Reinforcement Learning
or,
Learning and Planning with Markov Decision Processes

295 Seminar, Winter 2018
Rina Dechter

Slides will follow David Silver's, and Sutton's book

Goals: To learn together the basics of RL.
Some lectures and classic and recent papers from the literature

Students will be active learners and teachers

[Class page]

[Demo]

[Detailed demo]

Topics

1. Introduction and Markov Decision Processes: Basic concepts. S&B chapters 1, 3. (myslides 2)

2. Planning Dynamic Programming – Policy Iteration, Value Iteration, S&B chapter 4, (myslides 3)

3. Monte-Carlo(MC) and Temporal Differences (TD): S&B chapters 5 and 6, (myslides 4, myslides 5)

4. Multi-step bootstrapping: S&B chapter 7, (myslides 4, last part, slides 6 Sutton)

5. Bandit algorithms: S&B chapter 2, (myslides 7 , sutton-based)

6. Exploration exploitation. (Slides: silver 9, Brunskill)

7. Planning and learning MCTS: S&B chapter 8, (slides Brunskill)

8. function approximations S&B chapter 9,10,11, (slides: silver 6, Sutton 9,10,11)

9. Policy gradient methods: S&B chapter 13, (slides: silver 7, Sutton 13)
10. Deep RL ???

Resources

- Book: Reinforcement Learning: An Introduction
  Richard S. Sutton and Andrew G. Barto
- UCL Course on Reinforcement Learning
  David Silver
- RealLife Reinforcement Learning
  Emma Brunskill
- Udacity course on Reinforcement Learning:
  Isbell, Littman and Pryby

References

Bertsekas, D. P. (2007a). *Dynamic Programming and Optimal Control*, volume 1. Athena Scientific, Belmont, MA, 3 edition.

Bertsekas, D. P. (2007b). *Dynamic Programming and Optimal Control*, volume 2. Athena Scientific, Belmont, MA, 3 edition.

Bertsekas, D. P. and Shreve, S. (1978). *Stochastic Optimal Control (The Discrete Time Case)*. Academic Press, New York.

Puterman, M. (1994). *Markov Decision Processes — Discrete Stochastic Dynamic Programming*. John Wiley & Sons, Inc., New York, NY.

Singh, S. P. and Yee, R. C. (1994). An upper bound on the loss from approximate optimal-value functions. *Machine Learning*, 16(3):227–233.

Course Outline, Silver

- Part I: Elementary Reinforcement Learning
  1. Introduction to RL
  2. Markov Decision Processes
  3. Planning by Dynamic Programming
  4. Model-Free Prediction
  5. Model-Free Control
- Part II: Reinforcement Learning in Practice
  1. Value Function Approximation
  2. Policy Gradient Methods
  3. Integrating Learning and Planning
  4. Exploration and Exploitation
  5. Case study - RL in games

Introduction to Reinforcement Learnintg

Chapter 1 S&B

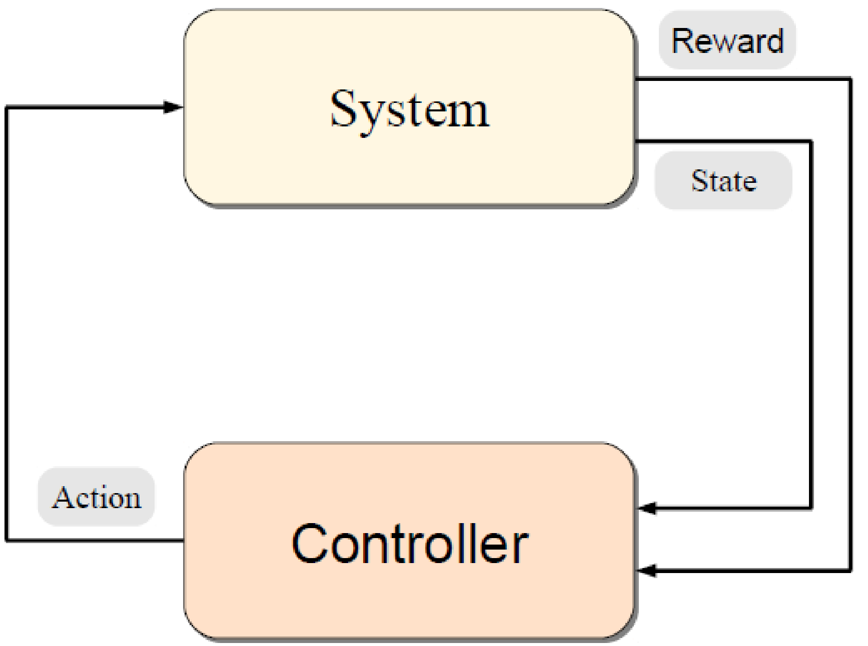
Reinforcement Learning

Learn a behavior strategy (policy) that maximizes the long term
Sum of rewards **in an unknown and stochastic environment (**Emma Brunskill:)

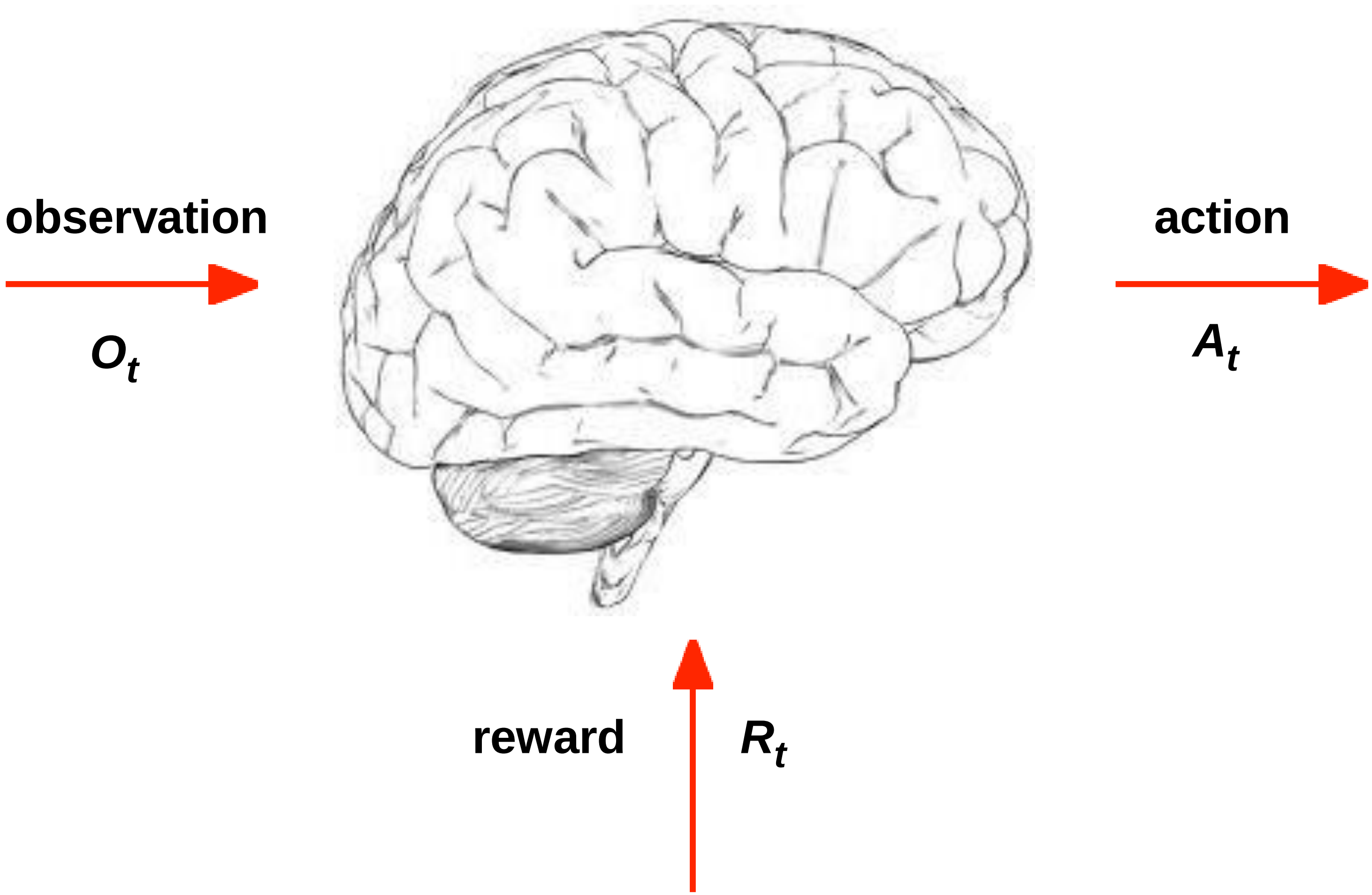
Planning under Uncertainty

Learn a behavior strategy (policy) that maximizes the long term
Sum of rewards **in a known stochastic environment (**Emma Brunskill:)

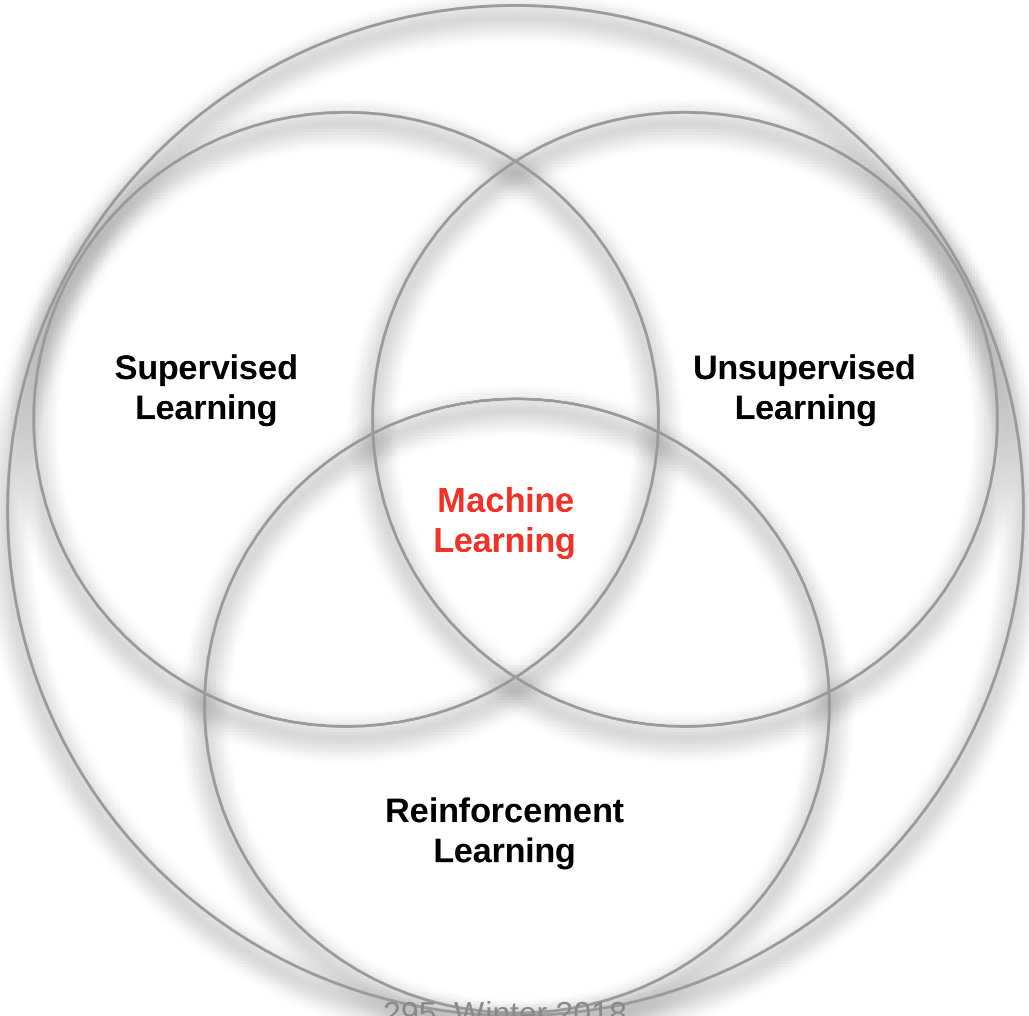
Reinforcement Learning



Agent and Environment



Branches of Machine Learning



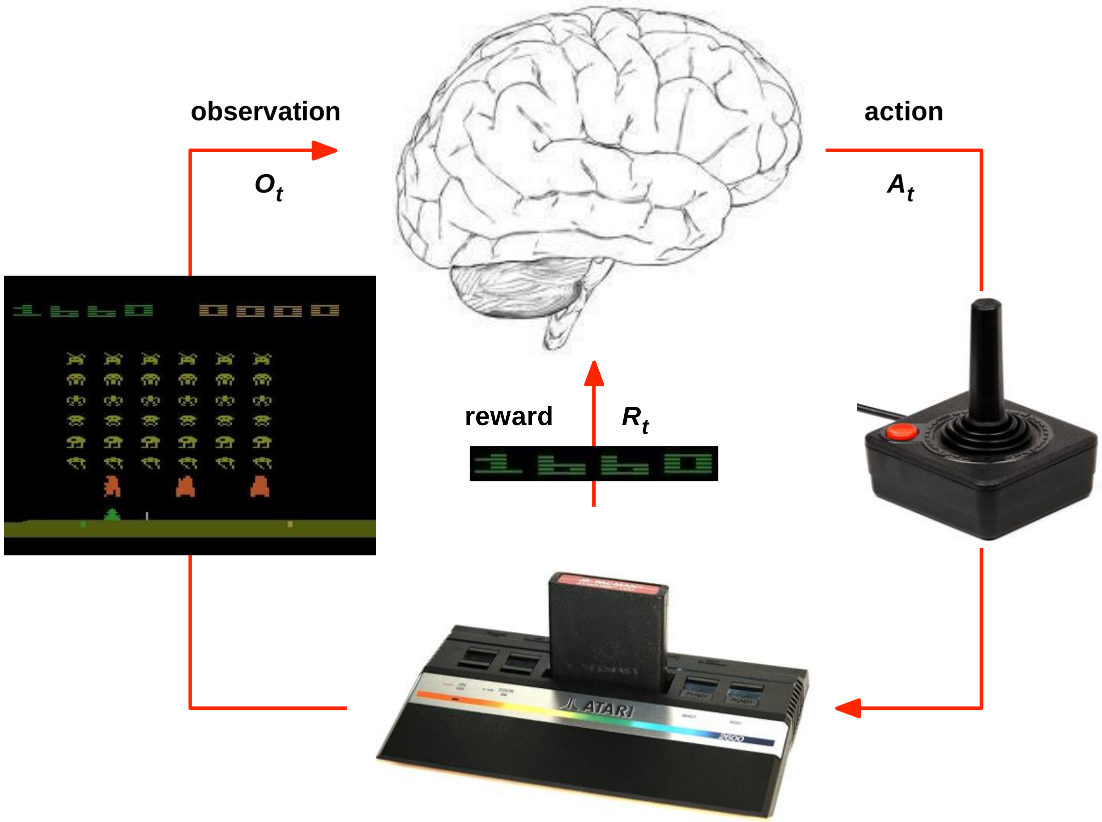
Sequential Decision Making

Reward

- Goal: *select actions to maximise total future reward*
- Actions may have long term consequences
- Reward may be delayed
- It may be better to sacrifice immediate reward to gain more long-term reward
- Examples:
  - A financial investment (may take months to mature)
  - Refuelling a helicopter (might prevent a crash in several hours)
  - Blocking opponent moves (might help winning chances many moves from now)
    - My pet project: **The academic commitment problem**. Given outside requests (committees, reviews, talks, teach…) what to accept and what to reject today?

