Fast Gradient-Descent Methods for Temporal-Difference Learning with Linear Function Approximation

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a breakthrough in RL

- function approximation in TD learning is now straightforward
  - as straightforward as it is in supervised learning
- TD learning can now be done as gradient-descent in a novel Bellman error

limitations (for this paper)

- linear function approximation
- one-step TD methods ($\lambda = 0$)
- prediction (policy evaluation), not control

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all of these are being removed in current work

keys to the breakthrough

- a new Bellman error objective function
- an algorithmic trick—a second set of weights
  - to estimate one of the sub-expectations
  - and avoid the need for double sampling
  - introduced in prior work (Sutton, Szepesvari & Maei, 2008)

outline

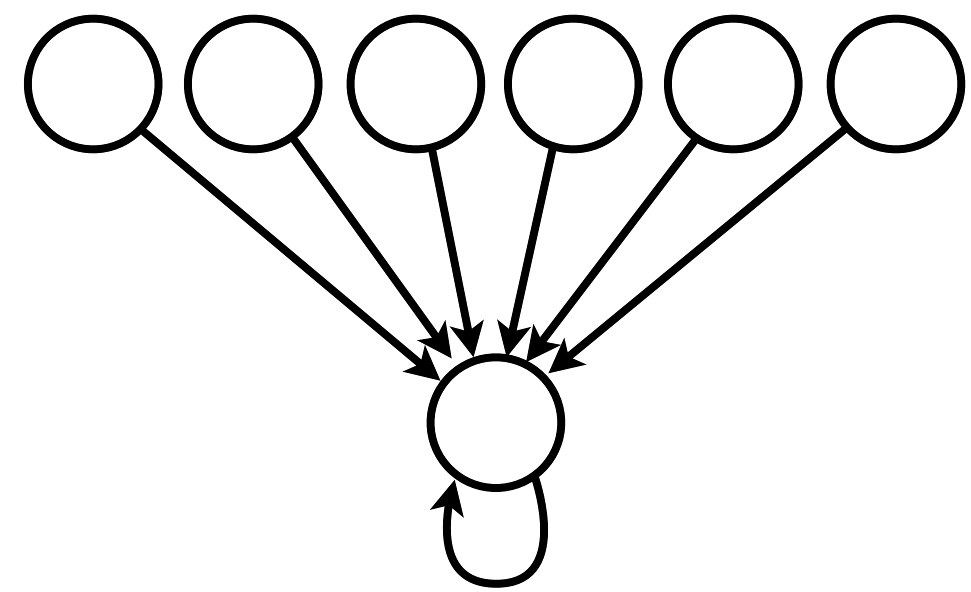
- ways in which TD with FA has not been straightforward
- the new Bellman error objective function
- derivation of new algorithms (the trick)
- results (theory and experiments)

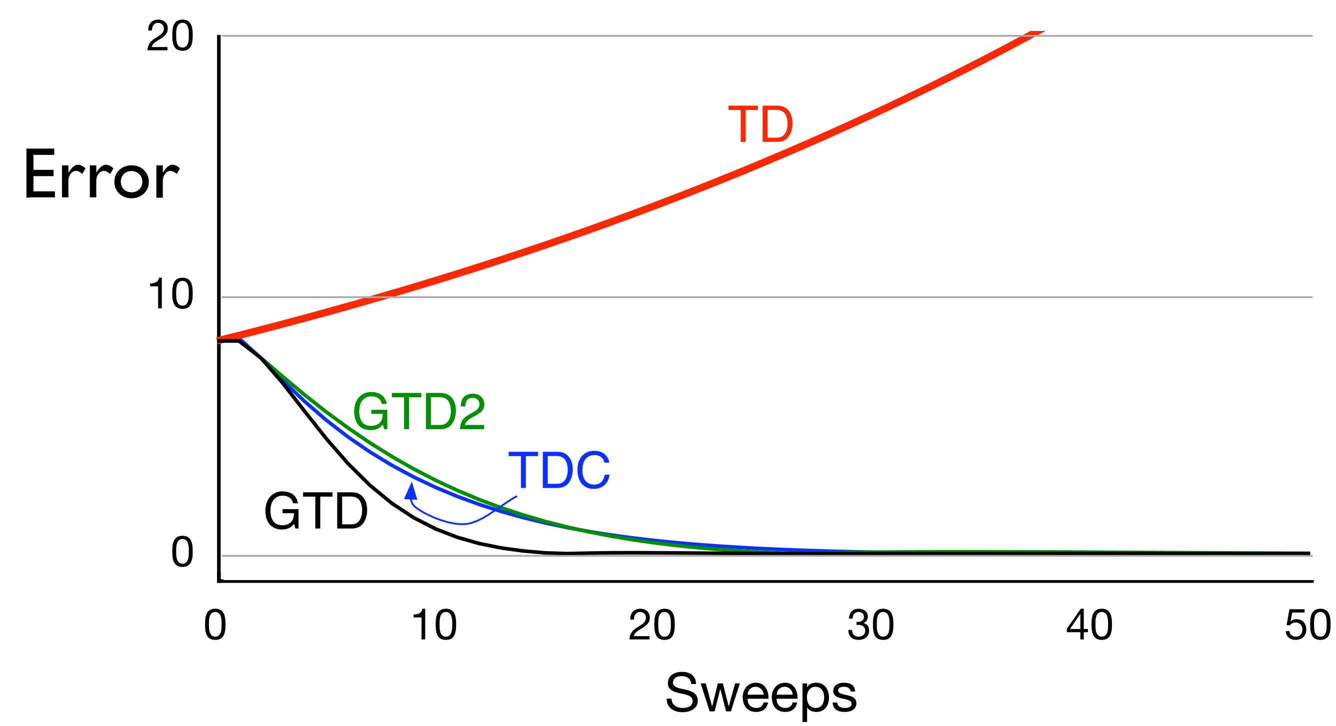
TD+FA was not straightforward

- with linear FA, off-policy methods such as Q-learning diverge on some problems (Baird, 1995)
- with nonlinear FA, even on-policy methods can diverge (Tsitsiklis & Van Roy, 1997)
- convergence guaranteed only for one very important special case—linear FA, learning about the policy being followed
- second-order or importance-sampling methods are complex, slow or messy
- no true gradient-descent methods

Baird's counterexample

- a simple Markov chain
- linear FA, all rewards zero
- deterministic, expectation-based full backups (as in DP)
- each state updated once per sweep (as in DP)
- weights can diverge to ±∞





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e.g. linear value-function approximation in Computer Go

