Algorithms for Reasoning with graphical models

Class3

Rina Dechter

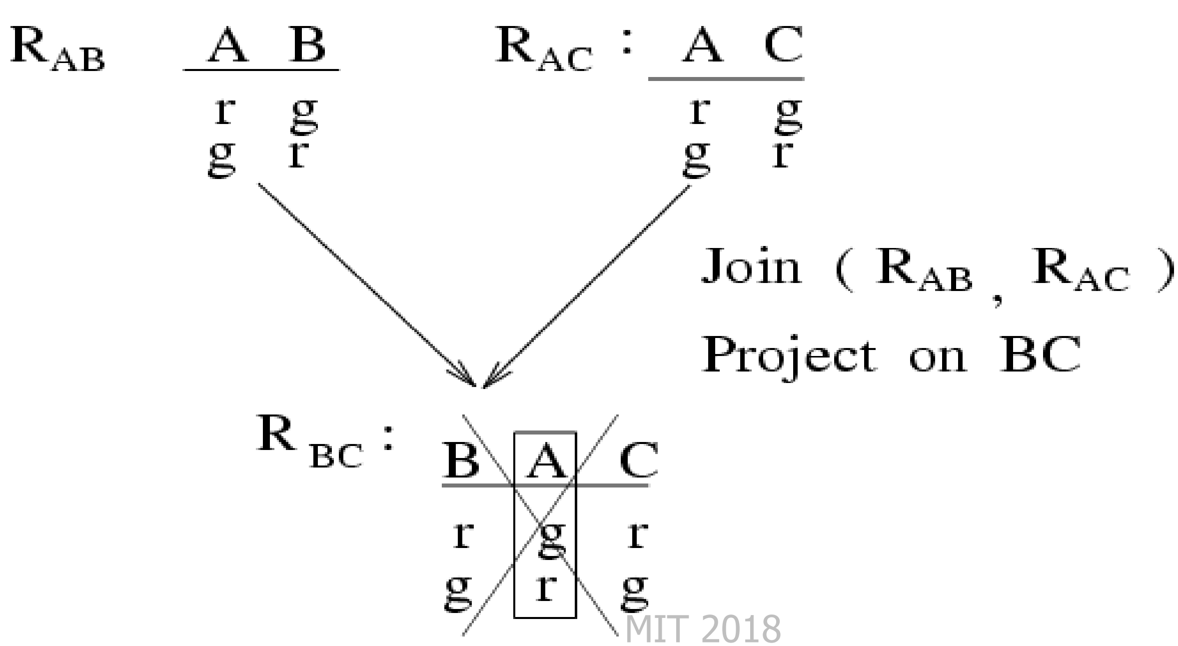
Road Map

- Graphical models
- Constraint networks Model
- Inference
  - Variable elimination for Constraints
  - Variable elimination for CNFs
  - Variable elimination for Linear Inequalities
  - Constraint propagation
- Search
- Probabilistic Networks

Inference: Join and Project

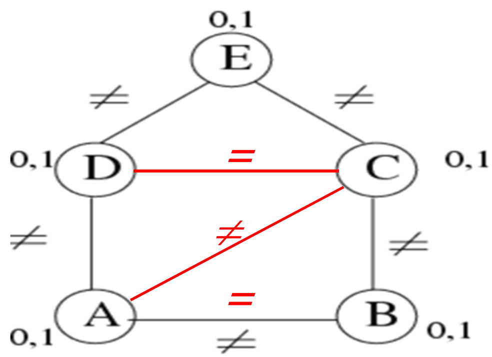
- Given 2 constraints we can deduce a new one by join and then project, via variable-elimination

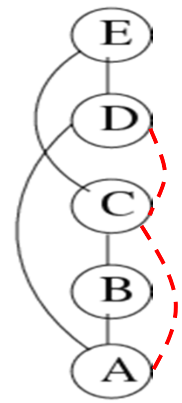
Join operation ⋈ *over A finds all solutions satisfying constraints that involve A*



Bucket Elimination

Adaptive Consistency (Dechter & Pearl, 1987)





Bucket E: $E \neq D,\ E \neq C$

Bucket D: $D \neq A$ → $\boldsymbol{D = C}$

Bucket C: $C \neq B$ → $\boldsymbol{A \neq C}$

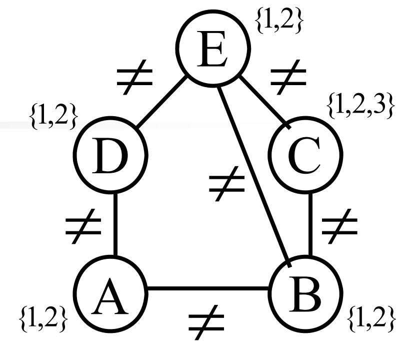
Bucket B: $B \neq A$ → $\boldsymbol{B = A}$