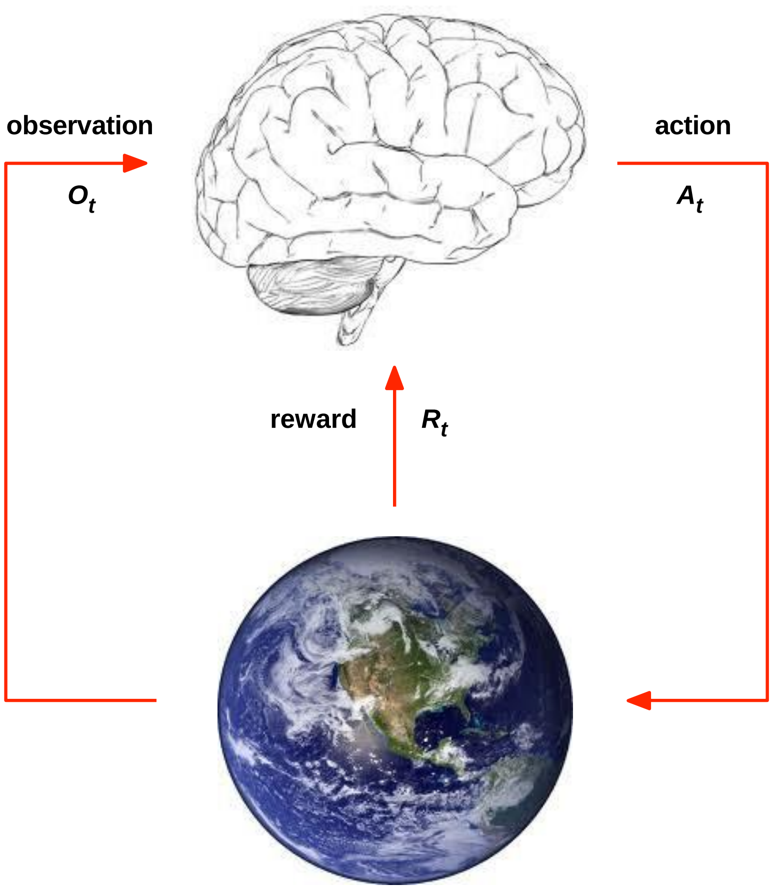
Examples: Robotics

Atari Example: Reinforcement Learning



- Rules of the game are unknown
- Learn directly from interactive game-play
- Pick actions on joystick, see pixels and scores

Agent and Environment



- At each step t the agent:
  - Executes action A_t
  - Receives observation O_t
  - Receives scalar reward R_t
- The environment:
  - Receives action A_t
  - Emits observation O_{t+1}
  - Emits scalar reward R_{t+1}
- t increments at env. step

Markov Decision Processes

In a nutshell:

MDP is a tuple (S,A,P,R,γ)

- Set of states S
- Start state s_0
- Set of actions A
- Transitions P(s'|s,a) (or T(s,a,s'))
- Rewards R(s,a,s') (or R(s) or R(s,a)
- Discount γ
- Policy = Choice of action for each state
- Utility / Value = sum of (discounted) rewards

Policy: $\pi(s) \rightarrow a$

Value and Q Functions

Most of the story in a nutshell:

- Value of a Policy
- $$V^{\pi}(s) = \sum_{s' \in S} p(s' \mid s, \pi(s)) \left[R(s, \pi(s), s') + \gamma V^{\pi}(s') \right.$$

$$Q^{\pi}(s, a) = \sum_{s' \in S} p(s' \mid s, a) \left[R(s, a, s') + \gamma V^{\pi}(s') \right]$$

- Optimal Value & Optimal Policy

$$V^*(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s, \pi(s), s') + \gamma V^*(s_j) \right] \right)$$

$$= \max_a Q^*(s, a)$$

$$\pi^*(s) = \operatorname{argmax}_a Q^*(s, a)$$

Most of the story in a nutshell:

Bellman Equation

$$V^*(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s, \pi(s), s') + \gamma V^*(s_j) \right] \right)$$

- Holds for V*
- Inspires an update rule

Most of the story in a nutshell:

Value Iteration

1. Initialize $V_1(s_i)$ for all states s_i
2. k=2
3. While k < desired horizon or (if infinite horizon) values have converged
   - For all s,

$$V_k(s_i) = \max_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s, \pi(s), s') + \gamma V_{k-1}(s_j) \right] \right)$$

$$\pi_k(s_i) = \operatorname{argmax}_a \left(\sum_{s_j \in S} p(s_j \mid s_i, a) \left[R(s, \pi(s), s') + \gamma V_{k-1}(s_j) \right] \right)$$

Most of the story in a nutshell:

Will Value Iteration Converge?

- Yes, if discount factor is < 1 or end up in a terminal state with probability 1

- Bellman equation is a contraction
- If apply it to two different value functions, distance between value functions shrinks after apply Bellman equation to each

Most of the story in a nutshell:

Bellman Operator is a Contraction

|| V-V'|| = Infinity norm
(find max diff
Over all states)

$$\|BV - BV'\| = \left\| \begin{array}{l} \max_a \left[R(s,a) + \gamma \sum_{s_j \in S} p(s_j \mid s_i, a) V(s_j) \right] \\ - \max_{a'} \left[R(s,a') - \gamma \sum_{s_j \in S} p(s_j \mid s_i, a') V'(s_j) \right] \end{array} \right\|$$

$$\leq \left\| \max_a \left[R(s,a) + \gamma \sum_{s_j \in S} p(s_j \mid s_i, a) V(s_j) - R(s,a) + \gamma \sum_{s_j \in S} p(s_j \mid s_i, a) V'(s_j) \right] \right\|$$

$$\leq \gamma \left\| \max_a \left[\sum_{s_j \in S} p(s_j \mid s_i, a) V(s_j) - \sum_{s_j \in S} p(s_j \mid s_i, a) V'(s_j) \right] \right\|$$

$$= \gamma \max_a \left\| \left[\sum_{s_j \in S} p(s_j \mid s_i, a) (V(s_j) - V'(s_j)) \right] \right\|$$

$$\leq \gamma \max_{a, s_i} \sum_{s_j \in S} p(s_j \mid s_i, a) \left| V(s_j) - V'(s_j) \right|$$

$$\leq \gamma \max_{a, s_i} \sum_{s_j \in S} p(s_j \mid s_i, a) \|V - V'\|$$

$$= \gamma \|V - V'\|$$

Most of the story in a nutshell:

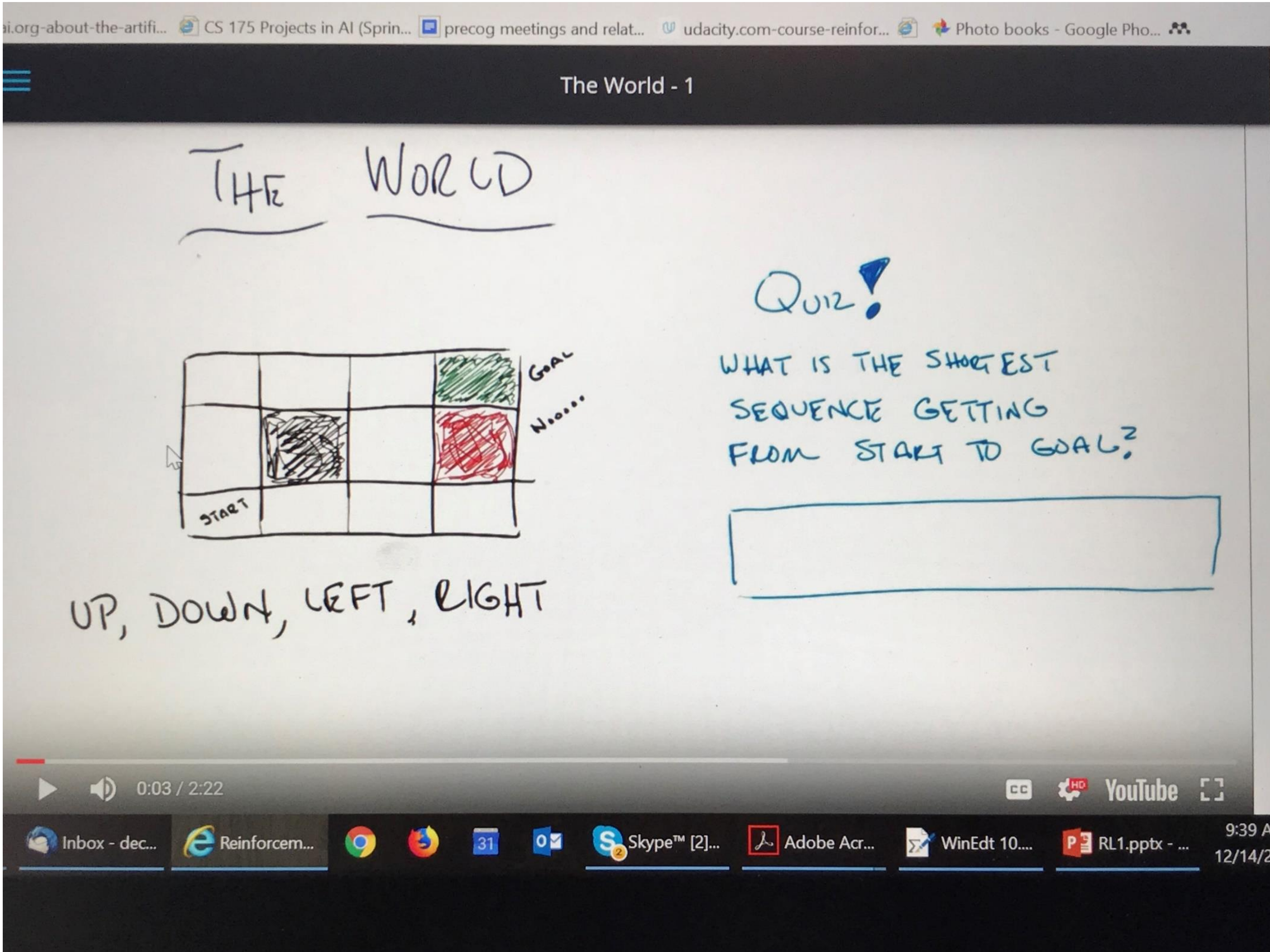
Properties of Contraction

- Only has 1 fixed point
  - If had two, then would not get closer when apply contraction function, violating definition of contraction
- When apply contraction function to any argument, value must get closer to fixed point
  - Fixed point doesn't move
  - Repeated function applications yield fixed point

Most of the story in a nutshell:

Value Iteration Converges

- If discount factor < 1
- Bellman is a contraction
- Value iteration converges to unique solution which is optimal value function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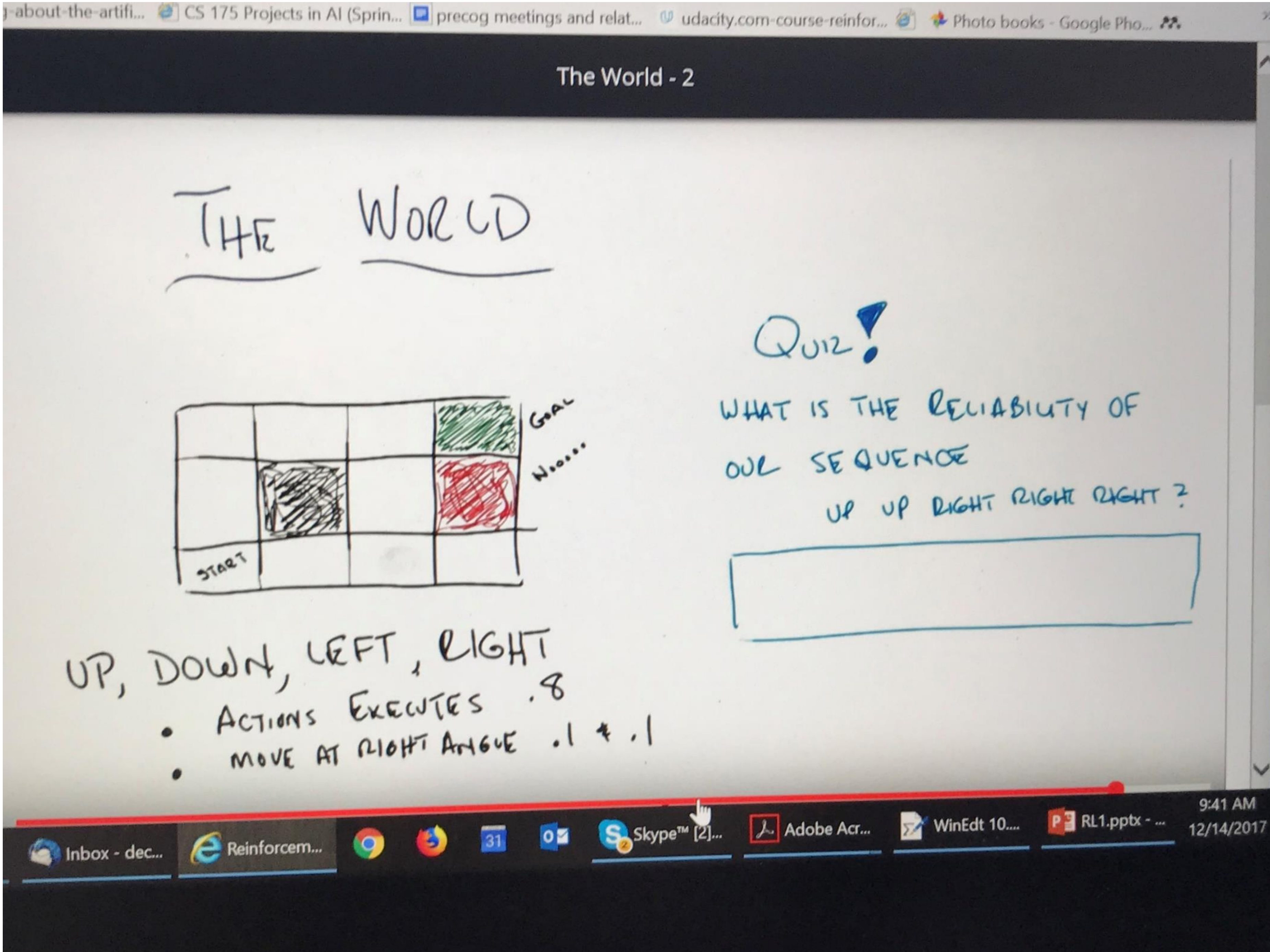

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History and State

- The history is the sequence of observations, actions, rewards

$$H_t = O_1, R_1, A_1, ..., A_{t-1}, O_t, R_t$$

- i.e. all observable variables up to time t
- i.e. the sensorimotor stream of a robot or embodied agent
- What happens next depends on the history:
  - The agent selects actions
  - The environment selects observations/rewards
- State is the information used to determine what happens next
- Formally, state is a function of the history:

$$S_t = f(H_t)$$

Information State

An information state (a.k.a. Markov state) contains all useful information from the history.

Definition

A state S_t is Markov if and only if

$$\mathbb{P}[S_{t+1} \mid S_t] = \mathbb{P}[S_{t+1} \mid S_1, ..., S_t]$$

- "The future is independent of the past given the present"

$$H_{1:t} \rightarrow S_t \rightarrow H_{t+1:\infty}$$

- Once the state is known, the history may be thrown away
- i.e. The state is a sufficient statistic of the future
- The environment state S_t is Markov
- The history H_t is Markov

Major Components of an RL Agent

- An RL agent may include one or more of these components:
  - Policy: agent's behaviour function
  - Value function: how good is each state and/or action
  - Model: agent's representation of the environment

Policy

- A policy is the agent's behaviour
- It is a map from state to action, e.g.
- Deterministic policy: $a = \pi(s)$
- Stochastic policy: $\pi(a|s) = \mathbb{P}[A_t = a | S_t = s]$

Value Function

- Value function is a prediction of future reward
- Used to evaluate the goodness/badness of states
- And therefore to select between actions, e.g.