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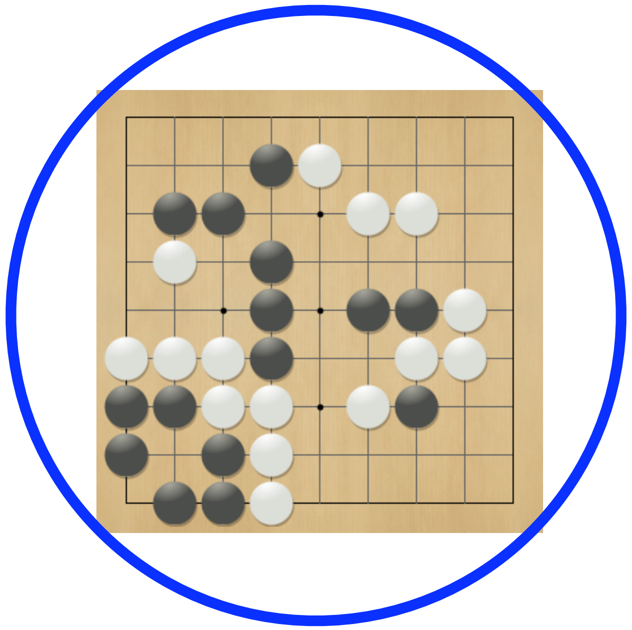
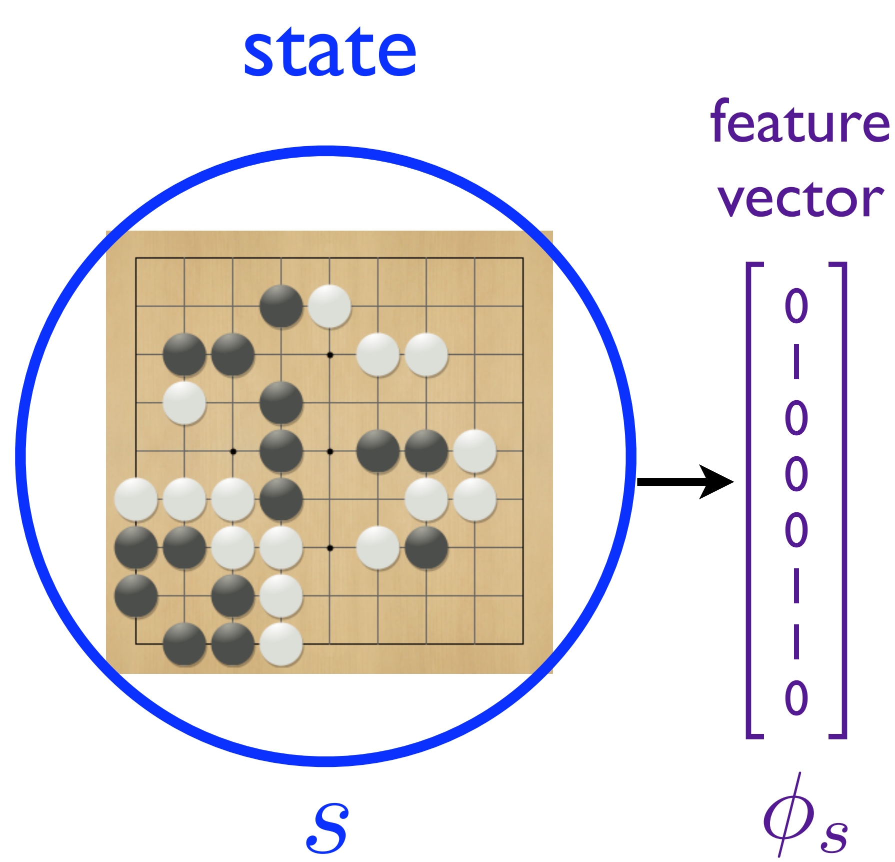


s

10^{35} states

e.g. linear value-function approximation in Computer Go



10^{35} states $\quad$ 10^{5} binary features

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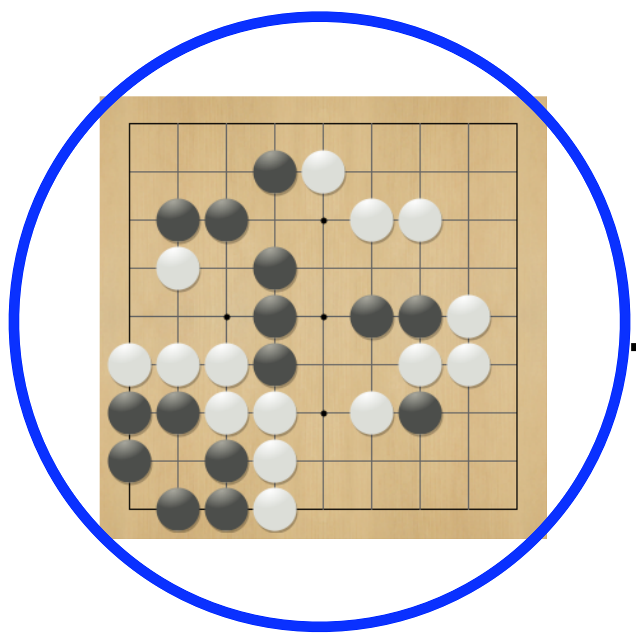

s

feature vector | parameter vector

$$\phi_s = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \times \theta = \begin{bmatrix} 0.1 \\ -2 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 5 \\ -.4 \end{bmatrix}$$

10^{35} states

10^{5} binary features and parameters

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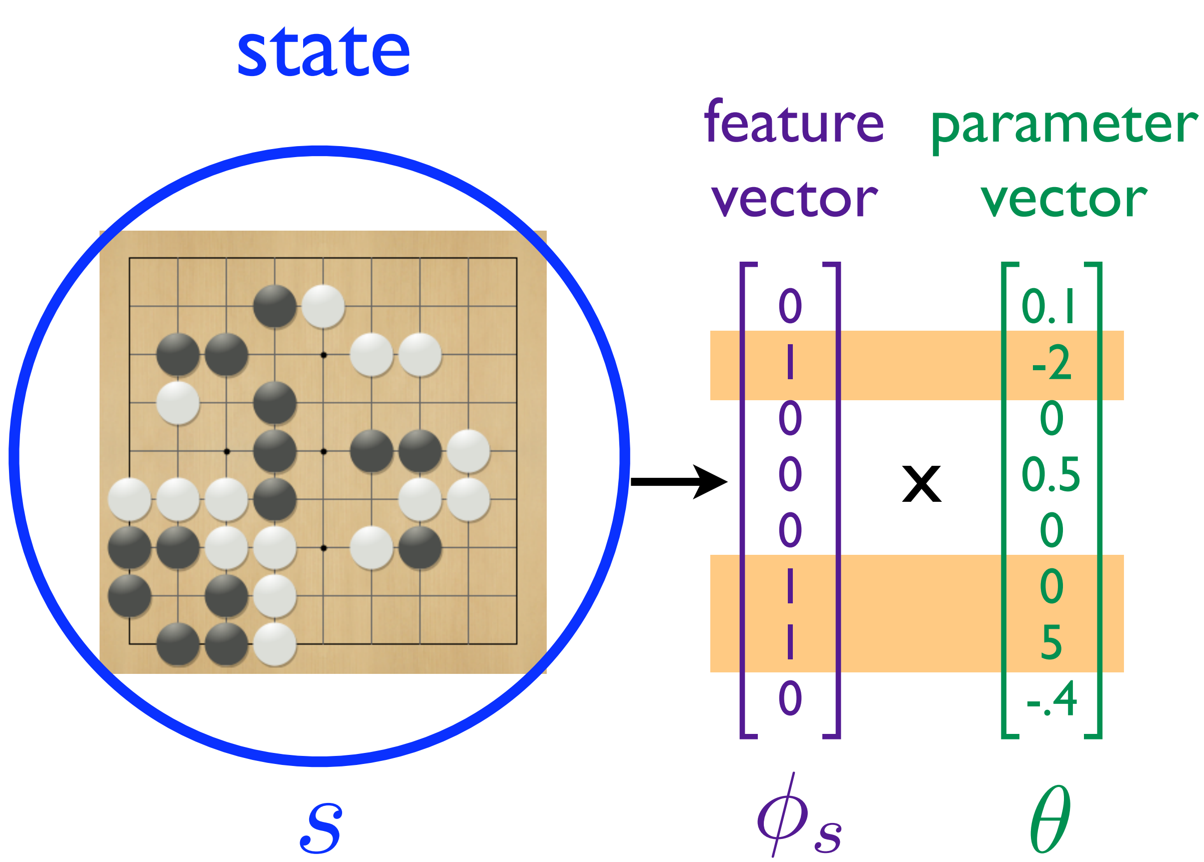