Bucket A: → ***contradiction***

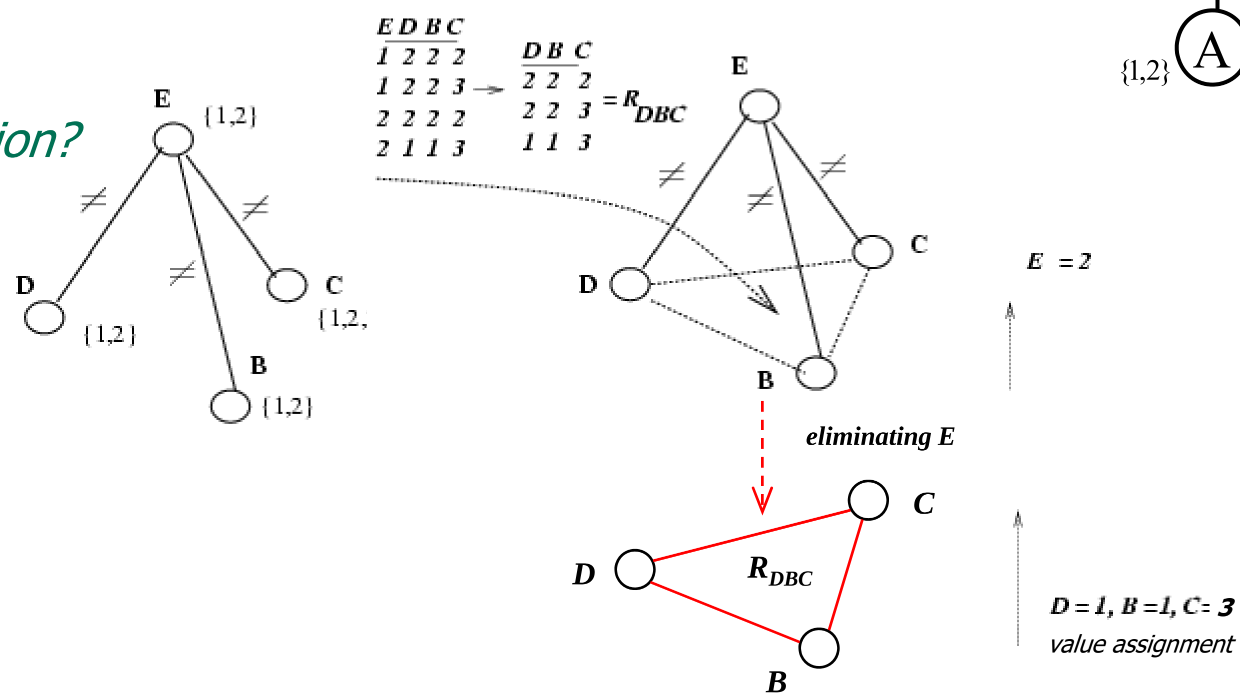
Complexity : $O(n\ exp(w^*))$

w^* - ***induced width***

The Idea of Elimination



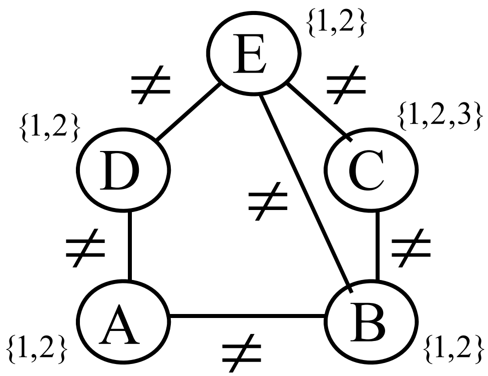
What is the Inferred function?



$$R_{DBC} = \prod_{DBC} R_{ED} \bowtie R_{EB} \bowtie R_{EC}$$

Eliminate variable E $\Leftrightarrow$ join and project

Bucket-Elimination



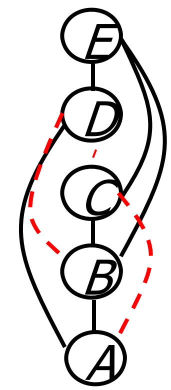
$Bucket(E):\ \mathrm{E} \neq \mathrm{D},\ \mathrm{E} \neq \mathrm{C},\ \mathrm{E} \neq \mathrm{B}$

$Bucket(D):\ \mathrm{D} \neq \mathrm{A}\ \|\ R_{DCB}$

$Bucket(C):\ \mathrm{C} \neq \mathrm{B}\ \|\ R_{ACB}$

$Bucket(B):\ \mathrm{B} \neq \mathrm{A}\ \|\ R_{AB}$

$Bucket(A):\ R_{A}$



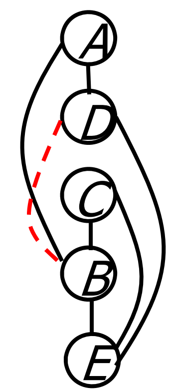
$Bucket(A):\ \mathrm{A} \neq \mathrm{D},\ \mathrm{A} \neq \mathrm{B}$