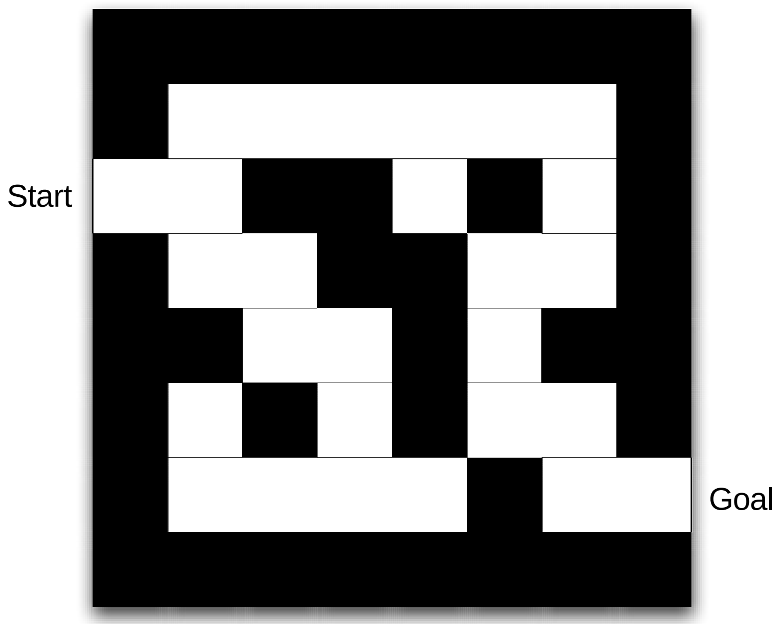
$$v_\pi(s) = \mathrm{E}_\pi\, R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots \mid S_t = s$$

Model

- A model predicts what the environment will do next
- $\mathcal{P}$ predicts the next state
- $\mathcal{R}$ predicts the next (immediate) reward, e.g.

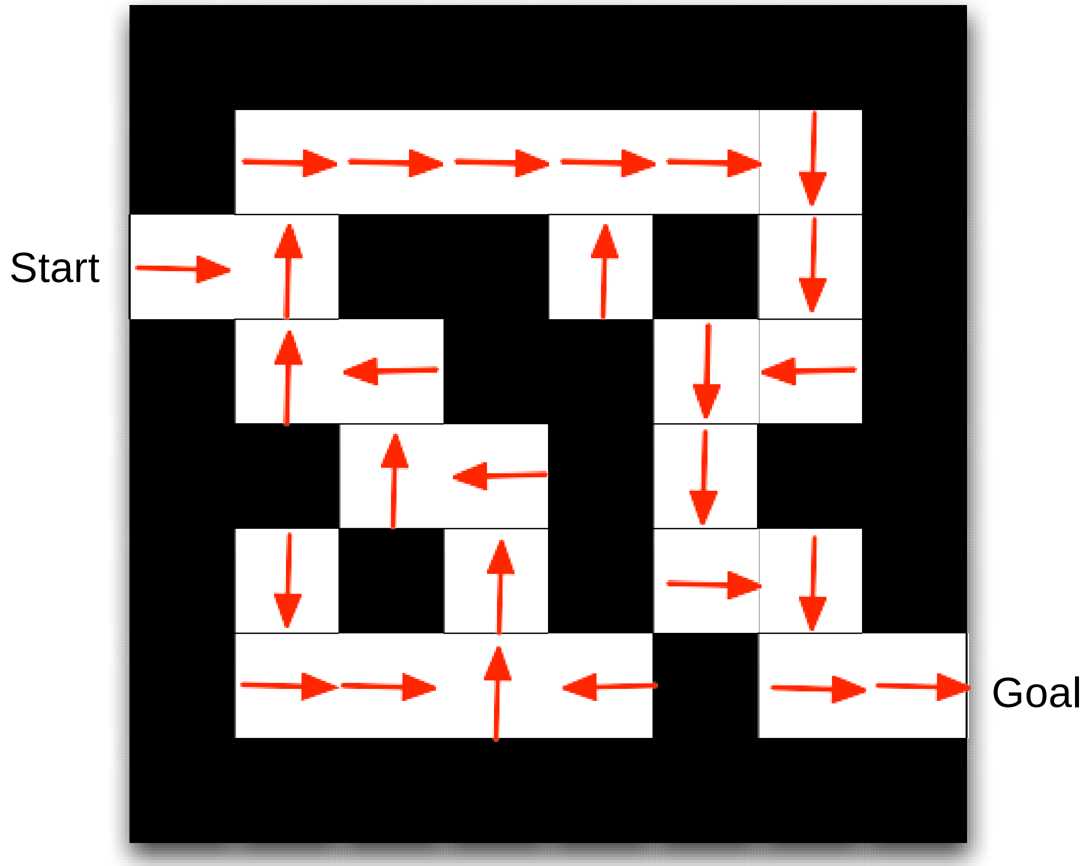
$$\mathcal{P}^a_{ss'} = \mathbb{P}[S_{t+1} = s' \mid S_t = s, A_t = a]$$
$$\mathcal{R}^a_s = \mathbb{E}\left[R_{t+1} \mid S_t = s, A_t = a\right]$$

Maze Example



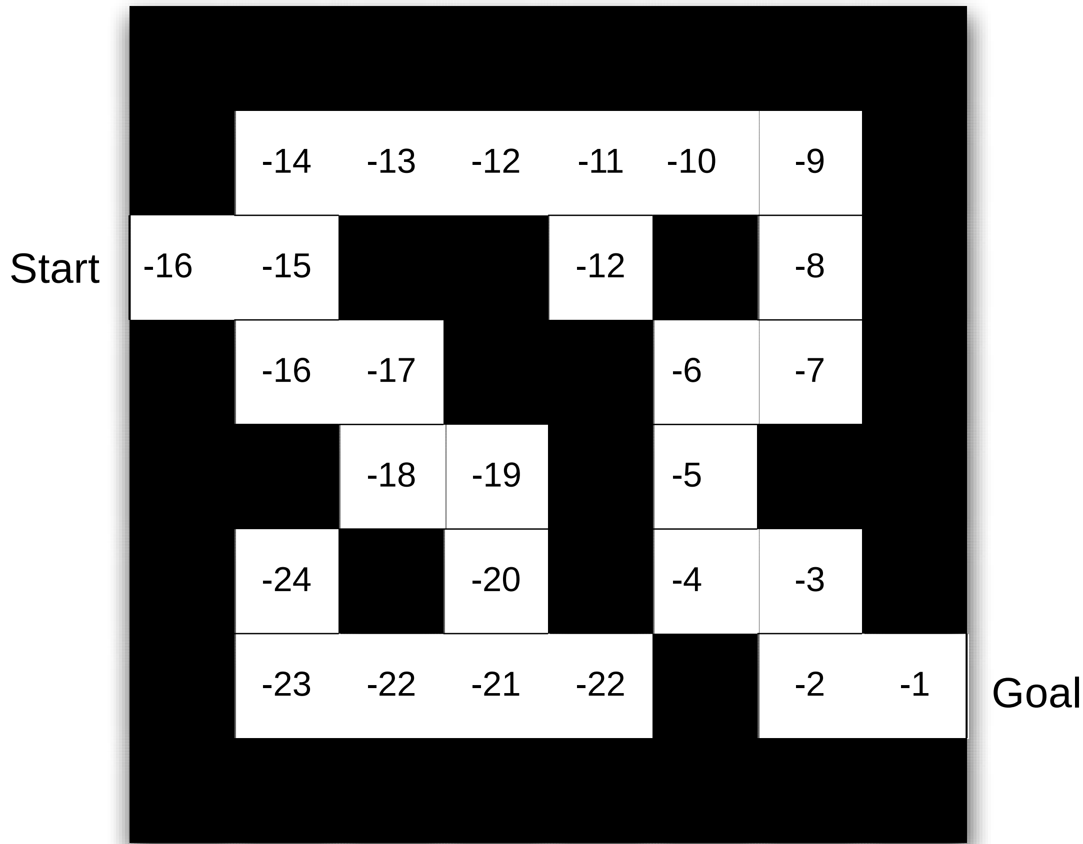
- Rewards: -1 per time-step
- Actions: N, E, S, W
- States: Agent's location

Maze Example: Policy



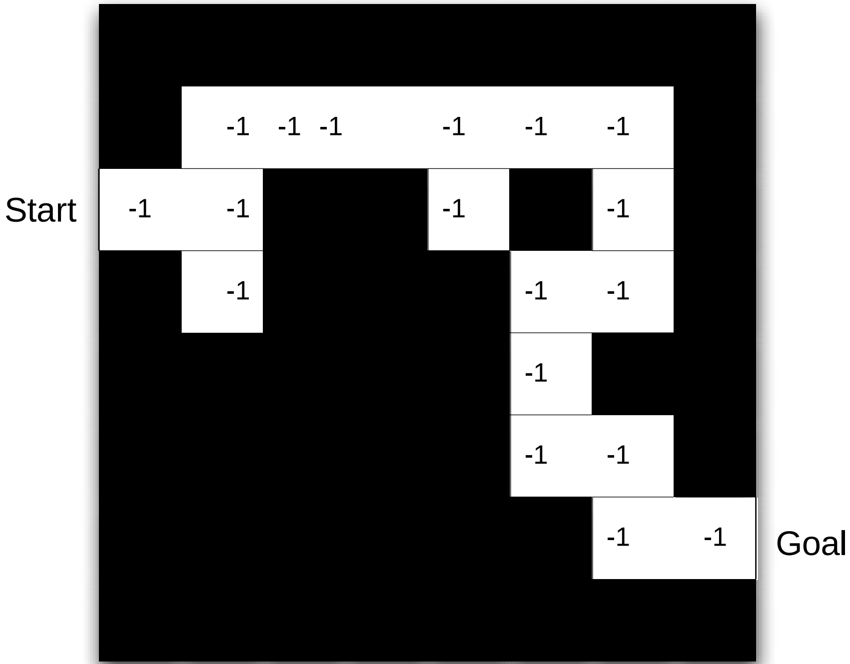
- Arrows represent policy $\pi(s)$ for each state s

Maze Example: Value Function



- Numbers represent value $v_\pi(s)$ of each state s

Maze Example: Model



- Agent may have an internal model of the environment
- Dynamics: how actions change the state
- Rewards: how much reward from each state
- The model may be imperfect

- Grid layout represents transition model $P^a_{ss'}$
- Numbers represent immediate reward R^a_s from each state *s* (same for all *a*)

Learning and Planning

Two fundamental problems in sequential decision making

- Reinforcement Learning:
  - The environment is initially unknown
  - The agent interacts with the environment
  - The agent improves its policy
- Planning:
  - A model of the environment is known
  - The agent performs computations with its model (without any external interaction)
  - The agent improves its policy
  - a.k.a. deliberation, reasoning, introspection, pondering, thought, search

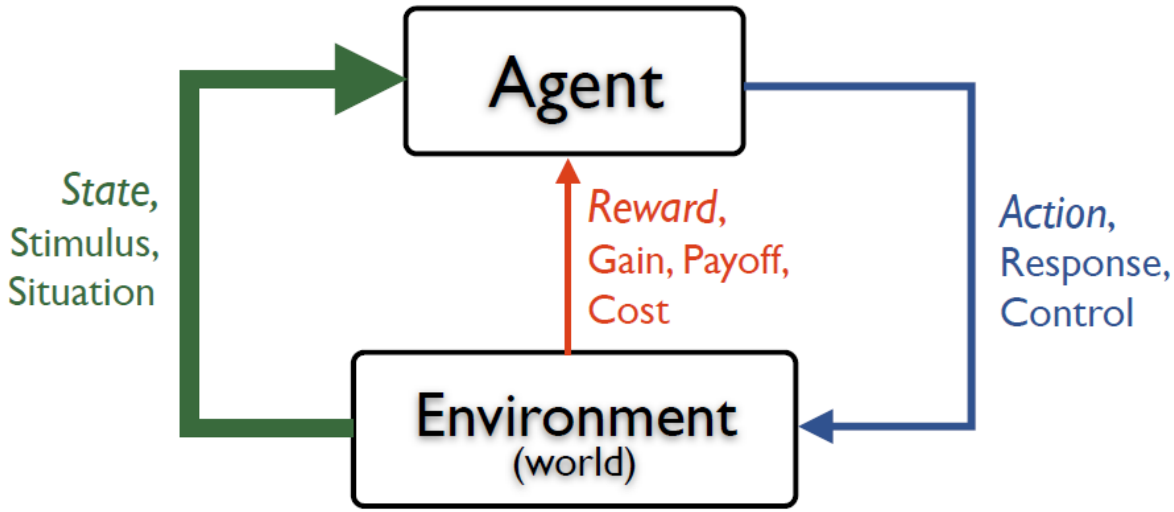
Prediction and Control

- Prediction: evaluate the future
  - Given a policy
- Control: optimise the future
  - Find the best policy

Markov Decision Processes

Chapter 3 S&B

The RL Interface



- Environment may be unknown, nonlinear, stochastic and complex
- Agent learns a policy mapping states to actions
  - Seeking to maximize its cumulative reward in the long run

MDPs

- The world is an MDP (combining the agent and the world): give rise to a trajectory

  S0,A0,R1,S1,A1,R2,S2,A3,R3,S3,…

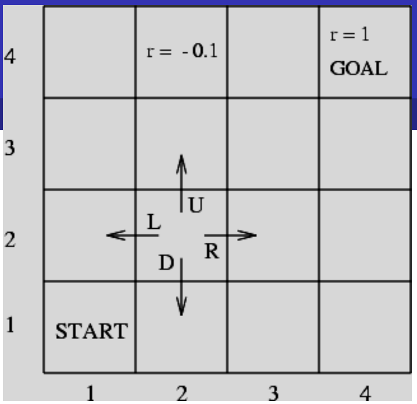
- The process is governed by a transition function

$$p(s', r | s, a) \doteq \Pr\{S_t = s', R_t = r \mid S_{t-1} = s, A_{t-1} = a\},$$

- Markov Process (MP)
- Markov Reward Process (MRP)
- Markov Decision Process (MDP)

Markov Property

Markov Property



"The future is independent of the past given the present"

Definition

A state S_t is *Markov* if and only if

$$P[S_{t+1} \mid S_t] = P[S_{t+1} \mid S_1, ..., S_t]$$

- The state captures all relevant information from the history
- Once the state is known, the history may be thrown away
- i.e. The state is a sufficient statistic of the future

State Transition Matrix

For a Markov state s and successor state s', the *state transition probability* is defined by

$$\mathcal{P}_{ss'} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s\right]$$

State transition matrix $\mathcal{P}$ defines transition probabilities from all states s to all successor states s',

$$\mathcal{P} = \textit{from} \overset{\textit{to}}{\begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \dots & \mathcal{P}_{nn} \end{bmatrix}}$$

where each row of the matrix sums to 1.

Markov Process

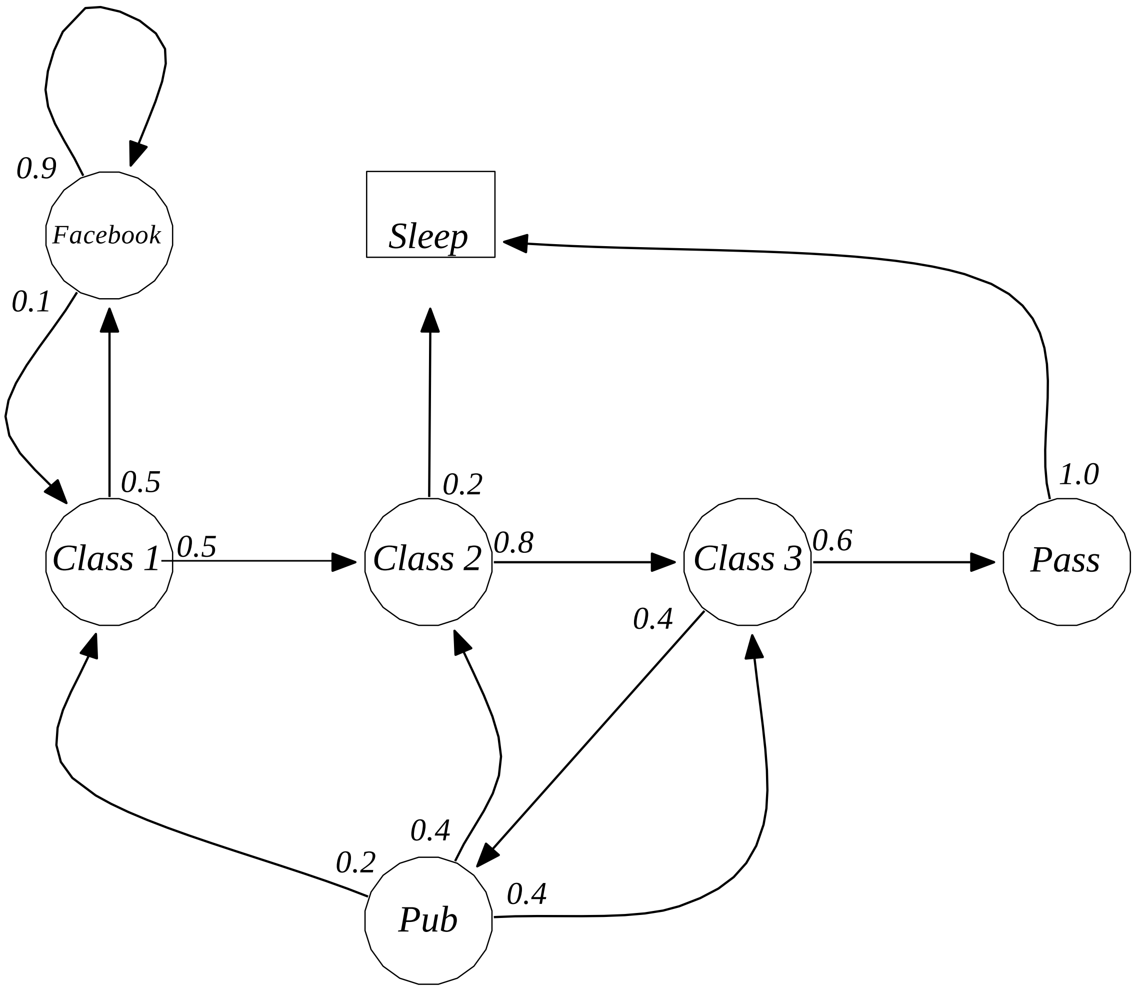
A Markov process is a memoryless random process, i.e. a sequence of random states $S_1, S_2, \ldots$ with the Markov property.