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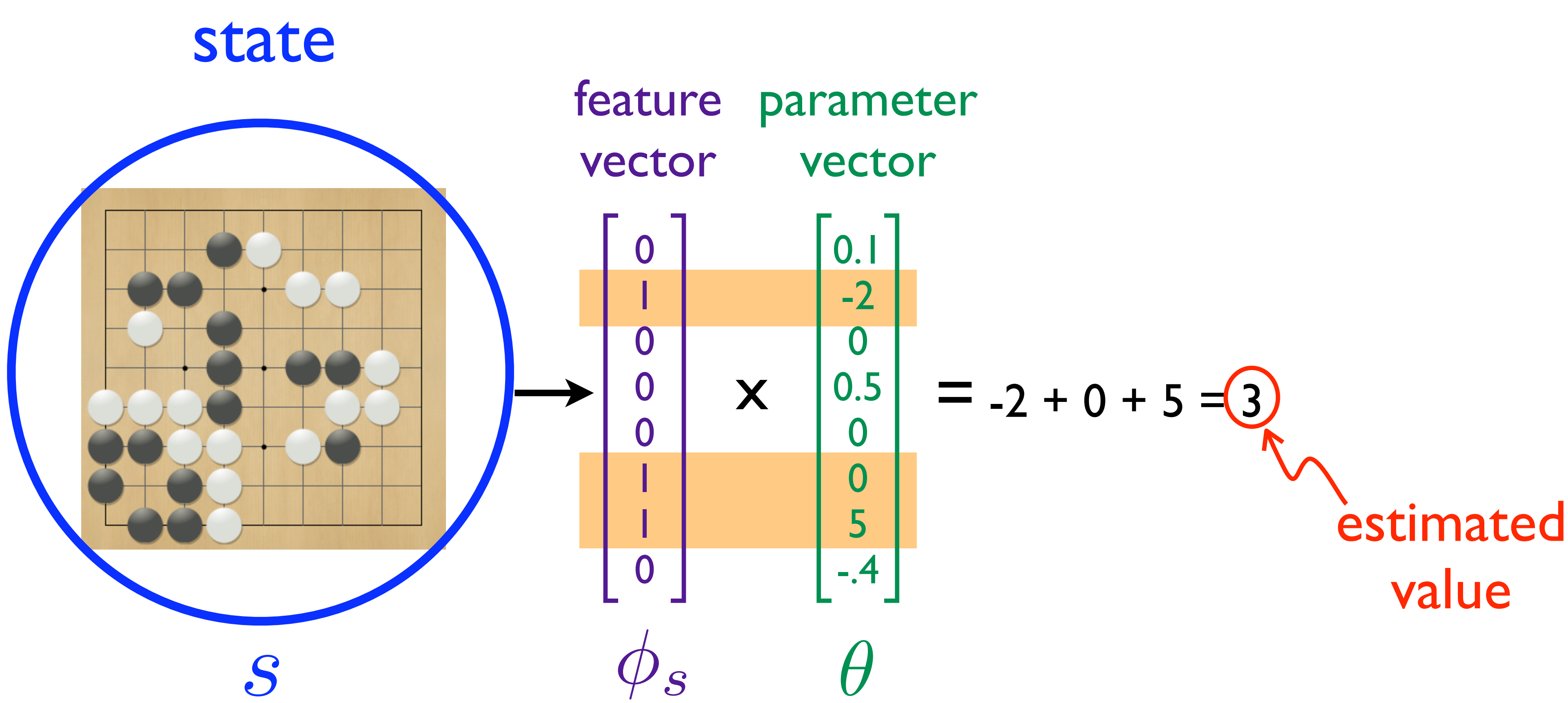

10^{35} states

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Notation

- state transitions: $s \xrightarrow{r} s'$
- feature vectors: $\phi \quad \phi' \in \Re^n \qquad n \ll \#\text{states}$
- approximate values: $V_\theta(s) = \theta^\top \phi \qquad \theta \in \Re^n$ parameter vector

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- TD error: $\delta = r + \gamma\theta^\top\phi' - \theta^\top\phi$ $\qquad \gamma \in [0, 1)$
- TD(0) algorithm: $\Delta\theta = \alpha\delta\phi$ $\qquad \alpha > 0$

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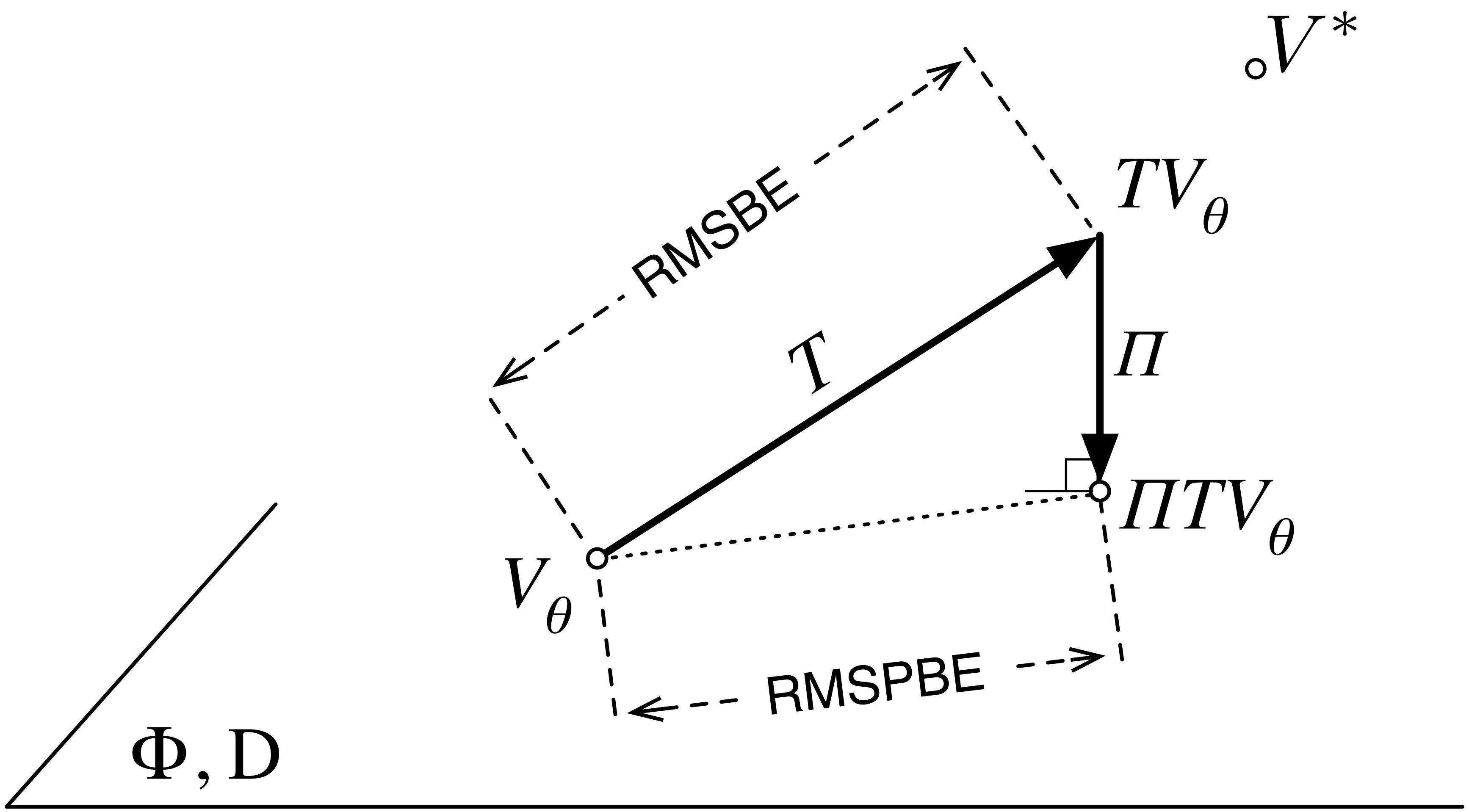
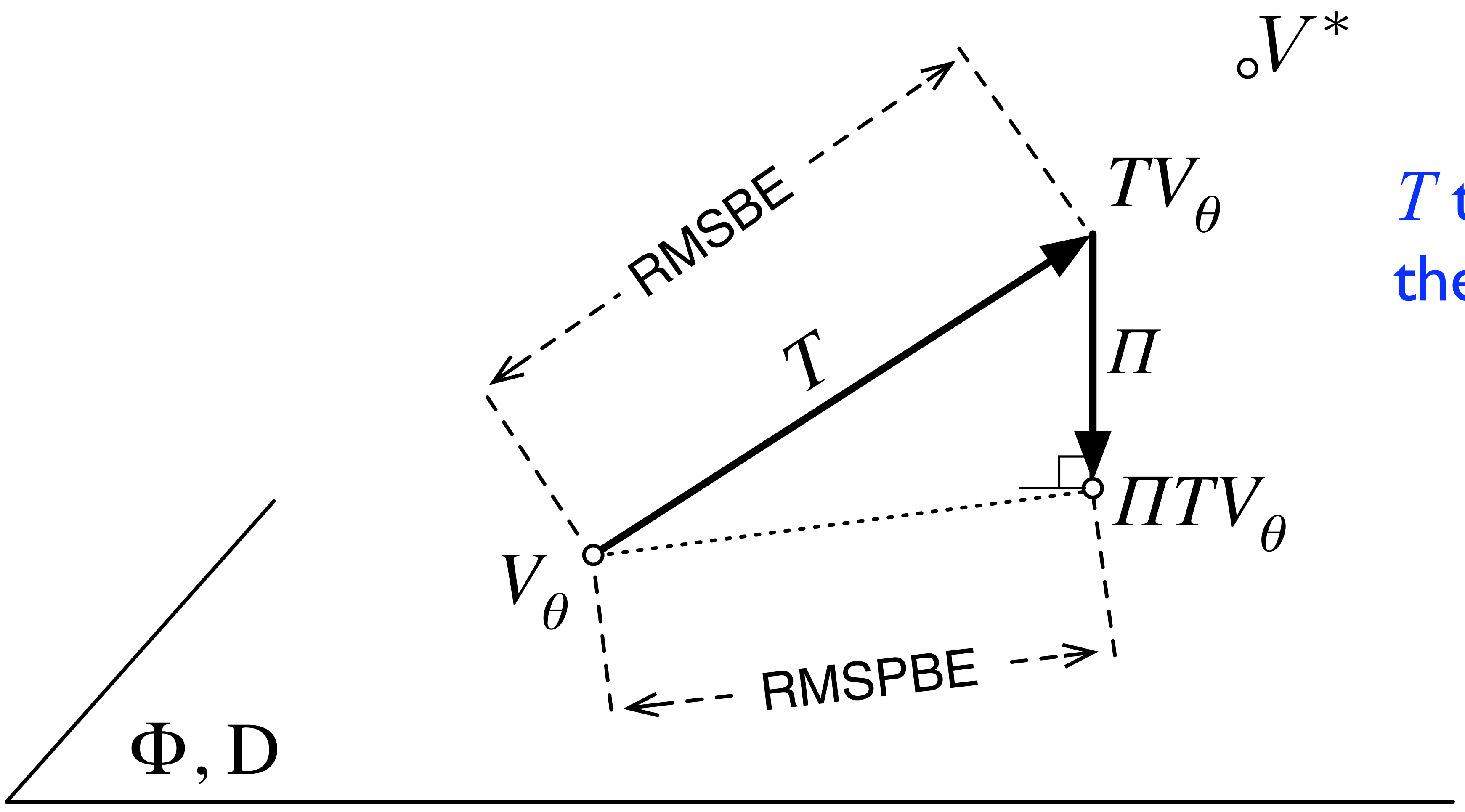
- true values: $V^*(s) = \mathbb{E}[r|s] + \gamma \sum_{s'} P_{ss'} V^*(s')$
- Bellman operator: $TV = R + \gamma PV$ $\qquad V^* = TV^*$
  over per-state vectors