$Bucket(D):\ \mathrm{D} \neq \mathrm{E}\ \|\ R_{DB}$

$Bucket(C):\ \mathrm{C} \neq \mathrm{B},\ \mathrm{C} \neq \mathrm{E}$

$Bucket(B):\ \mathrm{B} \neq \mathrm{E}\ \|\ R^{D}_{BE},\ R^{C}_{BE}$

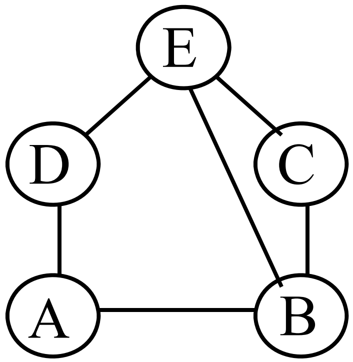
$Bucket(E):\ \|\ R_{E}$

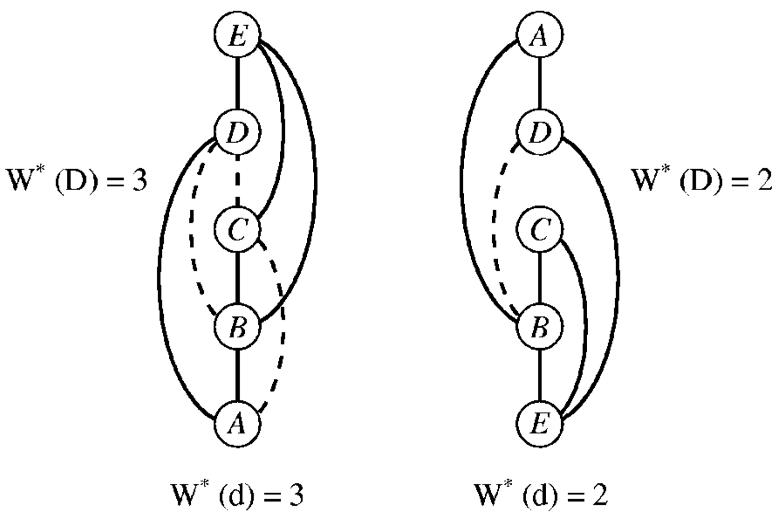


Complexity : $O(n\ exp(w^*(d)))$,

$w^*(d)$ - ***induced width along ordering d***

The Induced-Width





- Width along *d*, w(d):
  - max # of previous parents
- Induced width w*(d):
  - The width in the ordered induced graph
- Induced-width w*:
  - Smallest induced-width over all orderings
- Finding w*
  - NP-complete *(Arnborg, 1985) but greedy heuristics (min-fill).*

Adaptive Consistency, Bucket-Elimination

Initialize: partition constraints into $bucket_1, \ldots, bucket_n$

For *i=n* down to *1* along d // process in reverse order

for all relations $R_1, \ldots, R_m \in bucket_i$ **do**

join and "project-out" X_i

$$R_{new} \leftarrow \prod_{(-X_i)} (\quad {}_j R_j)$$

If R_{new} *is not empty, add it to* $bucket_k, k < i,$ *where k is the largest variable index in* R_{new}

Else *problem is unsatisfiable*

Return *the set of all relations (old and new) in the buckets*

Properties of Adaptive-Consistency

- Adaptive consistency generates a constraint network that is backtrack-free (can be solved without dead-ends).

Definition 3.1.2 (partial solution) *Given a constraint network $\mathcal{R}$, we say that an assignment of values to a subset of the variables $S = \{X_1, ..., X_j\}$ given by $\bar{a} = (< X_1, a_1 >, < X_2, a_2 >, ..., < X_j, a_j >)$ is consistent relative to $\mathcal{R}$ iff it satisfies every constraint whose scope is subsumed in S. The assignment $\bar{a}$ is also called a partial solution of $\mathcal{R}$.*

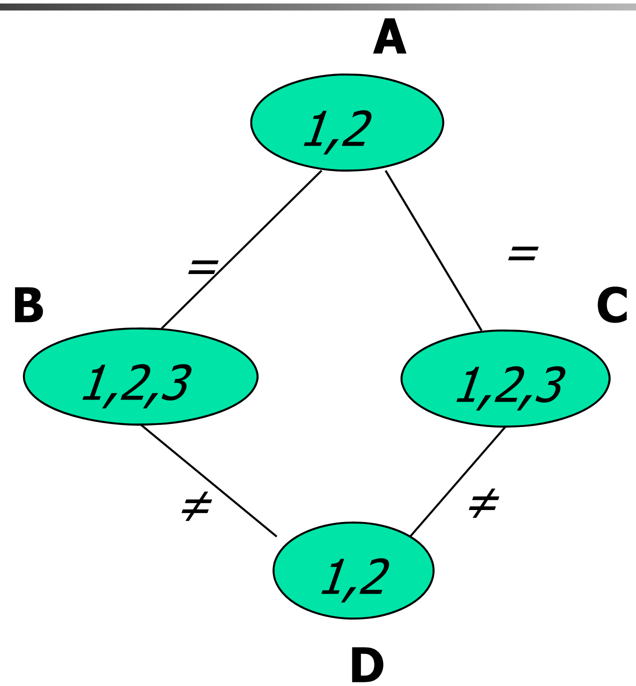
Definition 3.1.3 (backtrack-free search) *A constraint network is backtrack-free relative to a given ordering $d = (X_1, ..., X_n)$ if for every $i \leq n$, every partial solution over $(X_1, ..., X_i)$ can be consistently extended to include X_{i+1}.*

Properties of Adaptive-Consistency (AC)

- Adaptive consistency generates a constraint network that is backtrack-free (can be solved without dead-ends).
- The time and space complexity of AC along ordering *d* is exponential in w*(d) .