Definition

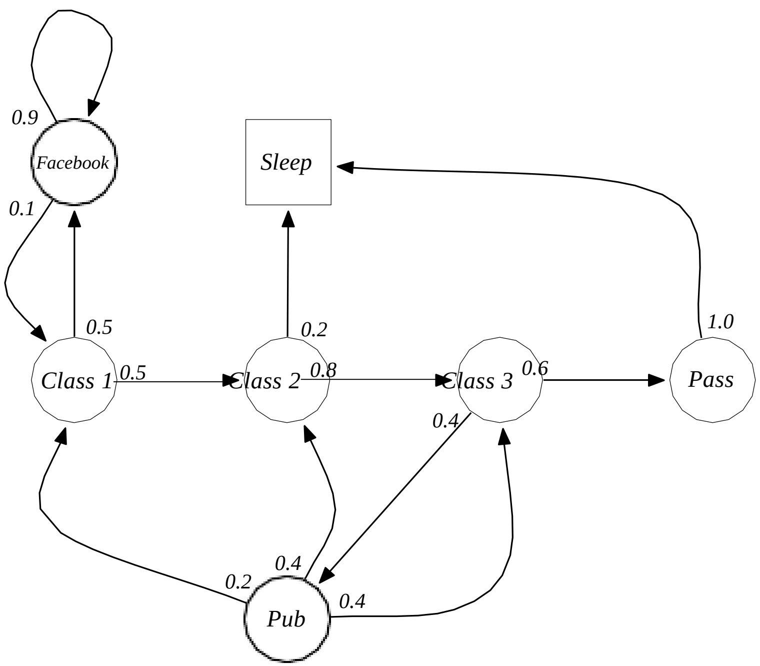
A *Markov Process* (or *Markov Chain*) is a tuple (S, P)

- S is a (finite) set of states
- P is a state transition probability matrix,
  $P_{ss'} = P\,[S_{t+1} = s' \mid S_t = s]$

Example: Student Markov Chain, a transition graph



Example: Student Markov Chain Episodes

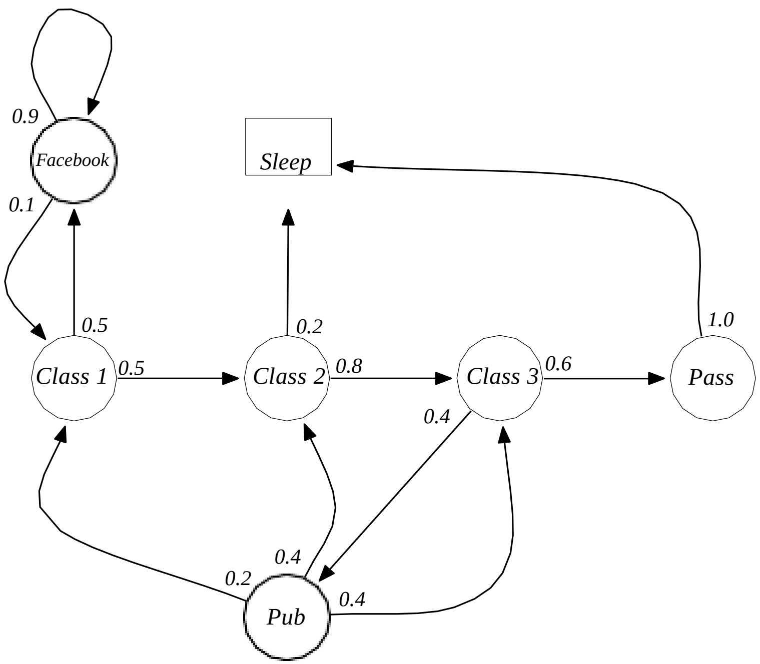


Sample episodes for Student Markov Chain starting from $S_1 = C1$

$$S_1, S_2, \ldots, S_T$$

- C1 C2 C3 Pass Sleep
- C1 FB FB C1 C2 Sleep
- C1 C2 C3 Pub C2 C3 Pass Sleep
- C1 FB FB C1 C2 C3 Pub C1 FB FB FB C1 C2 C3 Pub C2 Sleep

Example: Student Markov Chain Transition Matrix



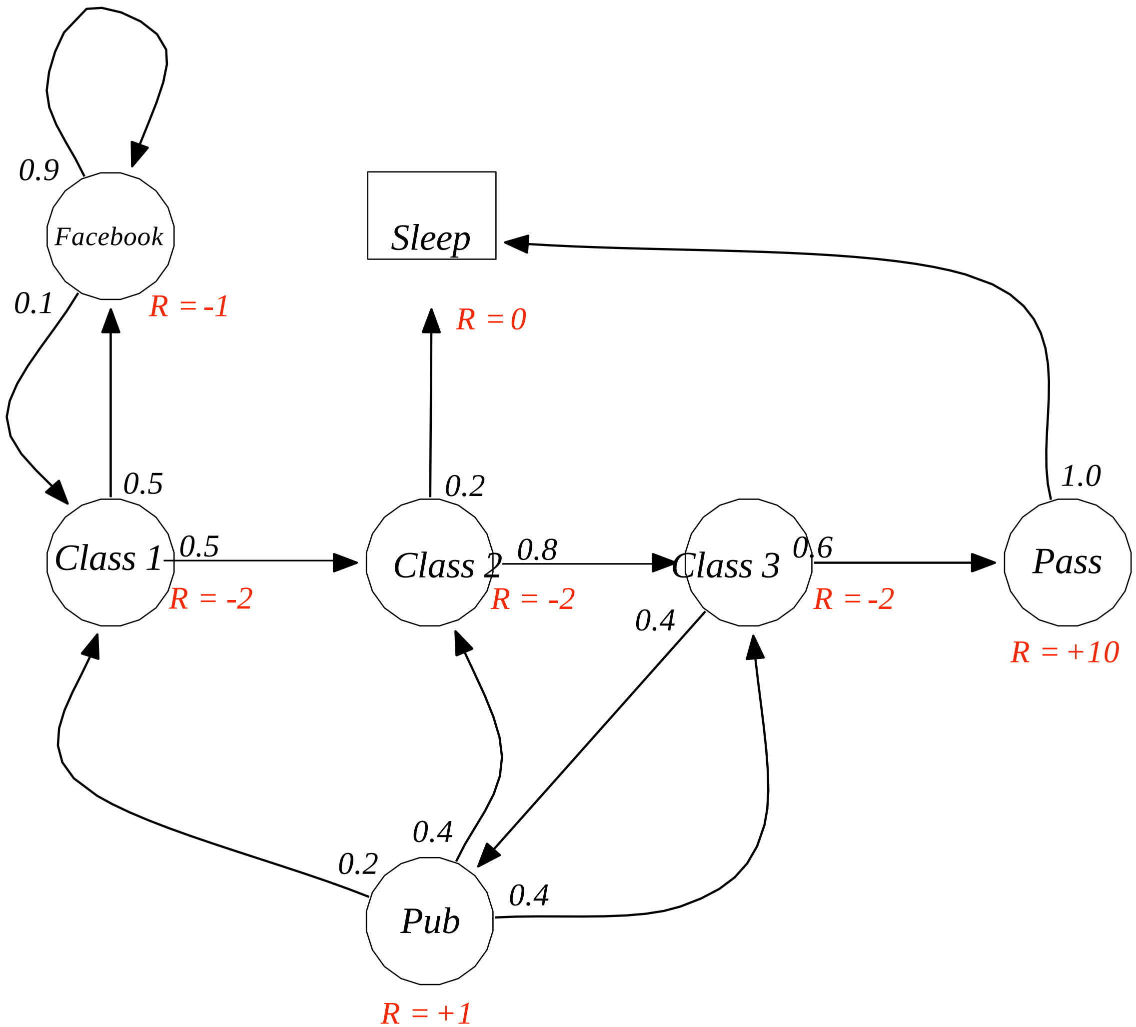
$\mathcal{P} =$

	C1	C2	C3	Pass	Pub	FB	Sleep
C1		0.5				0.5	
C2			0.8				0.2
C3				0.6	0.4		
Pass							1.0
Pub	0.2	0.4	0.4				
FB	0.1					0.9	
Sleep							1

Markov Decision Processes

- States: S
- Model: T(s,a,s’) = P(s’|s,a)
- Actions: A(s), A
- Reward: R(s), R(s,a), R(s,a,s’)
- Discount: γ
- Policy: $\pi(s) \to a$
- Utility/Value: sum of discounted rewards.
- We seek optimal policy that maximizes the expected total (discounted) reward

Example: Student MRP



Goals, Returns and Rewards

- The agent’s goal is to maximize the total amount of rewards it gets (not immediate ones), relative to the long run.
- Reward is -1 typically in mazes for every time step
- Deciding how to associate rewards with states is part of the problem modelling. If T is the final step then the return is:

$$G_t \doteq R_{t+1} + R_{t+2} + R_{t+3} + \cdots + R_T,$$

Return

Definition

The *return* G_t is the total discounted reward from time-step t.

$$G_t = R_{t+1} + \gamma R_{t+2} + \ldots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- The *discount* $\gamma \in [0, 1]$ is the present value of future rewards
- The value of receiving reward R after $k + 1$ time-steps is $\gamma^k R$.
- This values immediate reward above delayed reward.
  - γ close to 0 leads to "myopic" evaluation
  - γ close to 1 leads to "far-sighted" evaluation

Why discount?

Most Markov reward and decision processes are discounted. Why?

- Mathematically convenient to discount rewards
- Avoids infinite returns in cyclic Markov processes
- Uncertainty about the future may not be fully represented
- If the reward is financial, immediate rewards may earn more interest than delayed rewards
- Animal/human behaviour shows preference for immediate reward
- It is sometimes possible to use *undiscounted* Markov reward processes (i.e. $\gamma = 1$), e.g. if all sequences terminate.

Value Function

Value Function

The value function $v(s)$ gives the long-term value of state s

Definition

The *state value function* $v(s)$ of an MRP is the expected return starting from state s

$$v(s) = \mathbb{E}[G_t \mid S_t = s]$$

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right], \text{ for all } s \in \mathcal{S},$$

Example: Student MRP Returns

Sample returns for Student MRP:
Starting from $S_1 = C1$ with $\gamma = \frac{1}{2}$

$$G_1 = R_2 + \gamma R_3 + ... + \gamma^{T-2} R_T$$

C1 C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 10 * \frac{1}{8}$	$= -2.25$
C1 FB FB C1 C2 Sleep	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16}$	$= -3.125$
C1 C2 C3 Pub C2 C3 Pass Sleep C1	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 1 * \frac{1}{8} - 2 * \frac{1}{16} \dots$	$= -3.41$
FB FB C1 C2 C3 Pub C1 ... FB	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16} \dots$	$= -3.20$
FB FB C1 C2 C3 Pub C2 Sleep		

Bellman Equation for MRPs

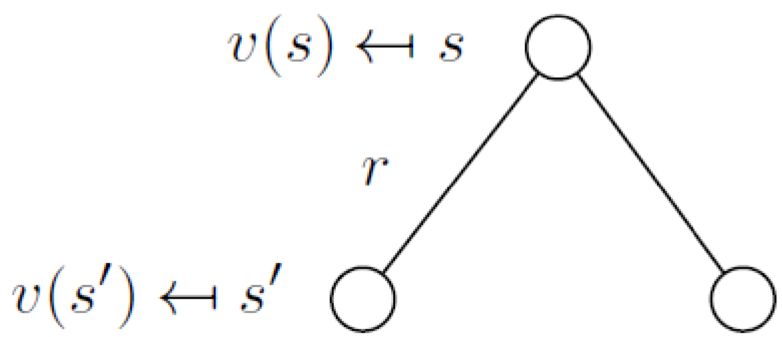
The value function can be decomposed into two parts:

- immediate reward R_{t+1}
- discounted value of successor state $\gamma v(S_{t+1})$

$$
\begin{aligned}
v(s) &= \mathbb{E}\left[G_t \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma \left(R_{t+2} + \gamma R_{t+3} + \dots\right) \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma G_{t+1} \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]
\end{aligned}
$$

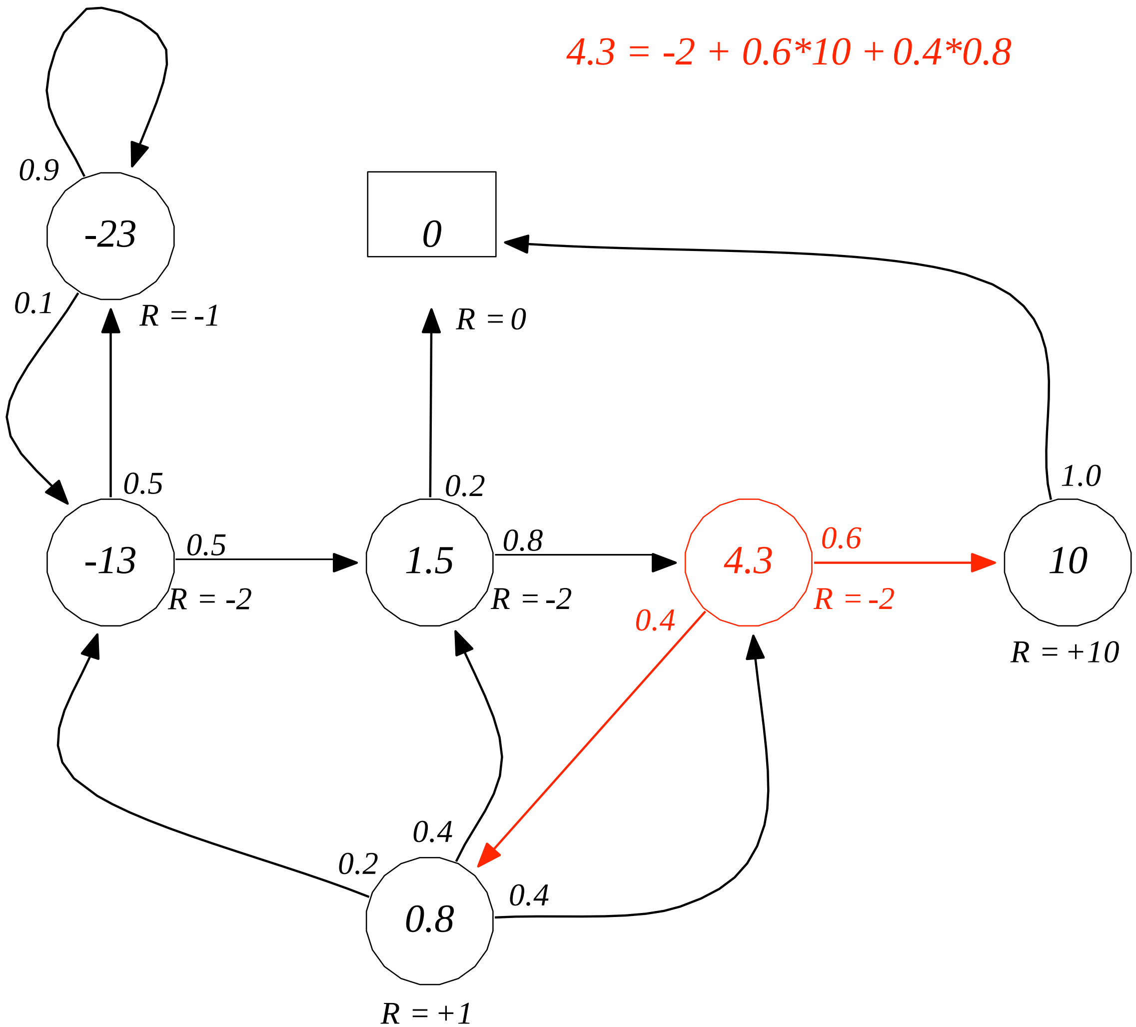
Bellman Equation for MRPs (2)

$$v(s) = \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]$$



$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

Example: Bellman Equation for Student MRP



Bellman Equation in Matrix Form

The Bellman equation can be expressed concisely using matrices,

$$v = \mathrm{R} + \gamma \mathrm{P} v$$

where v is a column vector with one entry per state

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{11} & \dots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix}$$