Value function geometry



The space spanned by the feature vectors,
weighted by the state visitation distribution

Value function geometry



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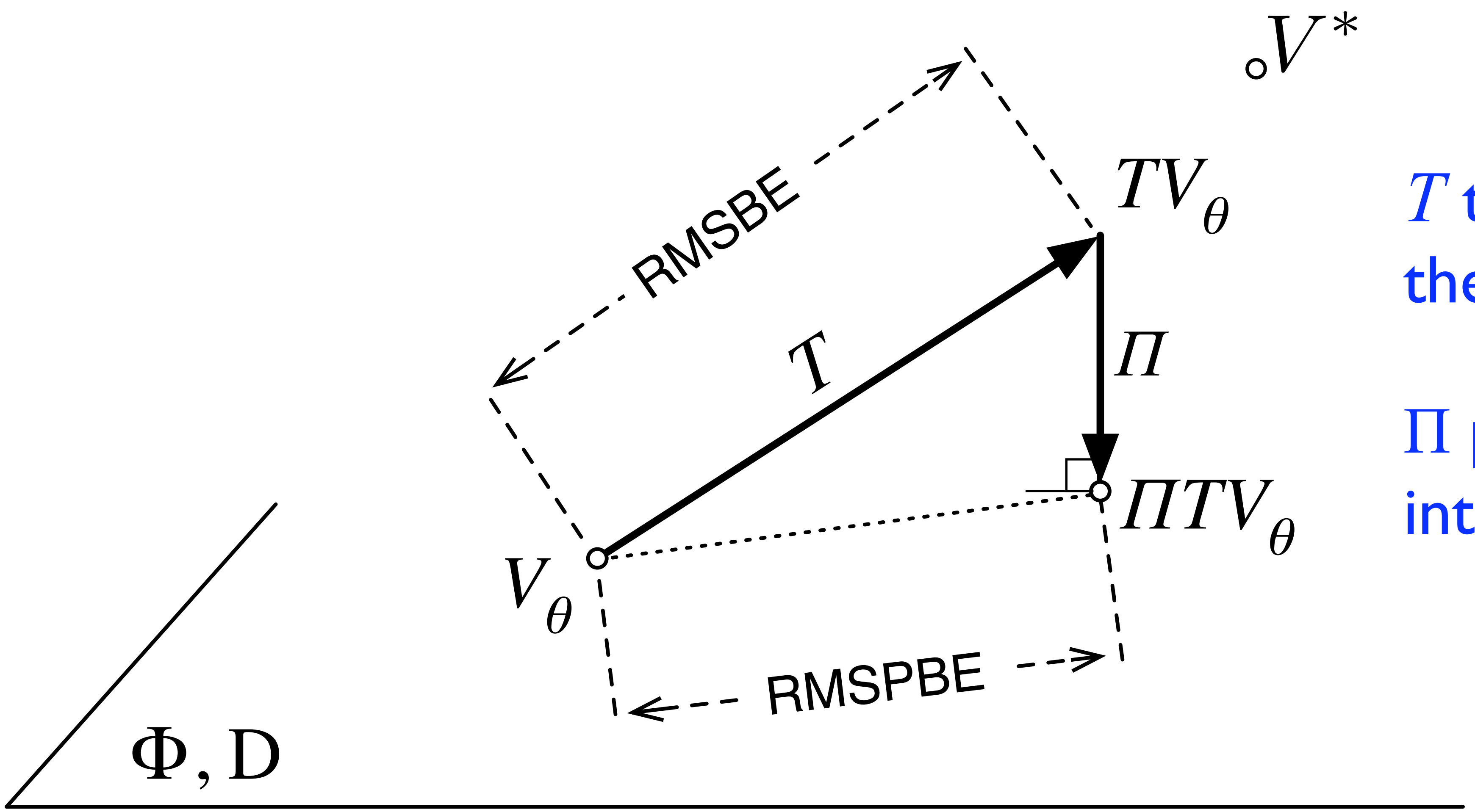