Theorem 3.9 *The time and space complexity of* ADAPTIVE-CONSISTENCY *is* $O((r+n)k^{w^*(d)+1})$ *and* $O(n \cdot k^{w^*(d)})$*, respectively, where n is the number of variables, k is the maximum domain size, and* $w^*(d)$ *is the induced-width along the order of processing d and r is the number of the problems' constraints.*

Example: deadends, backtrack-freeness



Assign values in the order
D,B,C,A before and after adaptive-consistence

Order A,B,C,D, order A,B,D,C

Properties of Adaptive-Consistency

- Adaptive-consistency generates a constraint network that is backtrack-free (can be solved without dead-ends).

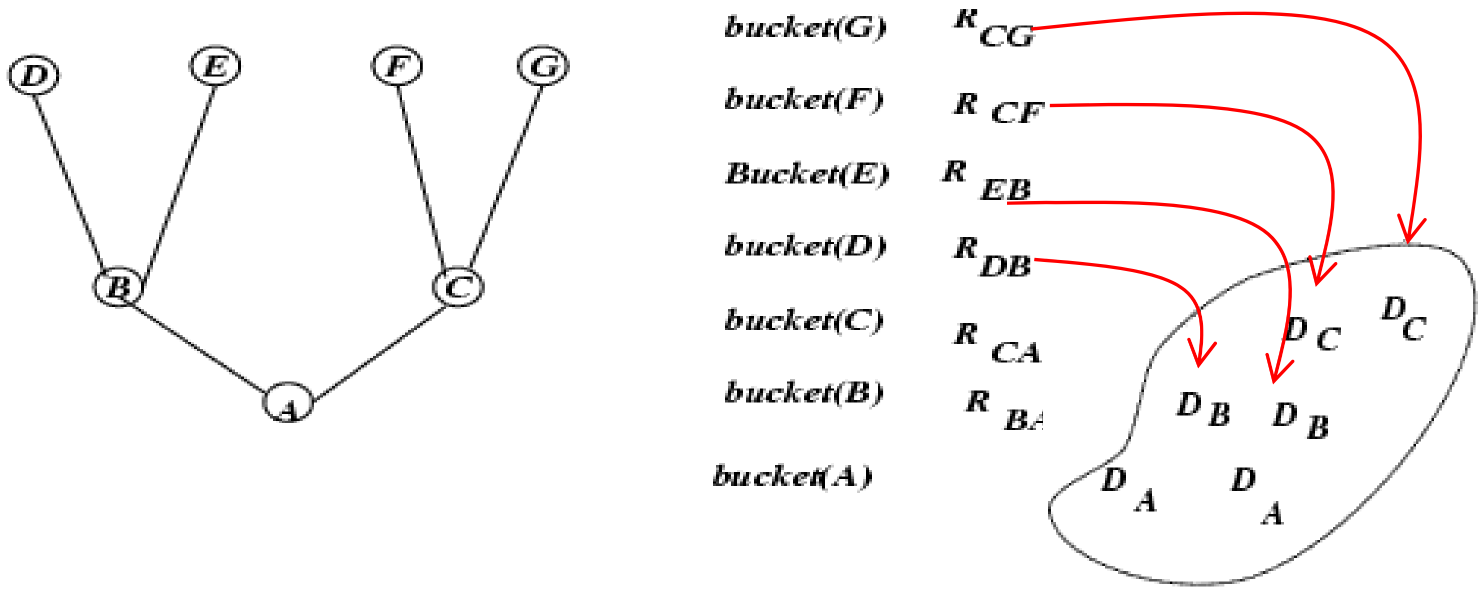
- The time and space complexity of adaptive-consistency along ordering *d* is time and memory exponential in w*(d) .

- Therefore, problems having bounded induced-width are tractable (solved in polynomial time).
  - *trees* (w*=1),
  - *series-parallel networks* (w*=2),
  - and in general *k-trees* (w*=k).