Solving the Bellman Equation

- The Bellman equation is a linear equation
- It can be solved directly:

$$v = R + \gamma P v$$
$$(I - \gamma P)\, v = R$$
$$v = (I - \gamma P)^{-1} R$$

- Computational complexity is $O(n^3)$ for n states
- Direct solution only possible for small MRPs
- There are many iterative methods for large MRPs, e.g.
  - Dynamic programming
  - Monte-Carlo evaluation
  - Temporal-Difference learning

Markov Decision Process

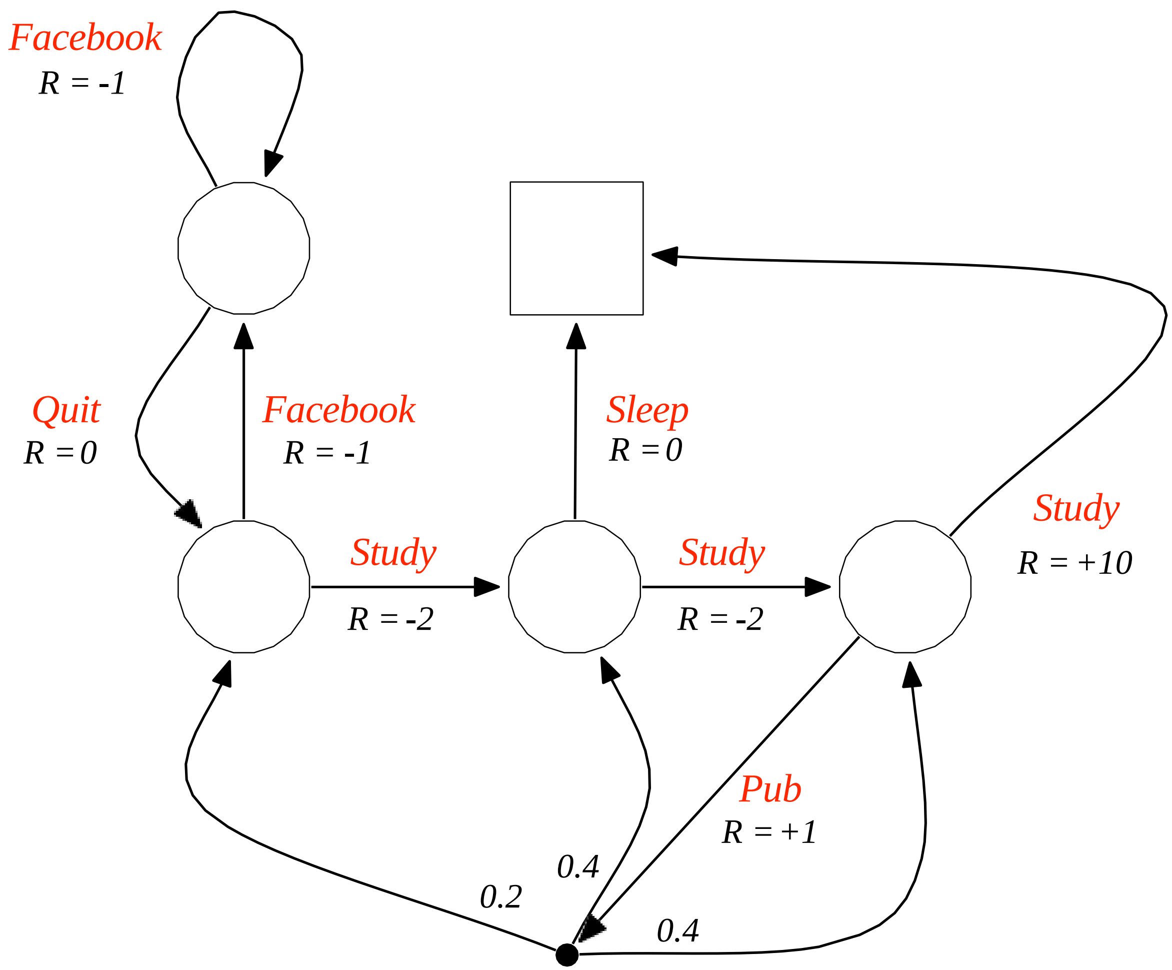
A Markov decision process (MDP) is a Markov reward process with decisions. It is an *environment* in which all states are Markov.

Definition

A *Markov Decision Process* is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{P}$ is a state transition probability matrix, $\mathcal{P}_{ss'}^{a} = \mathbb{P}\left[S_{t+1} = s' \mid S_t = s, A_t = a\right]$
- $\mathcal{R}$ is a reward function, $\mathcal{R}_s^a = \mathbb{E}\left[R_{t+1} \mid S_t = s, A_t = a\right]$
- γ is a discount factor $\gamma \in [0, 1]$.

Example: Student MDP



Policies and Value functions (1)

Definition

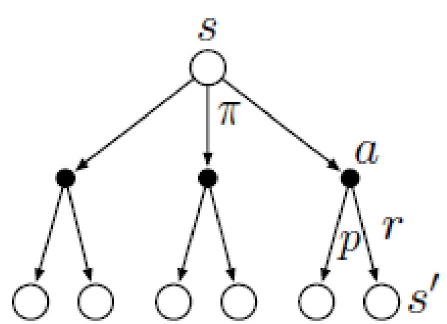
A *policy* π is a distribution over actions given states,

$$\pi(a|s) = \mathbb{P}[A_t = a \mid S_t = s]$$

- A policy fully defines the behaviour of an agent
- MDP policies depend on the current state (not the history)
- i.e. Policies are *stationary* (time-independent),
  $A_t \sim \pi(\cdot|S_t), \forall t > 0$

Policy's and Value functions

$$\begin{aligned}
v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \mid S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} \mid S_t = s] && \text{(by (3.9))} \\
&= \sum_a \pi(a|s) \sum_{s'} \sum_r p(s', r \mid s, a) \Big[r + \gamma \mathbb{E}_\pi[G_{t+1} | S_{t+1} = s'] \Big] \\
&= \sum_a \pi(a|s) \sum_{s', r} p(s', r \mid s, a) \Big[r + \gamma v_\pi(s') \Big], \quad \text{for all } s \in \mathcal{S}, && (3.14)
\end{aligned}$$



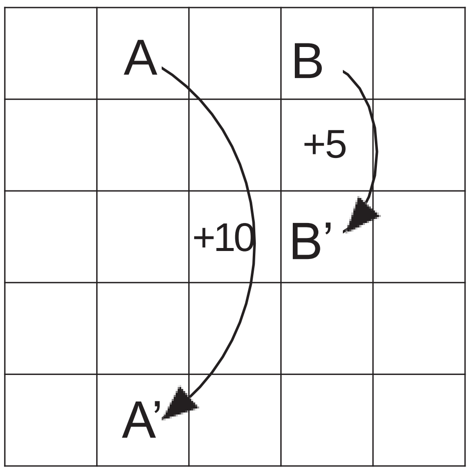
Backup diagram for v_π

Gridworld Example: Prediction

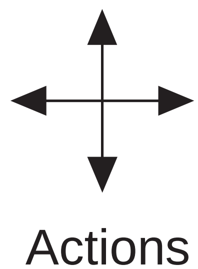
Actions: up, down, left, right. Rewards 0 unless off the grid with reward -1
From A to A', rewatd +10. from B to B' reward +5

Policy: actions are uniformly random.

Figure 3.3



(a)



3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

(b)

What is the value function for the uniform random policy?
Gamma=0.9. solved using EQ. 3.14

Exercise: show 3.14 holds for each state in Figure (b).

Value Function, Q Functions

Definition

The *state-value function* $v_\pi(s)$ of an MDP is the expected return starting from state s, and then following policy π

$$v_\pi(s) = \mathbb{E}_\pi[G_t \mid S_t = s]$$

Definition

The *action-value function* $q_\pi(s, a)$ is the expected return starting from state s, taking action a, and then following policy π

$$q_\pi(s, a) = \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a]$$

Bellman Expectation Equation

The state-value function can again be decomposed into immediate reward plus discounted value of successor state,

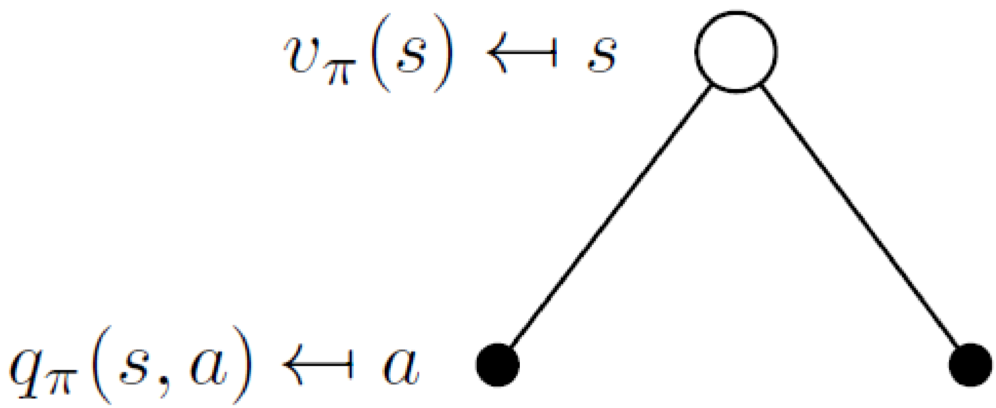
$$v_\pi(s) = \mathbb{E}_\pi\left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right]$$

The action-value function can similarly be decomposed,

$$q_\pi(s,a) = \mathbb{E}_\pi\left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a\right]$$

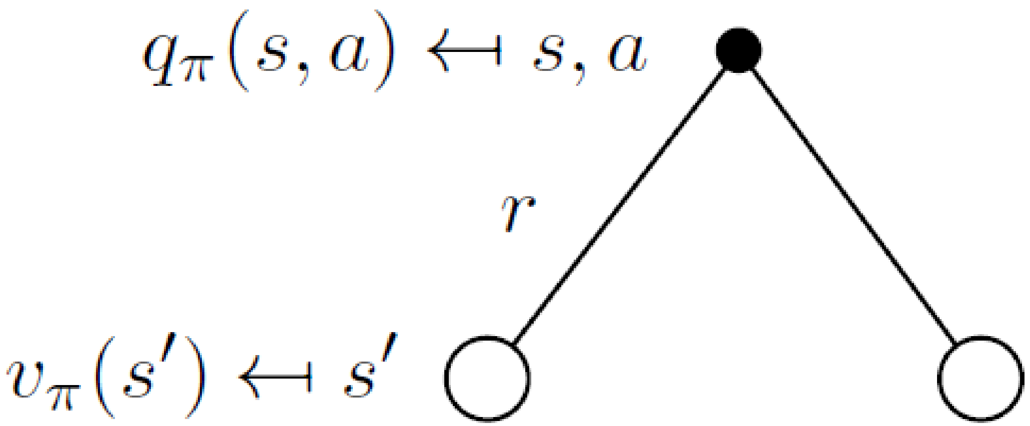
Expressing the functions recursively,
Will translate to one step look-ahead.

Bellman Expectation Equation for V^{π}



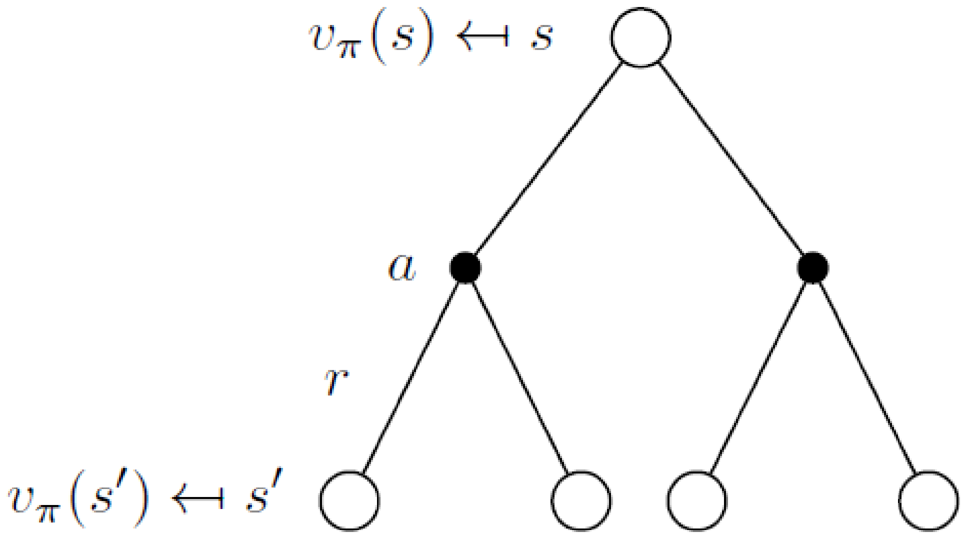
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s,a)$$

Bellman Expectation Equation for Q^{π}



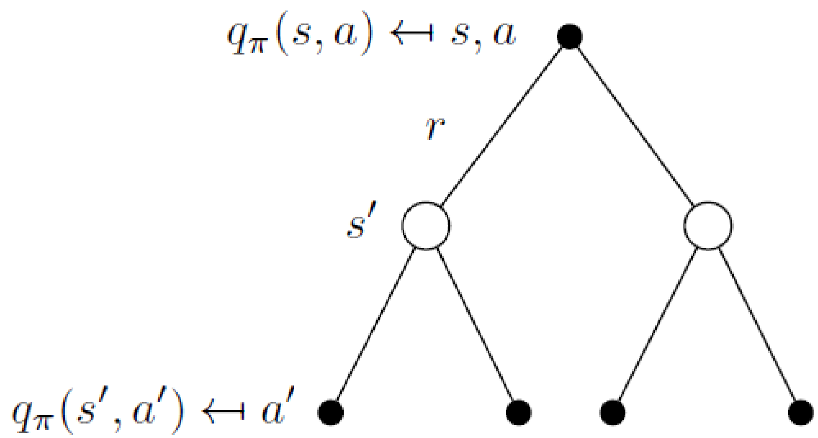
$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')$$

Bellman Expectation Equation for v_π (2)



$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$

Bellman Expectation Equation for q_π (2)



$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \sum_{a' \in \mathcal{A}} \pi(a'|s') q_\pi(s',a')$$

Optimal Policies and Optimal Value Function

Definition

The *optimal state-value function* $v_*(s)$ is the maximum value function over all policies