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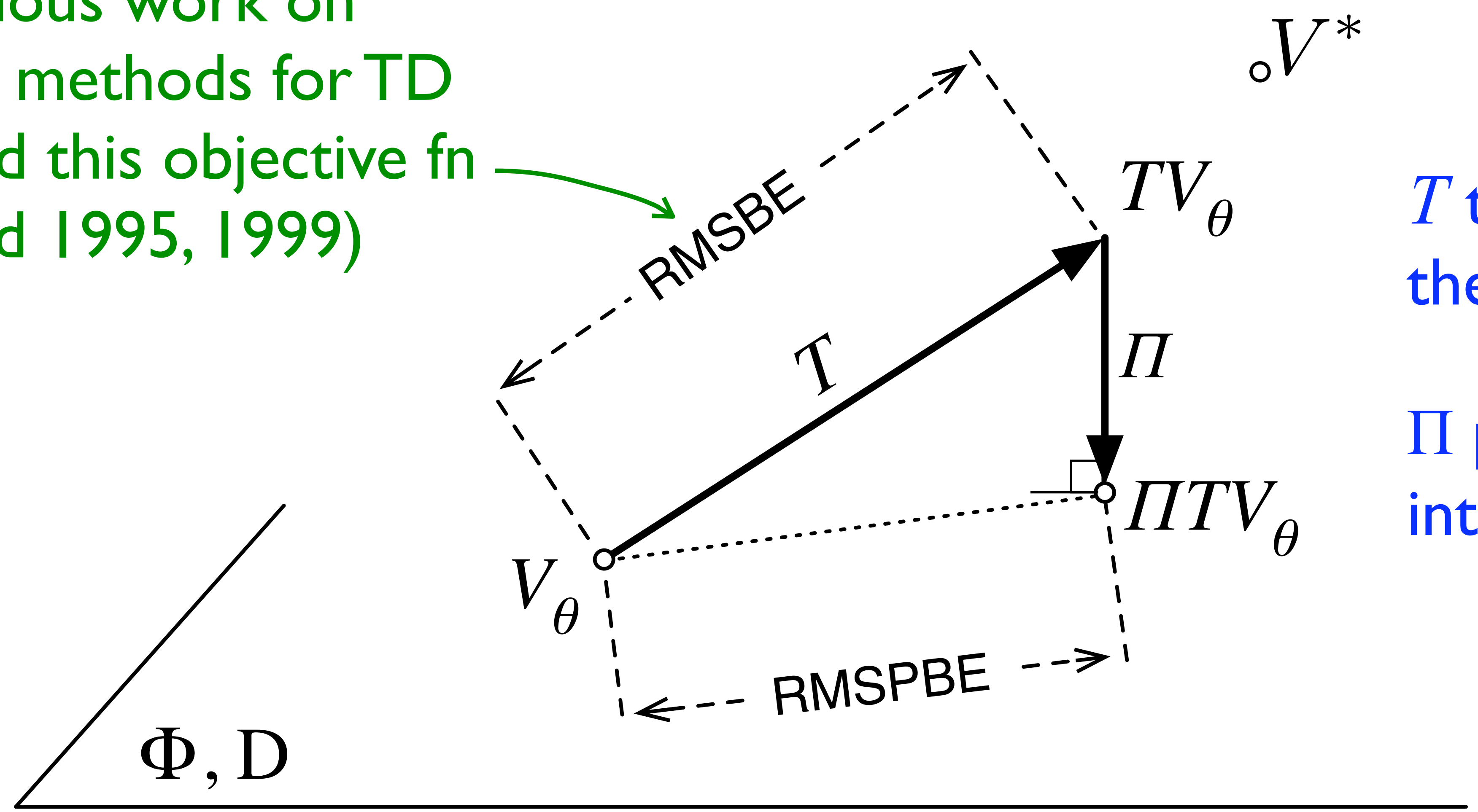
Previous work on gradient methods for TD minimized this objective fn (Baird 1995, 1999)



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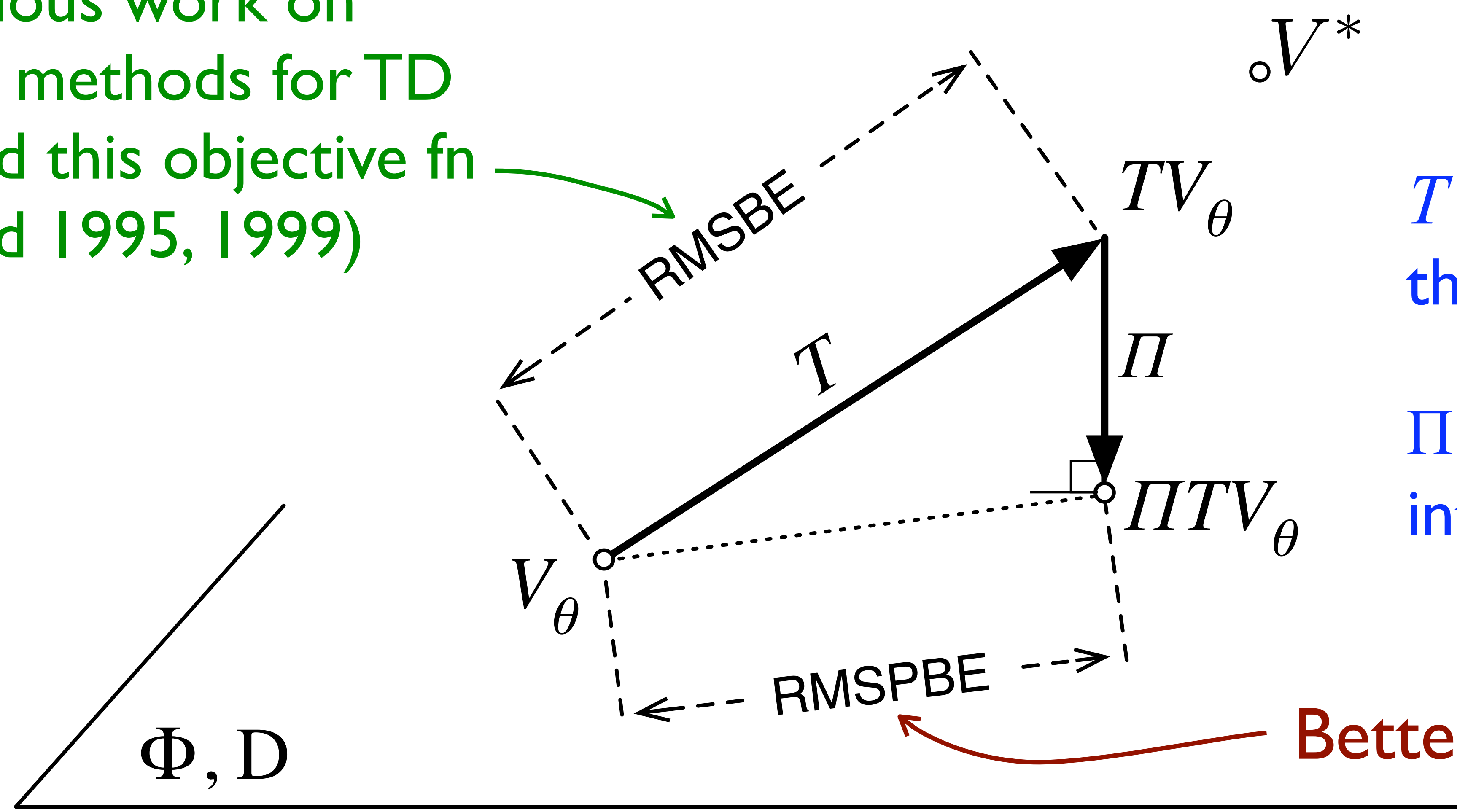

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Better objective fn?