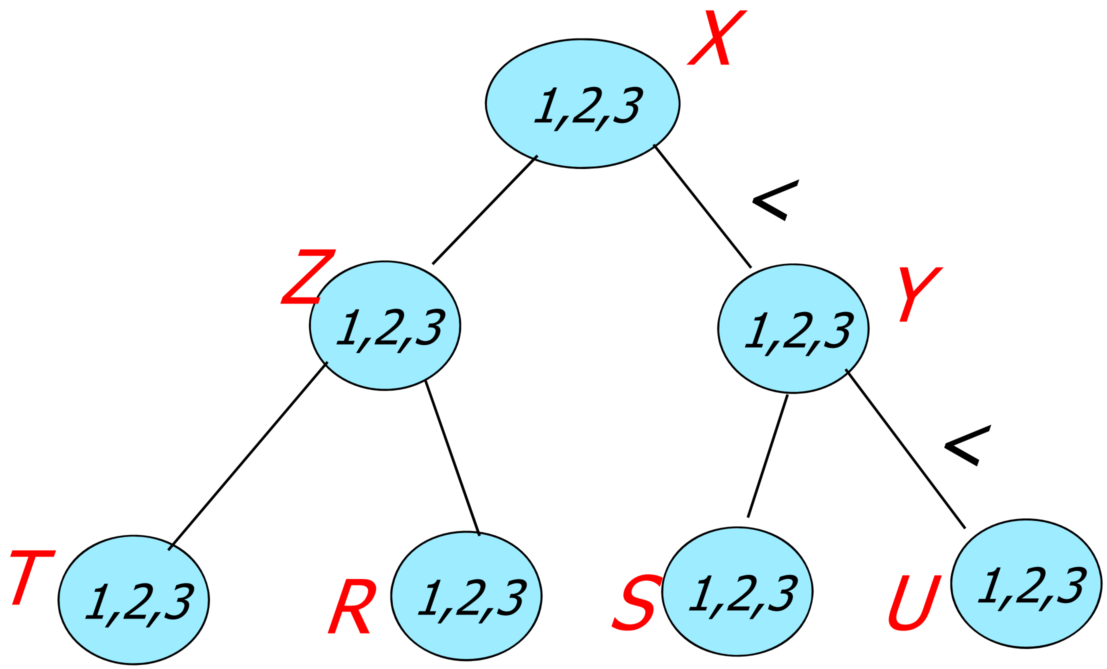
Solving Trees

(Mackworth and Freuder, 1985)

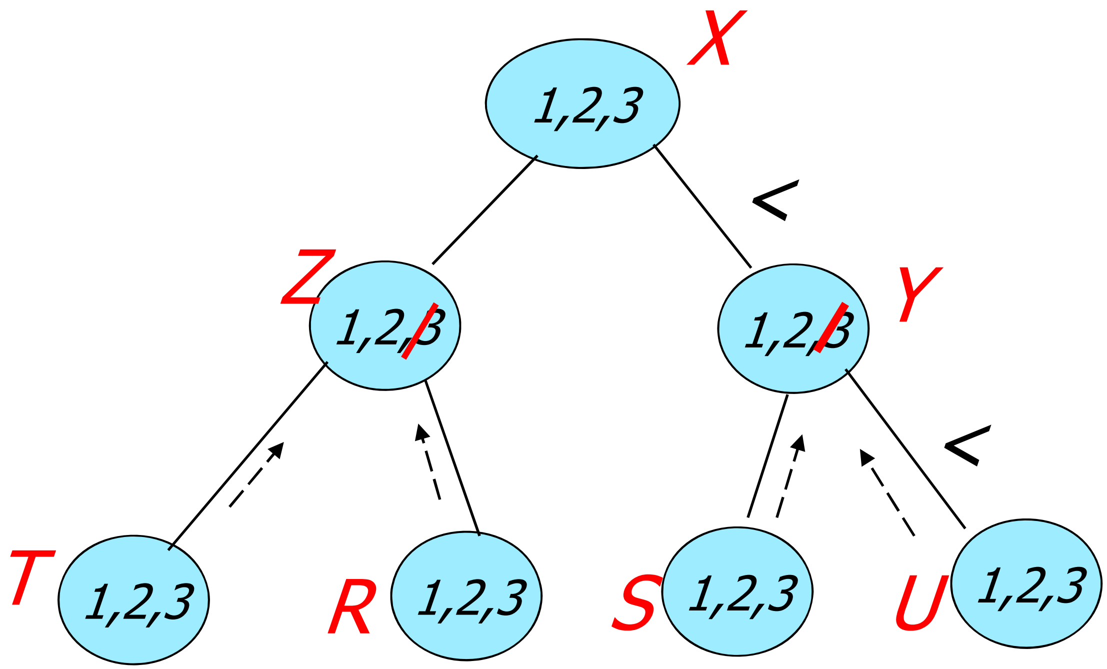
Adaptive consistency is linear for trees and equivalent to enforcing directional arc-consistency (recording only unary constraints)