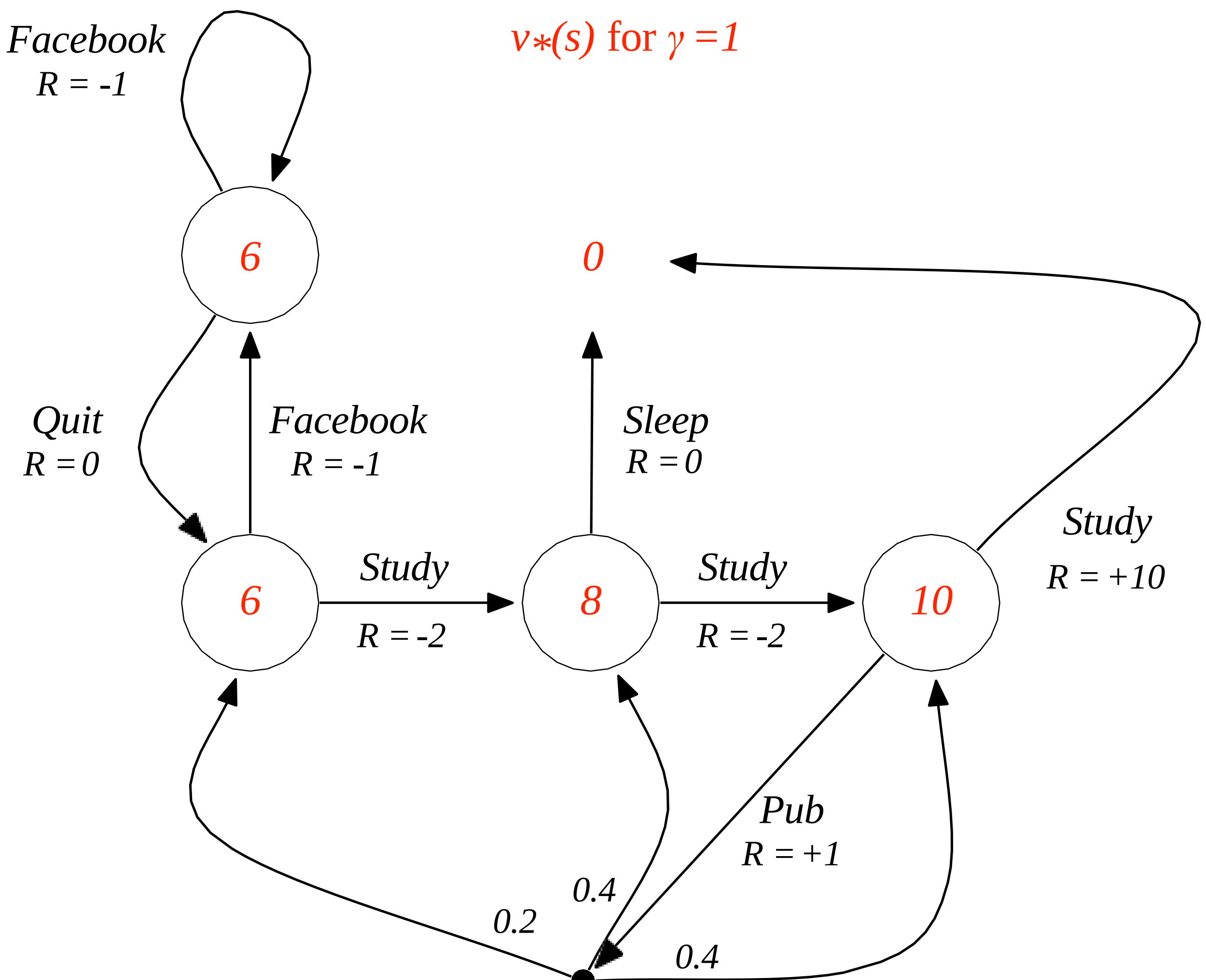
$$v_*(s) = \max_\pi v_\pi(s)$$

The *optimal action-value function* $q_*(s, a)$ is the maximum action-value function over all policies

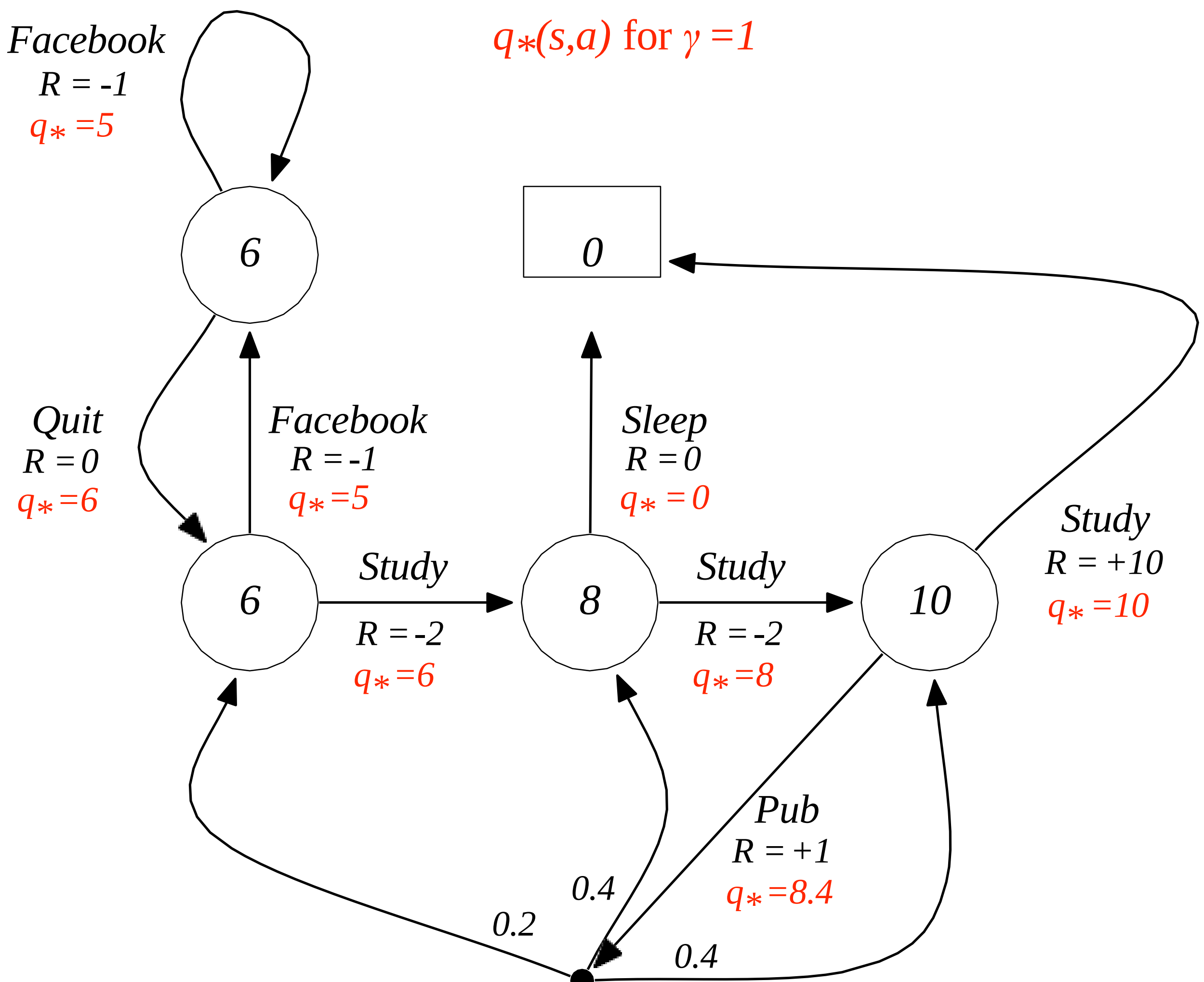
$$q_*(s, a) = \max_\pi q_\pi(s, a)$$

- The optimal value function specifies the best possible performance in the MDP.
- An MDP is “solved” when we know the optimal value function.

Optimal Value Function for Student MDP



Optimal Action-Value Function for Student MDP



Optimal Policy

Define a partial ordering over policies

$$\pi \geq \pi' \text{ if } v_{\pi}(s) \geq v_{\pi'}(s), \forall s$$

Theorem

For any Markov Decision Process

- *There exists an optimal policy π_* that is better than or equal to all other policies, $\pi_* \geq \pi, \forall \pi$*
- *All optimal policies achieve the optimal value function, $v_{\pi_*}(s) = v_*(s)$*
- *All optimal policies achieve the optimal action-value function, $q_{\pi_*}(s, a) = q_*(s, a)$*

Finding an Optimal Policy

An optimal policy can be found by maximising over $q_*(s, a)$,

$$\pi_*(a|s) = \begin{cases} 1 & \text{if } a = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\; q_*(s, a) \\ 0 & otherwise \end{cases}$$

- There is always a deterministic optimal policy for any MDP
- If we know $q_*(s, a)$, we immediately have the optimal policy

Bellman Equation for V* and Q*

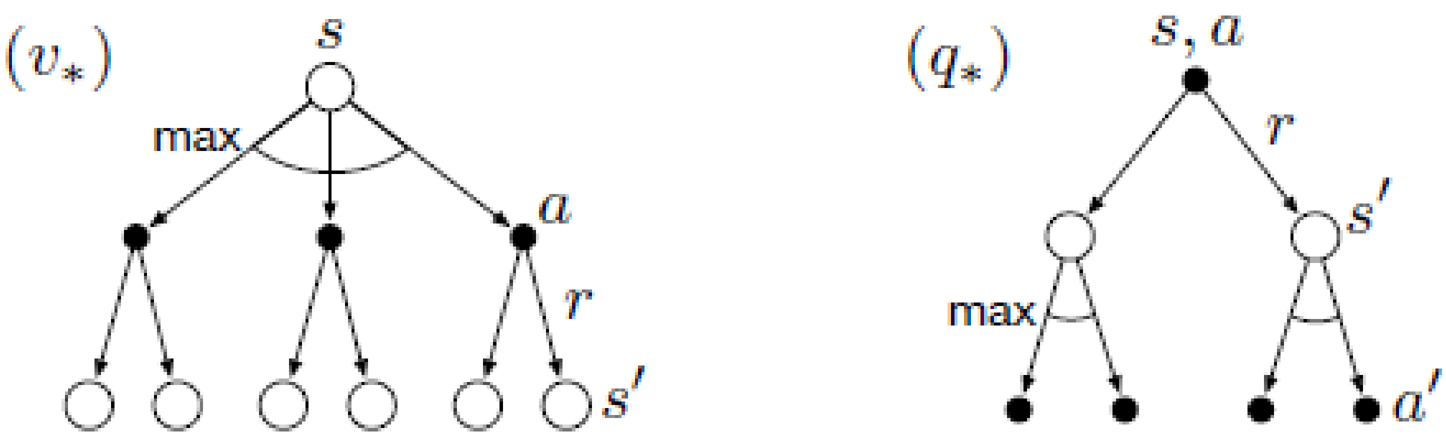
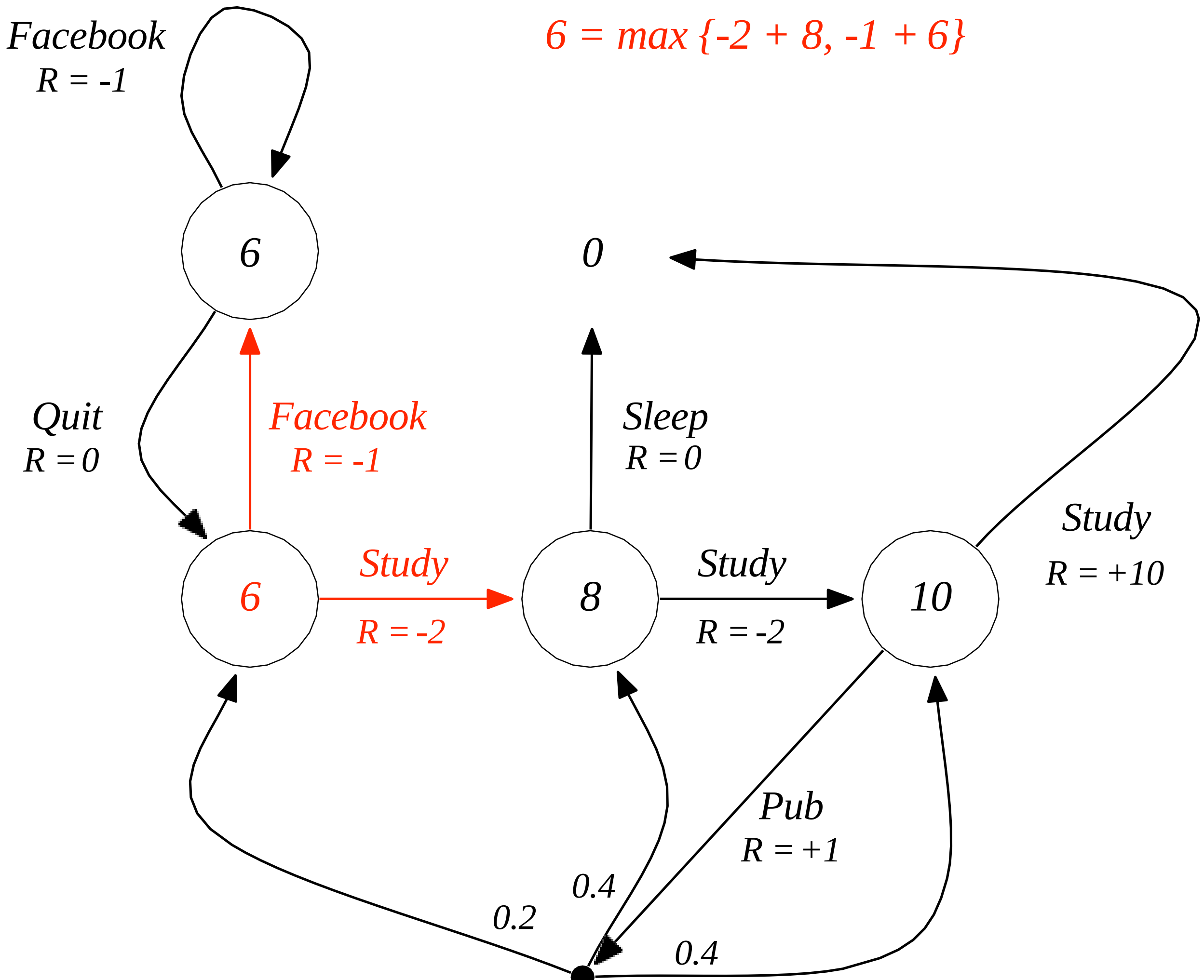


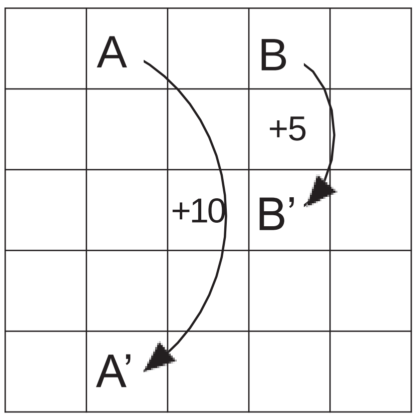
Figure 3.5: Backup diagrams for v_* and q_*

V*(s) $= \max_a \sum_{s',r} p(s', r | s, a)\big[r + \gamma v_*(s')\big].$

q*(s; a) $= \sum_{s',r} p(s', r | s, a)\Big[r + \gamma \max_{a'} q_*(s', a')\Big].$

Example: Bellman Optimality Equation in Student MDP

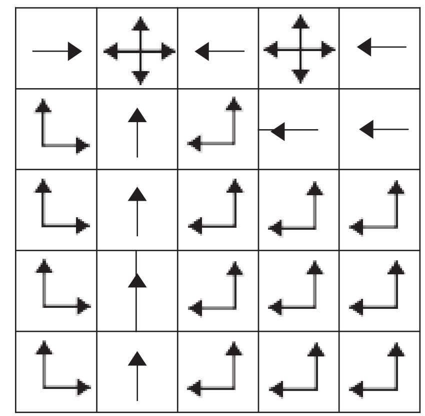




a) gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

b) **V**



c)

What is the optimal value function over all possible policies?

What is the optimal policy?