Mean Square ***Projected*** Bellman Error (MSPBE)

TD objective functions
(to be minimized)

- Error from the true values $\| V_\theta - V^* \|_D^2$
- Error in the Bellman equation (Bellman residual) $\| V_\theta - TV_\theta \|_D^2$
- Error in the Bellman equation *after projection* (MSPBE) $\| V_\theta - \Pi TV_\theta \|_D^2$

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- Error from the true values $\| V_\theta - V^* \|_D^2$ Not TD
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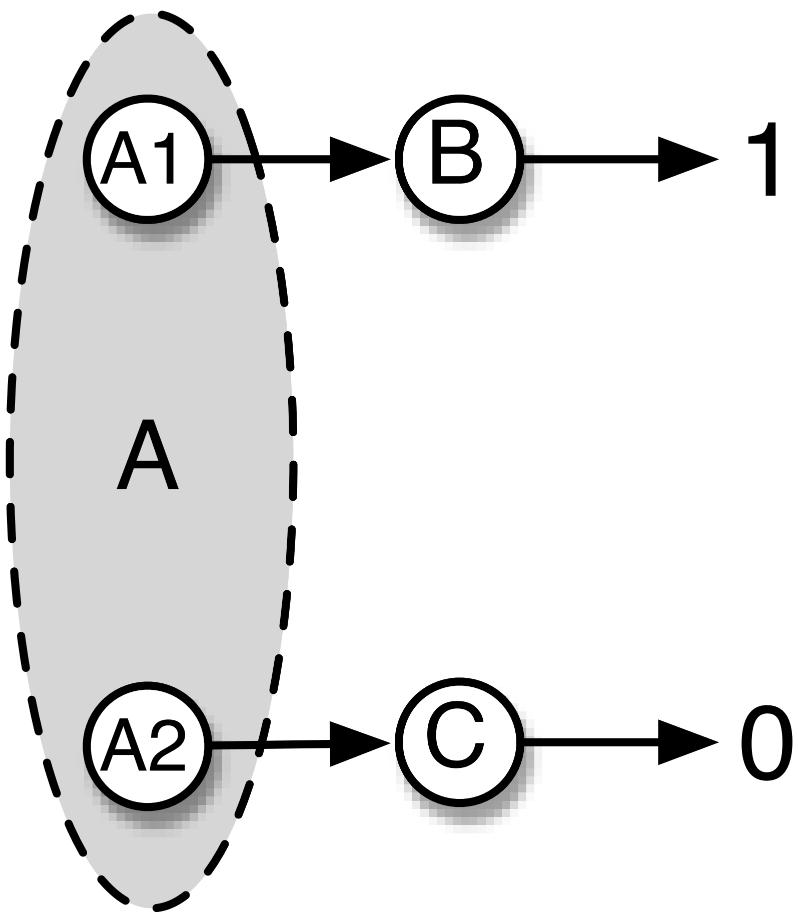
backwards-bootstrapping example



- The two 'A' states look the same; they share a single feature and must be given the same approximate value

$$V(A1) = V(A2) = \frac{1}{2}$$

- All transitions are deterministic; Bellman error = TD error
- Clearly, the right solution is

$$V(B) = 1, \; V(C) = 0$$

- But the solution the minimizes the Bellman error is

$$V(B) = \frac{3}{4}, \; V(C) = \frac{1}{4}$$

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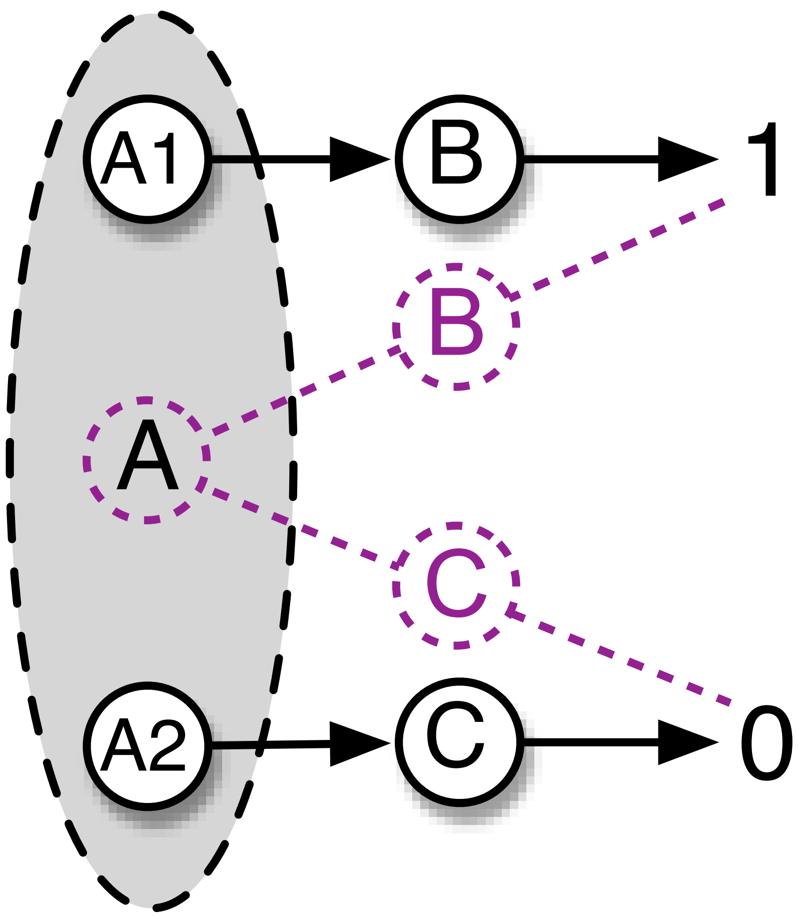

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Gradient-descent learning

1. Pick an objective function $J(\theta)$, a parameterized function to be minimized
2. Use calculus to analytically compute the gradient $\nabla_\theta J(\theta)$
3. Find a "sample gradient" that you can sample on every time step and whose expected value equals the gradient
4. Take small steps in θ proportional to the sample gradient:

$$\Delta\theta = -\alpha \nabla_\theta J_t(\theta)$$

Derivation of the TDC algorithm

$$\Delta\theta = -\frac{1}{2}\alpha\nabla_\theta J(\theta)$$

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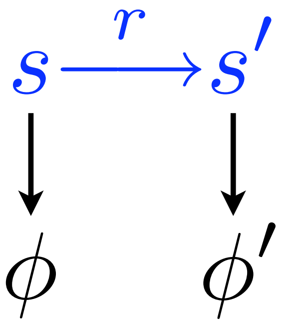
$$\Delta\theta = -\frac{1}{2}\alpha\nabla_\theta J(\theta) \quad = \quad -\frac{1}{2}\alpha\nabla_\theta \parallel V_\theta - \Pi T V_\theta \parallel_D^2$$

Derivation of the TDC algorithm



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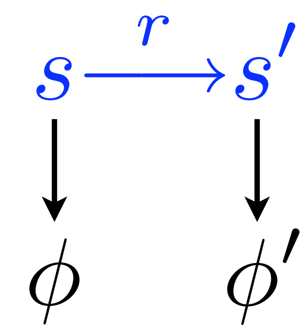

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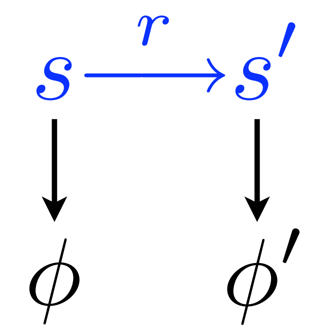

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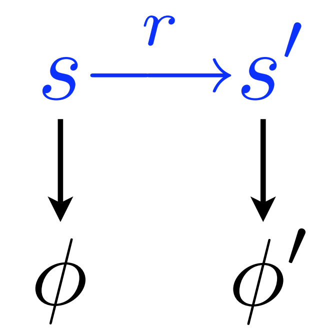

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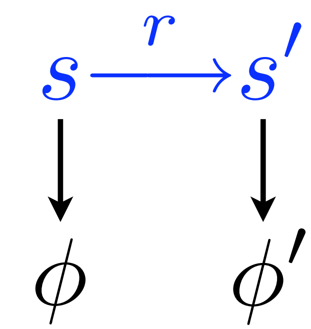

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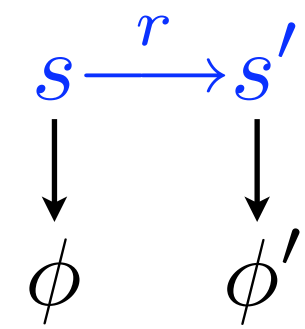