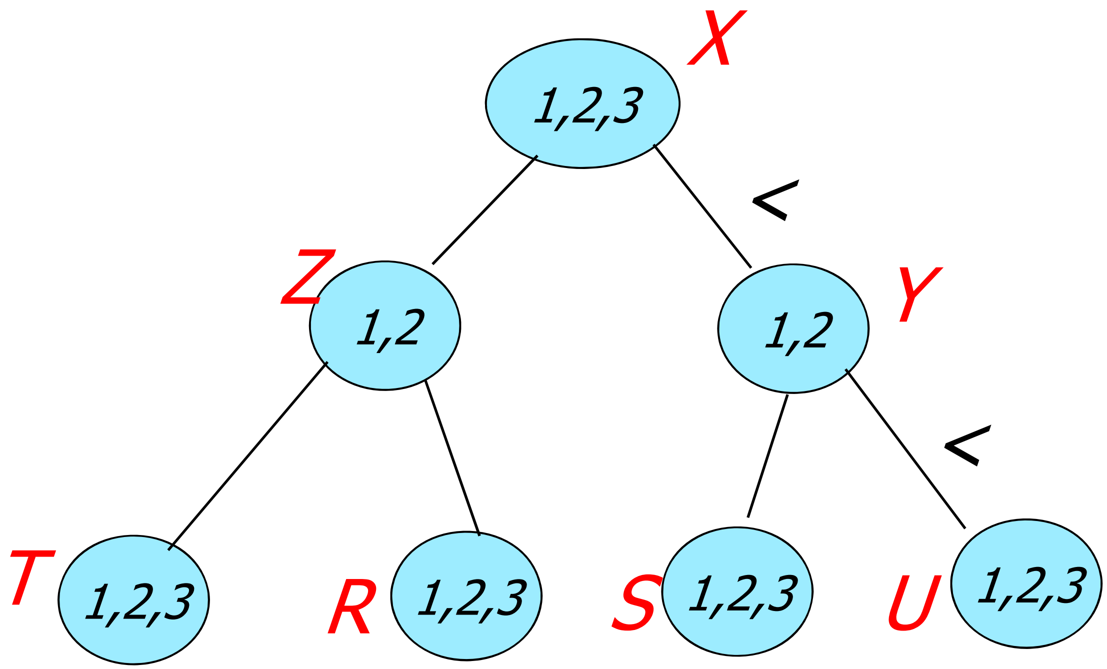
Tree Solving is Easy



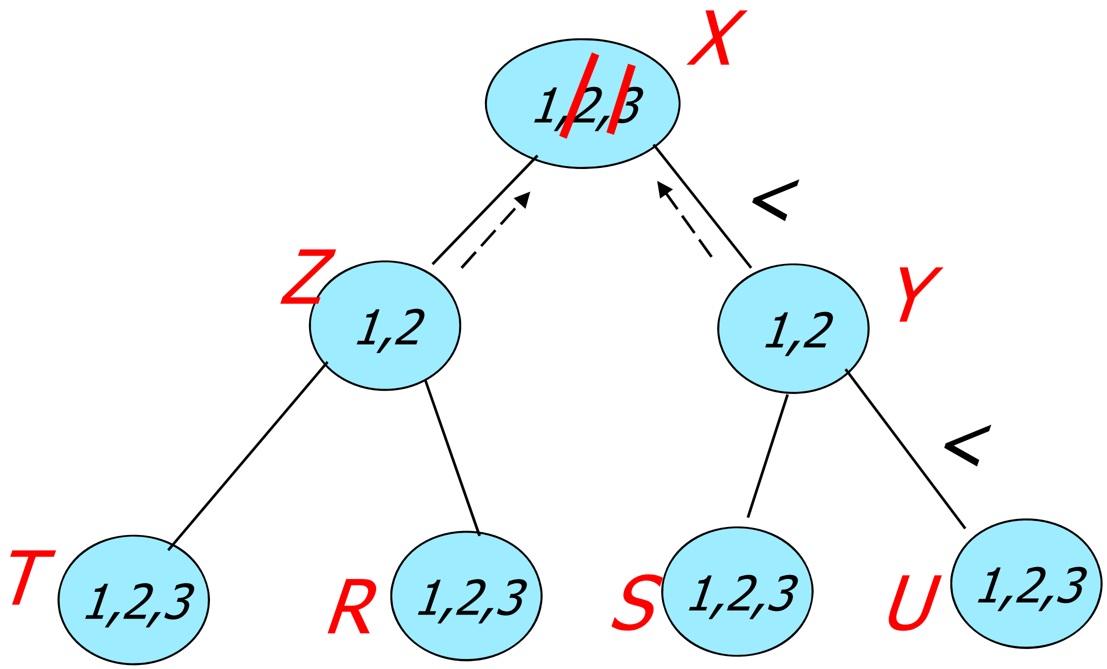
Tree Solving is Easy



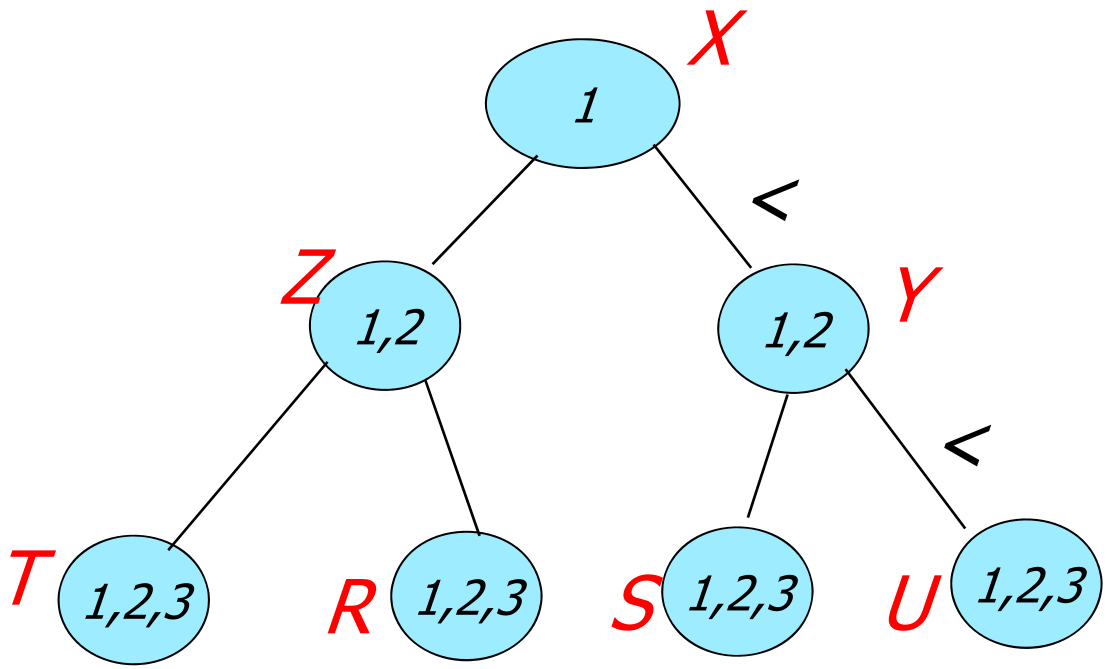
Tree Solving is Easy



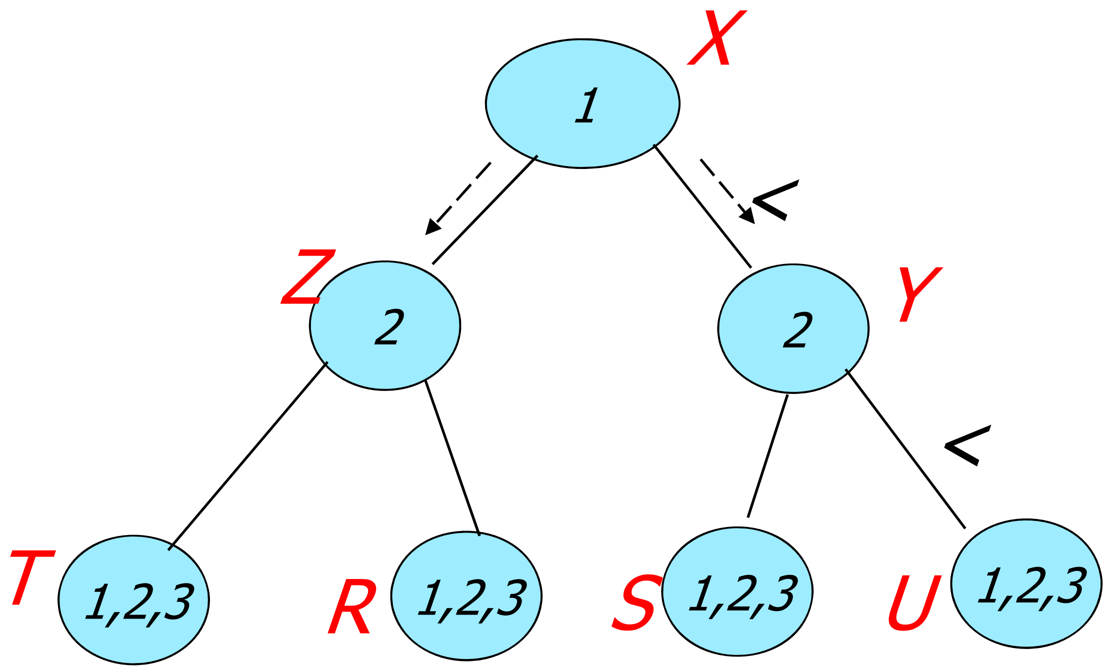
Tree Solving is Easy



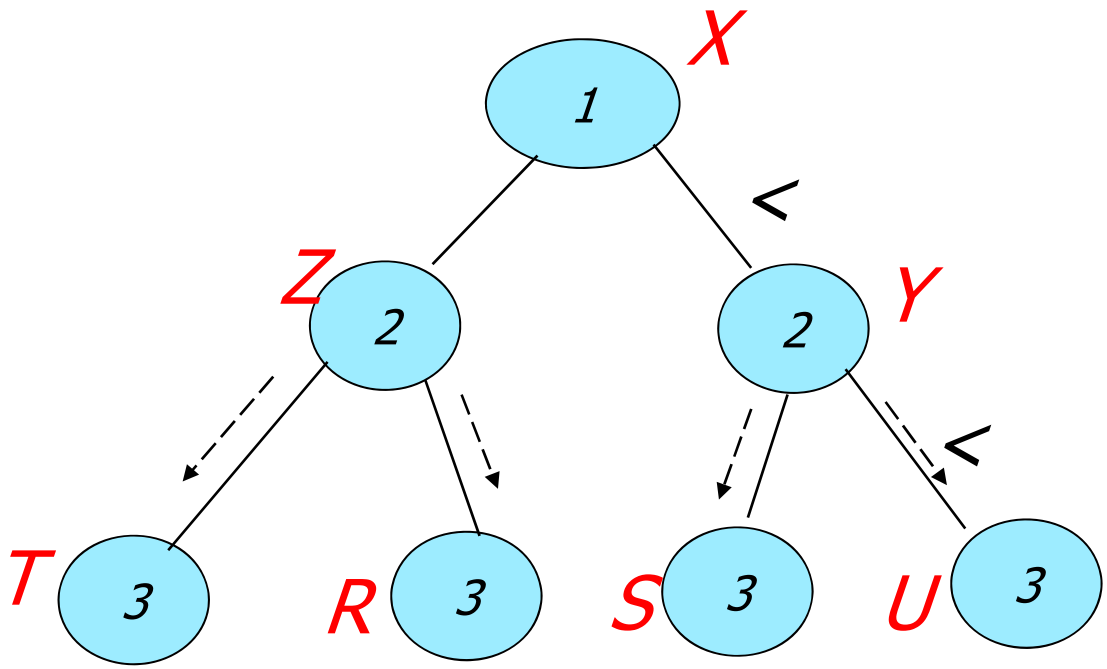
Tree Solving is Easy



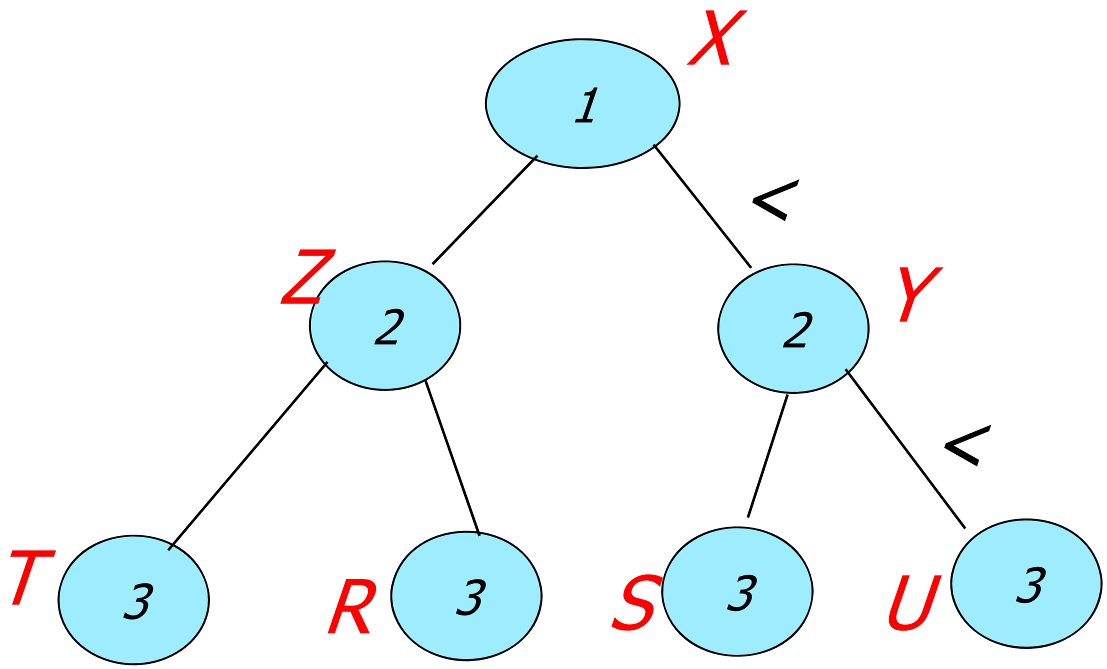
Tree Solving is Easy



Tree Solving is Easy



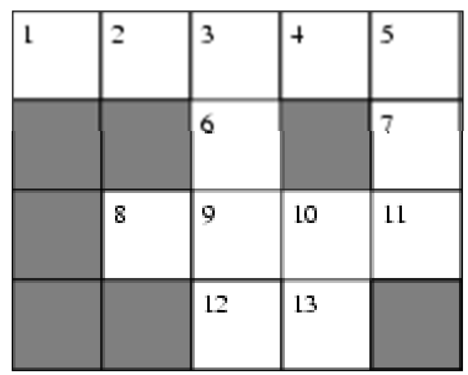
Tree Solving is Easy