Figure 3.6

Solving the Bellman Optimality Equation

Bellman Optimality Equation

- Bellman Optimality Equation is non-linear
- No closed form solution (in general) Many
- iterative solution methods
  - Value Iteration
  - Policy Iteration
  - Q-learning
  - Sarsa

Planning by Dynamic Programming

Sutton & Barto,
Chapter 4

Planning by Dynamic Programming

- Dynamic programming assumes full knowledge of the MDP
- It is used for *planning* in an MDP
- For prediction:
  - Input: MDP (S, A, P, R, γ) and policy π
  - or: MRP $(S, P^{\pi}, R^{\pi}, \gamma)$
  - Output: value function v_{π}
- Or for control:
  - Input: MDP (S, A, P, R, γ)
  - Output: optimal value function v_{*}
  - and: optimal policy π_{*}

Policy Evaluation (Prediction)

- Problem: evaluate a given policy π
- Solution: iterative application of Bellman expectation backup
- $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_\pi$
- Using *synchronous* backups,
  - At each iteration $k + 1$
  - For all states $s \in S$
  - Update $v_{k+1}(s)$ from $v_k(s')$
  - where s' is a successor state of s
- We will discuss *asynchronous* backups later
- Convergence to v_π will be proven at the end of the lecture

Iterative Policy Evaluations

$$
\begin{aligned}
v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \mid S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} \mid S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\
&= \sum_a \pi(a|s) \sum_{s',r} p(s', r \mid s, a)\Big[r + \gamma v_\pi(s')\Big],
\end{aligned}
$$

These is a simultaneous linear equations in ISI unknowns and can be solved.

Practically an iterative procedure until a foxed-point can be more effective

$$
\begin{aligned}
v_{k+1}(s) &\doteq \mathbb{E}_\pi[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s] \\
&= \sum_a \pi(a|s) \sum_{s',r} p(s', r \mid s, a)\Big[r + \gamma v_k(s')\Big],
\end{aligned}
$$

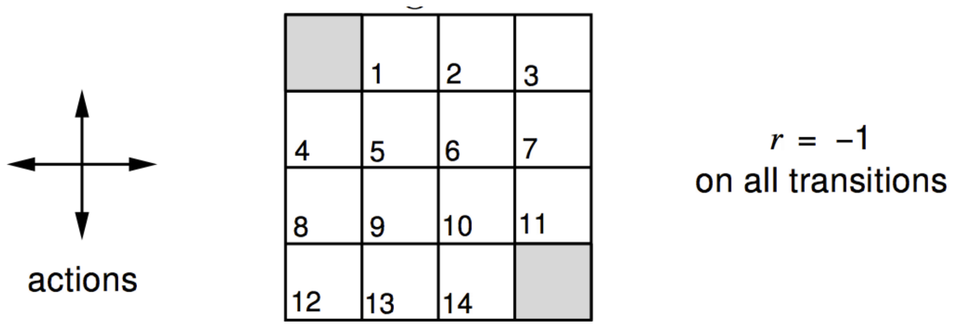
Iterative policy evaluation.

Iterative policy Evaluation

Iterative policy evaluation

Input π, the policy to be evaluated
Initialize an array $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
  $\Delta \leftarrow 0$
  For each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx v_\pi$

Evaluating a Random Policy in the Small Gridworld



- Undiscounted episodic MDP ($\gamma = 1$)
- Nonterminal states 1, ..., 14
- One terminal state (shown twice as shaded squares)
- Actions leading out of the grid leave state unchanged
- Reward is −1 until the terminal state is reached
- Agent follows uniform random policy

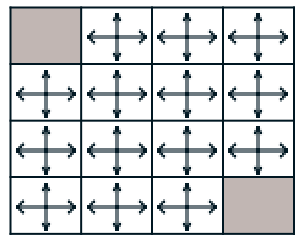
$$\pi(n|\cdot) = \pi(e|\cdot) = \pi(s|\cdot) = \pi(w|\cdot) = 0.25$$

Iterative Policy Evaluation in Small Gridworld

v_k for the Random Policy | Greedy Policy w.r.t. v_k

$k = 0$

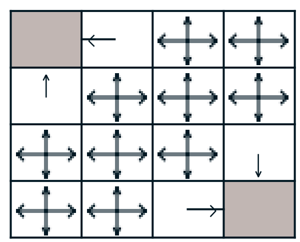
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



← random policy

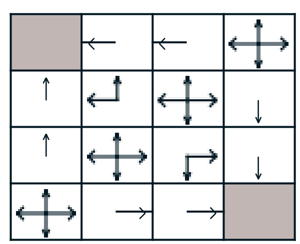
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0



$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0



Iterative Policy Evaluation in Small Gridworld (2)

$k = 3$

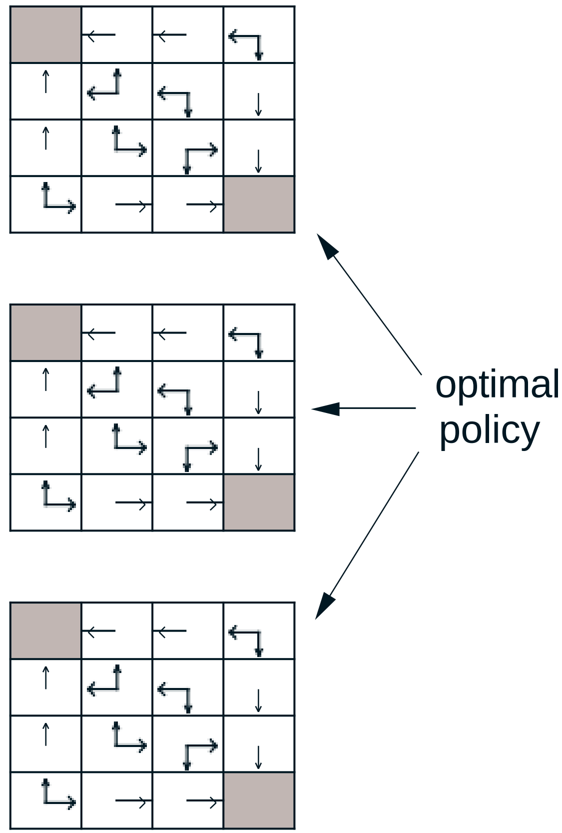
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



Policy Improvement

- Given a policy π
  - Evaluate the policy π

$$v_\pi(s) = \mathrm{E}\left[R_{t+1} + \gamma R_{t+2} + \ldots | S_t = s\right]$$

  - Improve the policy by acting greedily with respect to v_π

$$\pi' = \mathrm{greedy}(v_\pi)$$

- In Small Gridworld improved policy was optimal, $\pi' = \pi^*$
- In general, need more iterations of improvement / evaluation
- But this process of policy iteration always converges to $\pi*$

Policy Iteration

$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*,$$

where $\xrightarrow{\text{E}}$ denotes a policy *evaluation* and $\xrightarrow{\text{I}}$ denotes a policy *improvement*. Each policy is guaranteed to be a strict improvement over the previous one (unless it is already optimal). Because a finite MDP has only a finite number of policies, this process must converge to an optimal policy and optimal value function in a finite number of iterations.

Policy iteration (using iterative policy evaluation)

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

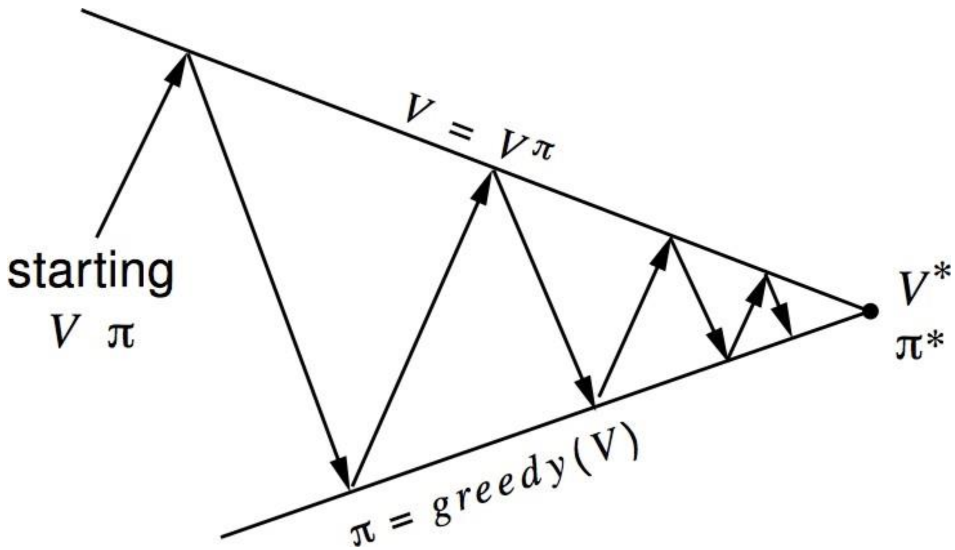
2. Policy Evaluation
   Repeat
   - $\Delta \leftarrow 0$
   - For each $s \in \mathcal{S}$:
     - $v \leftarrow V(s)$
     - $V(s) \leftarrow \sum_{s',r} p(s',r|s,\pi(s))\big[r + \gamma V(s')\big]$
     - $\Delta \leftarrow \max(\Delta, |v - V(s)|)$

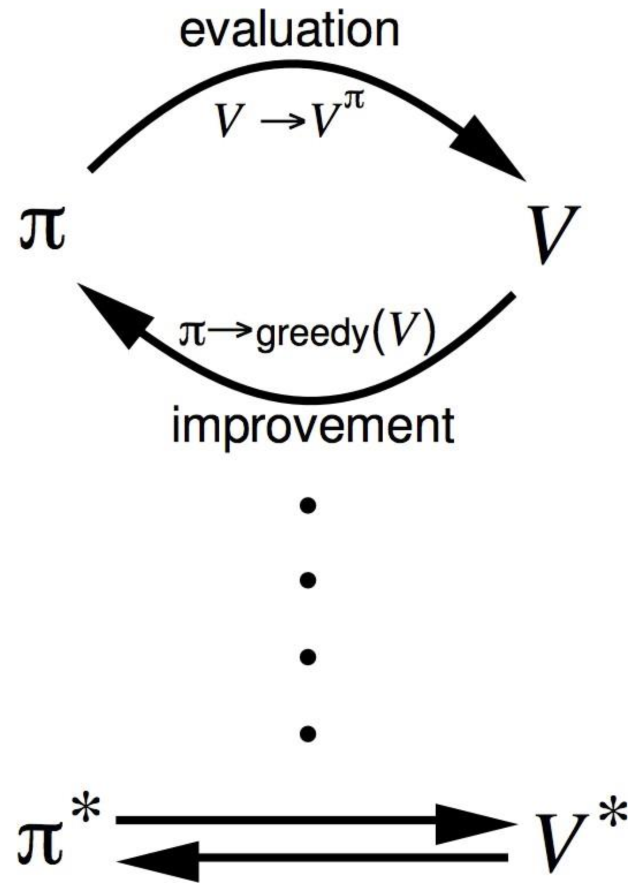
   until $\Delta < \theta$ (a small positive number)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   - *old-action* $\leftarrow \pi(s)$
   - $\pi(s) \leftarrow \text{argmax}_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
   - If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*

   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Policy Iteration





Policy evaluation Estimate v_π

Iterative policy evaluation

Policy improvement Generate $\pi' \geq \pi$

Greedy policy improvement

Policy Improvement

Policy Improvement

- Consider a deterministic policy, $a = \pi(s)$
- We can *improve* the policy by acting greedily

$$\pi'(s) = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\, q_\pi(s, a)$$

- This improves the value from any state s over one step,

$$q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$$

- It therefore improves the value function, $v_{\pi'}(s) \geq v_\pi(s)$

$$\begin{aligned} v_\pi(s) \leq q_\pi(s, \pi'(s)) &= \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right] \\ &\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s\right] \\ &\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s\right] \\ &\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right] = v_{\pi'}(s) \end{aligned}$$

Policy Improvement (2)

- If improvements stop,

$$q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) = q_\pi(s, \pi(s)) = v_\pi(s)$$

- Then the Bellman optimality equation has been satisfied

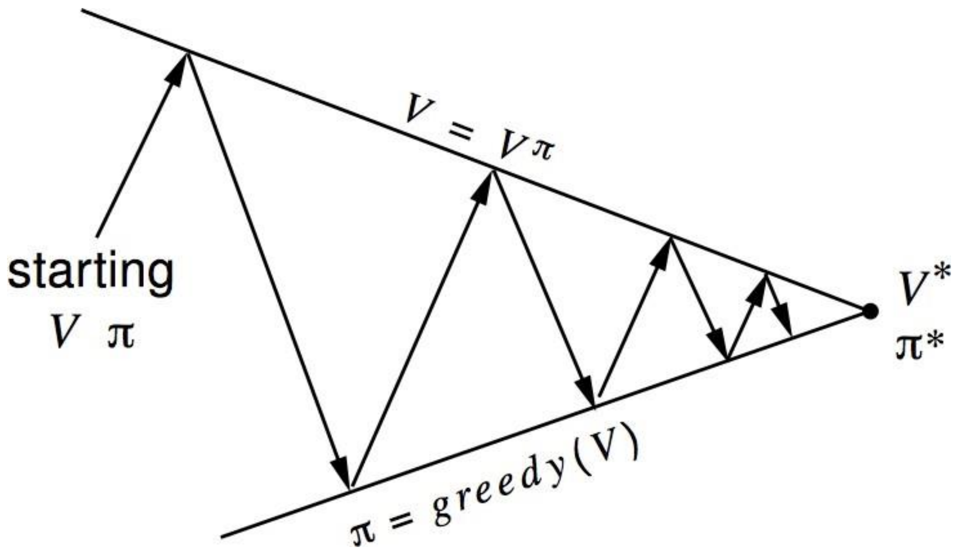
$$v_\pi(s) = \max_{a \in \mathcal{A}} q_\pi(s, a)$$

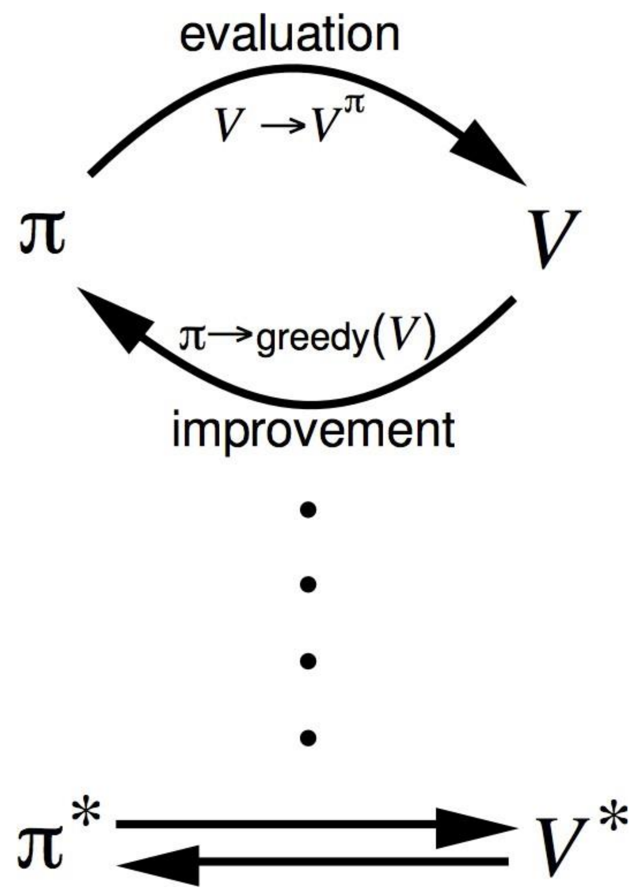
- Therefore $v_\pi(s) = v_*(s)$ for all $s \in \mathcal{S}$
- so π is an optimal policy

Modified Policy Iteration

- Does policy evaluation need to converge to v_π ?
- Or should we introduce a stopping condition
  - e.g. ϵ-convergence of value function
- Or simply stop after k iterations of iterative policy evaluation?
- For example, in the small gridworld $k = 3$ was sufficient to achieve optimal policy
- Why not update policy every iteration? i.e. stop after $k = 1$
  - This is equivalent to *value iteration* (next section)

Generalised Policy Iteration





Policy evaluation Estimate v_π

Any policy evaluation algorithm

Policy improvement Generate $\pi' \geq \pi$

Any policy improvement algorithm

Principle of Optimality

Any optimal policy can be subdivided into two components:

- An optimal first action A_*
- Followed by an optimal policy from successor state S'

Theorem (Principle of Optimality)

A policy $\pi(a|s)$ achieves the optimal value from state s, $v_\pi(s) = v_*(s)$, if and only if

- *For any state s' reachable from s*
- *π achieves the optimal value from state s', $v_\pi(s') = v_*(s')$*

Deterministic Value Iteration

- If we know the solution to subproblems $v_*(s')$
- Then solution $v_*(s)$ can be found by one-step lookahead

$$v_*(s) \leftarrow \max_{a \in \mathcal{A}} \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

- The idea of value iteration is to apply these updates iteratively
- Intuition: start with final rewards and work backwards
- Still works with loopy, stochastic MDPs

Value Iteration

$$
\begin{aligned}
v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_k(s') \Big], \qquad (4.10)
\end{aligned}
$$

for all $s \in \mathcal{S}$. For arbitrary v_0, the sequence $\{v_k\}$ can be shown to converge to v_* under the same conditions that guarantee the existence of v_*.