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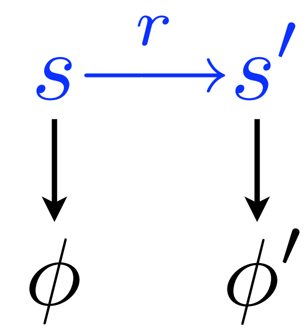

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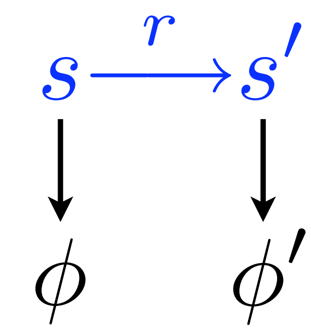

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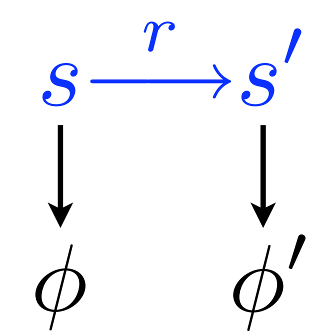
&= -\alpha\mathbb{E}\left[\left(\gamma\phi' - \phi\right)\phi\right]\mathbb{E}\left[\phi\phi^\top\right]^{-1}\mathbb{E}\left[\delta\phi\right] \\
&= \alpha\left(\mathbb{E}\left[\phi\phi^\top\right] - \gamma\mathbb{E}\left[\phi'\phi^\top\right]\right)\mathbb{E}\left[\phi\phi^\top\right]^{-1}\mathbb{E}\left[\delta\phi\right] \\
&= \alpha\mathbb{E}\left[\delta\phi\right] - \alpha\gamma\mathbb{E}\left[\phi'\phi^\top\right]\mathbb{E}\left[\phi\phi^\top\right]^{-1}\mathbb{E}\left[\delta\phi\right] \\
&\approx \alpha\mathbb{E}\left[\delta\phi\right] - \alpha\gamma\mathbb{E}\left[\phi'\phi^\top\right] w
\end{aligned}
$$

This is the trick!
$w \in \Re^n$ is a second set of weights

Derivation of the TDC algorithm

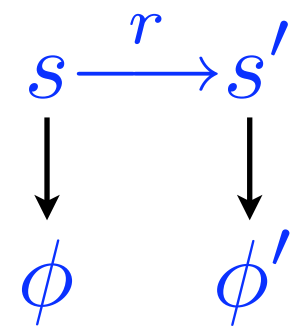


$$
\begin{aligned}
\Delta\theta = -\frac{1}{2}\alpha\nabla_\theta J(\theta) \quad &= \quad -\frac{1}{2}\alpha\nabla_\theta \parallel V_\theta - \Pi T V_\theta \parallel_D^2 \\
&= \quad -\frac{1}{2}\alpha\nabla_\theta \left(\mathbb{E}\left[\delta\phi\right]\mathbb{E}\left[\phi\phi^\top\right]^{-1}\mathbb{E}\left[\delta\phi\right]\right) \\
&= \quad -\alpha\left(\nabla_\theta\mathbb{E}\left[\delta\phi\right]\right)\mathbb{E}\left[\phi\phi^\top\right]^{-1}\mathbb{E}\left[\delta\phi\right] \\
&= \quad -\alpha\mathbb{E}\left[\nabla_\theta\left(r + \gamma\theta^\top\phi' - \theta^\top\phi\right)\phi\right]\mathbb{E}\left[\phi\phi^\top\right]^{-1}\mathbb{E}\left[\delta\phi\right] \\
&= \quad -\alpha\mathbb{E}\left[\left(\gamma\phi' - \phi\right)\phi\right]\mathbb{E}\left[\phi\phi^\top\right]^{-1}\mathbb{E}\left[\delta\phi\right] \\
&= \quad \alpha\left(\mathbb{E}\left[\phi\phi^\top\right] - \gamma\mathbb{E}\left[\phi'\phi^\top\right]\right)\mathbb{E}\left[\phi\phi^\top\right]^{-1}\mathbb{E}\left[\delta\phi\right] \\
&= \quad \alpha\mathbb{E}\left[\delta\phi\right] - \alpha\gamma\mathbb{E}\left[\phi'\phi^\top\right]\mathbb{E}\left[\phi\phi^\top\right]^{-1}\mathbb{E}\left[\delta\phi\right] \\
&\approx \quad \alpha\mathbb{E}\left[\delta\phi\right] - \alpha\gamma\mathbb{E}\left[\phi'\phi^\top\right]w \\
\text{(sampling)} \quad &\approx \quad \alpha\delta\phi - \alpha\gamma\phi'\phi^\top w
\end{aligned}
$$

This is the trick!
$w \in \Re^n$ is a second set of weights

The complete *TD with gradient correction* (TDC) algorithm

- on each transition



- update two parameters

$$\theta \leftarrow \theta + \alpha\delta\phi - \alpha\gamma\phi' \left(\phi^\top w\right)$$

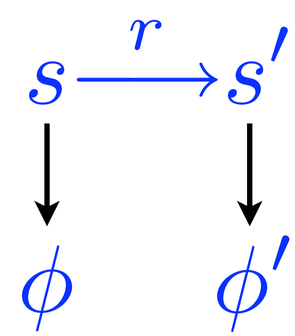
$$w \leftarrow w + \beta(\delta - \phi^\top w)\phi$$

- where

$$\delta = r + \gamma\theta^\top\phi' - \theta^\top\phi$$

The complete *TD with gradient correction* (TDC) algorithm

- on each transition



- update two parameters TD(0)

$$\theta \leftarrow \theta + \alpha\delta\phi - \alpha\gamma\phi' \left(\phi^\top w\right)$$

$$w \leftarrow w + \beta(\delta - \phi^\top w)\phi$$

- where

$$\delta = r + \gamma\theta^\top\phi' - \theta^\top\phi$$

The complete *TD with gradient correction* (TDC) algorithm

- on each transition

$$\begin{array}{ccc} s & \xrightarrow{r} & s' \\ \downarrow & & \downarrow \\ \phi & & \phi' \end{array}$$

- update two parameters

TD(0) — with gradient correction

$$\theta \leftarrow \theta + \alpha\delta\phi - \alpha\gamma\phi'\left(\phi^\top w\right)$$

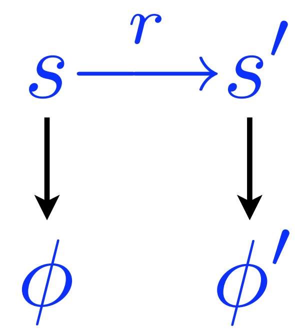
$$w \leftarrow w + \beta(\delta - \phi^\top w)\phi$$

- where

$$\delta = r + \gamma\theta^\top\phi' - \theta^\top\phi$$

The complete *TD with gradient correction* (TDC) algorithm

- on each transition



- update two parameters

$$\theta \leftarrow \theta + \alpha\delta\phi - \alpha\gamma\phi' \left(\phi^\top w\right)$$

$$w \leftarrow w + \beta(\delta - \phi^\top w)\phi$$

estimate of the TD error (δ) for the current state ϕ

- where

$$\delta = r + \gamma\theta^\top\phi' - \theta^\top\phi$$

outline

- ways in which TD with FA has not been straightforward
- the new Bellman error objective function
- derivation of new algorithms (the trick)
- results (theory and experiments)

Three new algorithms

- GTD, the original *gradient TD algorithm* (Sutton, Szepevari & Maei, 2008)
- GTD2, a second-generation GTD
- TDC

