the world in 2022.

12. Rosen, page 410, problem 17.
Please see *"Solutions to Odd-Numbered Exercises"* of Rosen.(Page S-36)

13. Rosen, page 411, problem 36.
a) First let a_n be the number of ways the bus driver can pay a toll of n cents by using nickels and dimes (where the order in which the coins are used matters).

- If the toll can be divided by 5, i.e. $n = 5k$ (where k is a nonnegative integer), then the number of ways the bus driver can pay a toll of n = 5k cents by using nickels and dimes is a_{5k}.
- If the toll can't be divided by 5, i.e. $n = 5k+1$, $5k+2$, $5k+3$ or $5k+4$ (where k is a nonnegative integer), then the number of ways the bus driver can pay n = $5k+1$, $5k+2$, $5k+3$ or $5k+4$ cents is $a_{5(k+1)}$, which means he has to pay a little more money.

So in convenience **from now on we denote a_k be the number of ways the bus driver can pay a toll of $n = 5k$ cents by using nickels and dimes** (where the order in which the coins are used matters), then no matter how the bus driver pays the toll, the last coin he throws into the toll collector is always either a nickel or a dime:

- Case 1: if the last coin is a nickel, then we need to find all ways of paying $5(k-1)$, which is a_{k-1}.
- Case2: If the last coin is a dime, it is preceeded by all the possible ways of paying $5(k-2)$ which is a_{k-2}.

So we got the recurrence relation: $a_k = a_{k-1} + a_{k-2}$, where $a_0 = 1$, $a_1 = 1$

b)**Answer**: 55 ways. $45 = 5 \times 9$, using $a_k = a_{k-1} + a_{k-2}$, where $a_0 = 1$, $a_1 = 1$
$a_0 = 1$
$a_1 = 1$
$a_2 = a_0 + a_1 = 1 + 1 = 2$
$a_3 = a_1 + a_2 = 1 + 2 = 3$
$a_4 = a_2 + a_3 = 2 + 3 = 5$
$a_5 = a_3 + a_4 = 3 + 5 = 8$
$a_6 = a_4 + a_5 = 5 + 8 = 13$
$a_7 = a_5 + a_6 = 8 + 13 = 21$
$a_8 = a_6 + a_7 = 13 + 21 = 34$
$a_9 = a_7 + a_8 = 21 + 34 = 55$ ways