Value Iteration

Value iteration

Initialize array V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat
  $\Delta \leftarrow 0$
  For each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \max_a \sum_{s',r} p(s', r | s, a)\big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

Output a deterministic policy, $\pi \approx \pi_*$, such that
  $\pi(s) = \arg\max_a \sum_{s',r} p(s', r | s, a)\big[r + \gamma V(s')\big]$

Example: Shortest Path

g			

Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

V_3

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_4

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-4
-3	-4	-4	-4

V_5

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-5

V_6

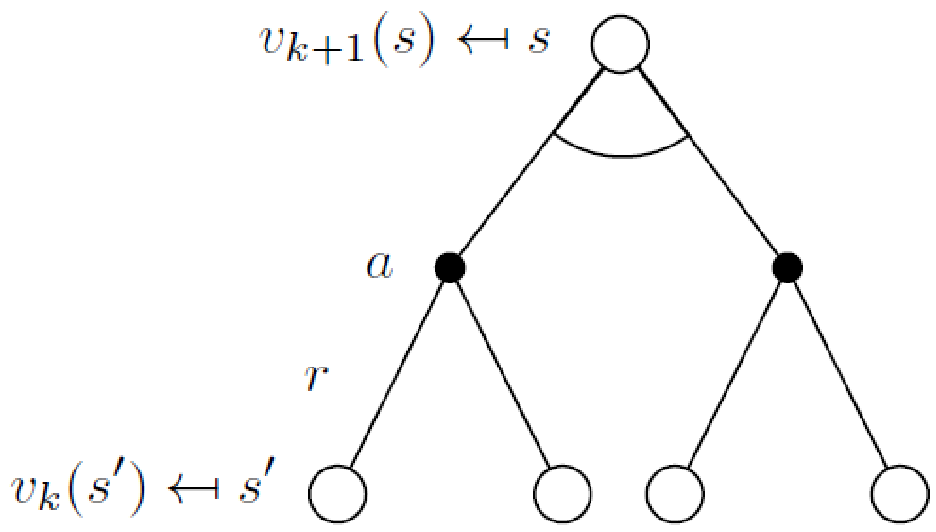
0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-6

V_7

Value Iteration

- Problem: find optimal policy π
- Solution: iterative application of Bellman optimality backup
- $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_*$
- Using synchronous backups
  - At each iteration $k + 1$
  - For all states $s \in \mathcal{S}$
  - Update $v_{k+1}(s)$ from $v_k(s')$
- Convergence to v_* will be proven later
- Unlike policy iteration, there is no explicit policy
- Intermediate value functions may not correspond to any policy

Value Iteration (2)



$$v_{k+1}(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

$$\mathbf{v}_{k+1} = \max_{a \in \mathcal{A}} \boldsymbol{\mathcal{R}}^{\mathbf{a}} + \gamma \boldsymbol{\mathcal{P}}^{\mathbf{a}} \mathbf{v}_k$$

Asynchronous Dynamic Programming

Asynchronous Dynamic Programming

- DP methods described so far used *synchronous* backups
- i.e. all states are backed up in parallel
- *Asynchronous DP* backs up states individually, in any order
- For each selected state, apply the appropriate backup
- Can significantly reduce computation
- Guaranteed to converge if all states continue to be selected

Asynchronous Dynamic Programming

Three simple ideas for asynchronous dynamic programming:

- *In-place* dynamic programming
- *Prioritised sweeping*
- *Real-time* dynamic programming

In-Place Dynamic Programming

- Synchronous value iteration stores two copies of value function

  for all s in $\mathcal{S}$

$$v_{new}(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_{old}(s') \right)$$

$$v_{old} \leftarrow v_{new}$$

- In-place value iteration only stores one copy of value function

  for all s in $\mathcal{S}$

$$v(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right)$$

Prioritised Sweeping

- Use magnitude of Bellman error to guide state selection, e.g.

$$\left| \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right) - v(s) \right|$$

- Backup the state with the largest remaining Bellman error
- Update Bellman error of affected states after each backup
- Requires knowledge of reverse dynamics (predecessor states)
- Can be implemented efficiently by maintaining a priority queue

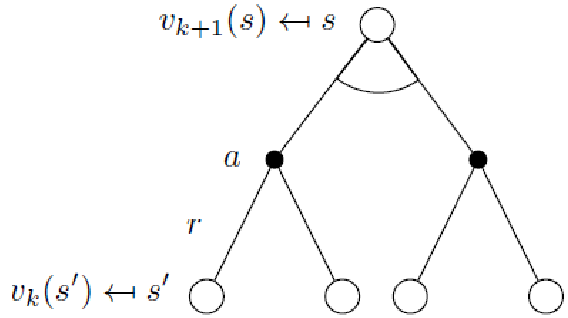
Real-Time Dynamic Programming

- Idea: only states that are relevant to agent
- Use agent's experience to guide the selection of states
- After each time-step S_t, A_t, R_{t+1}
- Backup the state S_t

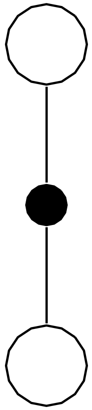
$$v(S_t) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}^a_{S_t} + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}^a_{S_t s'} v(s') \right)$$

Full-Width Backups

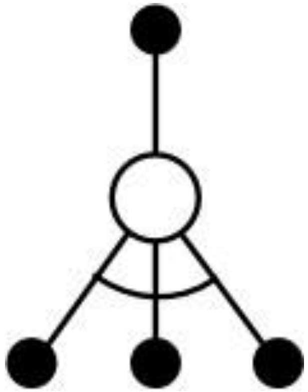
- DP uses *full-width* backups
- For each backup (sync or async)
  - Every successor state and action is considered
  - Using knowledge of the MDP transitions and reward function
- DP is effective for medium-sized problems (millions of states)
- For large problems DP suffers Bellman's *curse of dimensionality*
  - Number of states $n = |S|$ grows exponentially with number of state variables
- Even one backup can be too expensive



Sample Backups



- In subsequent lectures we will consider *sample backups*
- Using sample rewards and sample transitions $\langle S, A, R, S' \rangle$
- Instead of reward function $\mathcal{R}$ and transition dynamics $\mathcal{P}$
- Advantages:
  - Model-free: no advance knowledge of MDP required
  - Breaks the curse of dimensionality through sampling
  - Cost of backup is constant, independent of $n = |\mathcal{S}|$



Approximate Dynamic Programming

- Approximate the value function
- Using a *function approximator* $\hat{v}(s, \mathbf{w})$
- Apply dynamic programming to $\hat{v}(\cdot, \mathbf{w})$
- e.g. Fitted Value Iteration repeats at each iteration k,
  - Sample states $\tilde{\mathcal{S}} \subseteq \mathcal{S}$
  - For each state $s \in \tilde{\mathcal{S}}$, estimate target value using Bellman optimality equation,

$$\tilde{v}_k(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \hat{v}(s', \mathbf{w_k}) \right)$$

  - Train next value function $\hat{v}(\cdot, \mathbf{w_{k+1}})$ using targets $\{\langle s, \tilde{v}_k(s) \rangle\}$

Csaba slides,

The fundamental theorem and the Bellman (optimality) operator

Theorem

Assume that $|\mathcal{A}| < +\infty$. *Then the optimal value function satisfies*

$$V^*(x) = \max_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \sum_{y \in \mathcal{X}} \mathcal{P}(x,a,y) V^*(y) \right\}, \qquad x \in \mathcal{X}.$$

and if policy π *is such that in each state* x *it selects an action that maximizes the r.h.s. then* π *is an optimal policy.*

A shorter way to write this is

$$V^* = T^* V^*,$$

$$(T^* V)(x) = \max_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \sum_{y \in \mathcal{X}} \mathcal{P}(x,a,y) V(y) \right\}, \quad x \in \mathcal{X}.$$

Policy evaluation operator

Definition (Policy evaluation operator)

Let π be a stochastic stationary policy. Define

$$
\begin{aligned}
(T^\pi V)(x) &= \sum_{a \in \mathcal{A}} \pi(a|x) \left\{ r(x,a) + \gamma \sum_{y \in \mathcal{X}} \mathcal{P}(x,a,y) V(y) \right\} \\
&= \sum_{a \in \mathcal{A}} \pi(a|x) T_a V(x), \quad x \in \mathcal{X}.
\end{aligned}
$$

Corollary

T^π is a contraction, and V^π is the unique fixed point of T^π.

Greedy policy

Definition (Greedy policy)

Policy π is greedy w.r.t. V if

$$T^{\pi}V = T^{*}V,$$

or

$$\sum_{a \in \mathcal{A}} \pi(a|x) \left\{ r(x,a) + \gamma \sum_{y \in \mathcal{X}} \mathcal{P}(x,a,y) V(y) \right\} = \max_{a \in \mathcal{A}} \left\{ r(x,a) + \gamma \sum_{y \in \mathcal{X}} \mathcal{P}(x,a,y) V(y) \right\}$$

holds for all states x.

A restatement of the main theorem

Theorem

Assume that $|\mathcal{A}| < +\infty$. Then the optimal value function satisfies the fixed-point equation $V^* = T^* V^*$ and any greedy policy w.r.t. V^* is optimal.

Action-value functions

Corollary

Let Q^* *be the optimal action-value function. Then,*

$$Q^* = T^* Q^*$$

and if π *is a policy such that*

$$\sum_{a \in \mathcal{A}} \pi(a|x) Q^*(x,a) = \max_{a \in \mathcal{A}} Q^*(x,a)$$

then π *is optimal. Here,*

$$T^* Q\,(x,a) = r(x,a) + \gamma \sum_{y \in \mathcal{X}} \mathcal{P}(x,a,y) \max_{a' \in \mathcal{A}} Q(y,a'), \quad x \in \mathcal{X}, a \in \mathcal{A}.$$

Finding the action-value functions of policies

Theorem

Let π be a stationary policy, T^π be defined by

$$T^\pi Q\,(x,a) = r(x,a) + \gamma \sum_{y\in\mathcal{X}} \mathcal{P}(x,a,y) \sum_{a'\in\mathcal{A}} \pi(a'|y)\, Q(y,a'), \quad x\in\mathcal{X}, a\in\mathcal{A}.$$

Then Q^π is the unique solution of

$$T^\pi Q^\pi = Q^\pi.$$

Value iteration

Note

- *If V_t is the value-function computed in the t^{th} iteration of value iteration then*

$$V_{t+1} = T^* V_t.$$

- *The key is that T^* is a contraction in the supremum norm and Banach's fixed-point theorem gives the key to the proof the theorem mentioned before.*

Note

One can also use $Q_{t+1} = T^* Q_t$, or value functions with post-decision states. What is the advantage?

Policy iteration

```
function PolicyIteration(π)
  repeat
      π′ ← π
      V ← GetValueFunction(π′)
      π ← GetGreedyPolicy(V)
  until π ≠ π′
  return π
```

What if we stop early?

Theorem (e.g., Corollary 2 of Singh and Yee 1994)

Fix an action-value function Q and let π be a greedy policy w.r.t. Q. Then the value of policy π can be lower bounded as follows:

$$V^{\pi}(x) \geq V^{*}(x) - \frac{2}{1-\gamma} \|Q - Q^{*}\|_{\infty}, \quad x \in \mathcal{X}.$$

Value Function ∞-Norm

- We will measure distance between state-value functions u and v by the ∞-norm
- i.e. the largest difference between state values,

$$||u - v||_\infty = \max_{s \in S} |u(s) - v(s)|$$

Contraction Mapping Theorem

Theorem (Contraction Mapping Theorem)

For any metric space $\mathcal{V}$ *that is complete (i.e. closed) under an operator* $T(v)$*, where* T *is a* γ*-contraction,*

- T *converges to a unique fixed point*
- *At a linear convergence rate of* γ

Bellman Operator is a Contraction

|| V-V'|| = Infinity norm
(find max diff
Over all states)

$$\|BV - BV'\| = \left\| \begin{array}{l} \max_a \left[R(s,a) + \gamma \sum_{s_j \in S} p(s_j \mid s_i, a) V(s_j) \right] \\ - \max_{a'} \left[R(s,a') - \gamma \sum_{s_j \in S} p(s_j \mid s_i, a') V'(s_j) \right] \end{array} \right\|$$

$$\leq \left\| \max_a \left[R(s,a) + \gamma \sum_{s_j \in S} p(s_j \mid s_i, a) V(s_j) - R(s,a) + \gamma \sum_{s_j \in S} p(s_j \mid s_i, a) V'(s_j) \right] \right\|$$

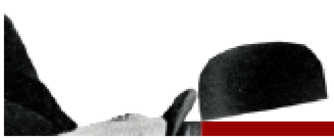
$$\leq \gamma \left\| \max_a \left[\sum_{s_j \in S} p(s_j \mid s_i, a) V(s_j) - \sum_{s_j \in S} p(s_j \mid s_i, a) V'(s_j) \right] \right\|$$

$$= \gamma \max_a \left\| \left[\sum_{s_j \in S} p(s_j \mid s_i, a) (V(s_j) - V'(s_j)) \right] \right\|$$

$$\leq \gamma \max_{a, s_i} \sum_{s_j \in S} p(s_j \mid s_i, a) \left| V(s_j) - V'(s_j) \right|$$

$$\leq \gamma \max_{a, s_i} \sum_{s_j \in S} p(s_j \mid s_i, a) \|V - V'\|$$

$$= \gamma \|V - V'\|$$



Convergence of Iter. Policy Evaluation and Policy Iteration

- The Bellman expectation operator T^{π} has a unique fixed point
- v_{π} is a fixed point of T^{π} (by Bellman expectation equation)
- By contraction mapping theorem
- Iterative policy evaluation converges on v_{π}
- Policy iteration converges on v_{*}

Bellman Optimality Backup is a Contraction

- Define the *Bellman optimality backup operator* T^*,

$$T^*(v) = \max_{a \in \mathcal{A}} \mathcal{R}^a + \gamma \mathcal{P}^a v$$

- This operator is a γ-contraction, i.e. it makes value functions closer by at least γ (similar to previous proof)

$$||T^*(u) - T^*(v)||_\infty \leq \gamma ||u - v||_\infty$$

Convergence of Value Iteration

- The Bellman optimality operator T^* has a unique fixed point
- v_* is a fixed point of T^* (by Bellman optimality equation) By
- contraction mapping theorem
- Value iteration converges on v_*

Will Value Iteration Converge?

- Yes, if discount factor is < 1 or end up in a terminal state with probability 1

- Bellman equation is a contraction
- If apply it to two different value functions, distance between value functions shrinks after apply Bellman equation to each

Properties of Contraction

- Only has 1 fixed point
  - If had two, then would not get closer when apply contraction function, violating definition of contraction
- When apply contraction function to any argument, value must get closer to fixed point
  - Fixed point doesn't move
  - Repeated function applications yield fixed point

Value Iteration Converges

- If discount factor < 1
- Bellman is a contraction
- Value iteration converges to unique solution which is optimal value function