Convergence theorems

- For arbitrary on- or off-policy training
- All algorithms converge w.p.1 to the TD fix-point:

$$\mathbb{E}\left[\delta\phi\right] \longrightarrow 0$$

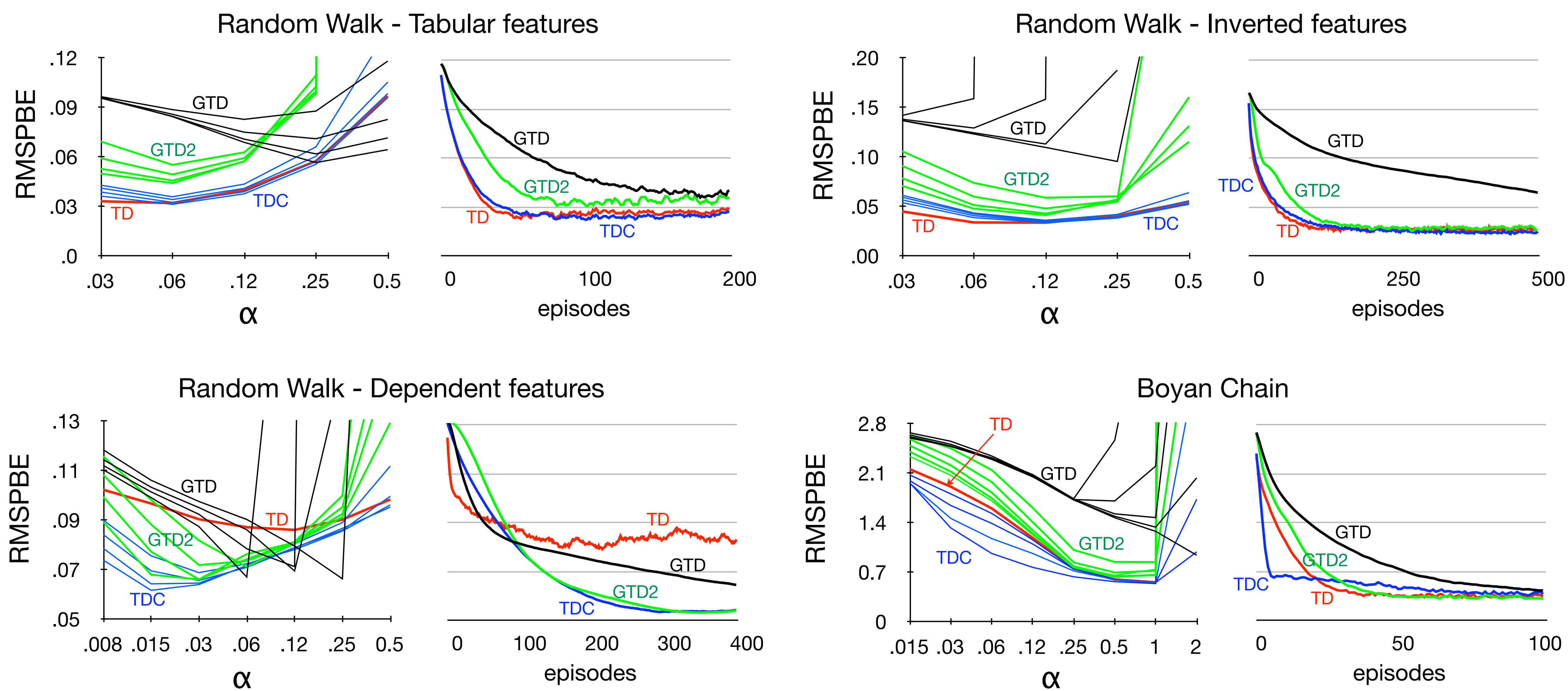
- GTD, GTD2 converge at one time scale

$$\alpha = \beta \longrightarrow 0$$

- TDC converges in a two-time-scale sense

$$\alpha, \beta \longrightarrow 0 \qquad \frac{\alpha}{\beta} \longrightarrow 0$$

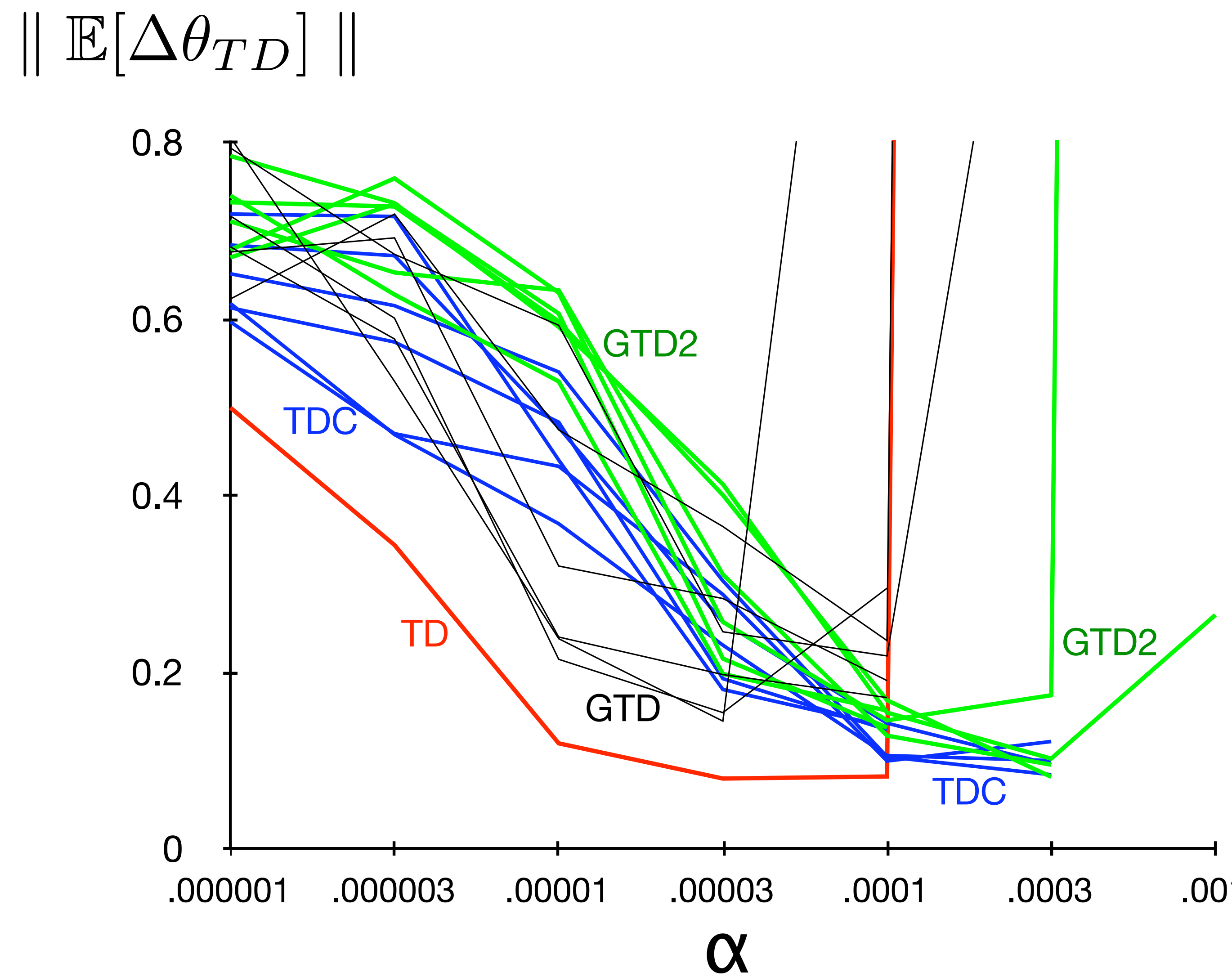
Summary of empirical results on small problems



Usually GTD << GTD2 < TD, TDC

Computer Go experiment

- Learn a linear value function (probability of winning) for 9x9 Go from self play
- One million features, each corresponding to a template on a part of the Go board
- An established experimental testbed



conclusions

- the new algorithms are roughly the same efficiency as conventional TD on on-policy problems
- but are guaranteed convergent under general off-policy training as well
- their key ideas appear to extend quite broadly, to control, general λ, non-linear settings, DP, intra-option learning, TD nets...
- *TD with FA is now straightforward*
- *the curse of dimensionality is removed*