Adaptive-consistency is linear time because induced-width is 1
(Constraint propagation Solves trees in linear time)

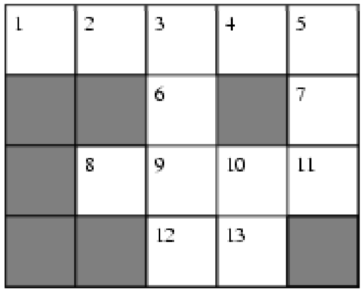
Example: Crossword Puzzle

$$
\begin{aligned}
R_{1,2,3,4,5} &= \{(H,O,S,E,S),(L,A,S,E,R),(S,H,E,E,T),\\
&\quad (S,N,A,I,L),(S,T,E,E,R)\}\\
R_{3,6,9,12} &= \{(H,I,K,E),(A,R,O,N),(K,E,E,T),(E,A,R,N),\\
&\quad (S,A,M,E)\}\\
R_{5,7,11} &= \{(R,U,N),(S,U,N),(L,E,T),(Y,E,S),(E,A,T),(T,E,N)\}\\
R_{8,9,10,11} &= R_{3,6,9,12}\\
R_{10,13} &= \{(N,O),(B,E),(U,S),(I,T)\}\\
R_{12,13} &= R_{10,13}
\end{aligned}
$$



Adaptive-Consistency on the Crossword Puzzle

$$R_{1,2,3,4,5} = \{(H,O,S,E,S),(L,A,S,E,R),(S,H,E,E,T),(S,N,A,I,L),(S,T,E,E,R)\}$$
$$R_{3,6,9,12} = \{(H,I,K,E),(A,R,O,N),(K,E,E,T),(E,A,R,N),(S,A,M,E)\}$$
$$R_{5,7,11} = \{(R,U,N),(S,U,N),(L,E,T),(Y,E,S),(E,A,T),(T,E,N)\}$$
$$R_{8,9,10,11} = R_{3,6,9,12}$$
$$R_{10,13} = \{(N,O),(B,E),(U,S),(I,T)\}$$
$$R_{12,13} = R_{10,13}$$



$bucket(x_1)$

$bucket(x_2)$

$bucket(x_3)$

$bucket(x_4)$

$bucket(x_5)$

$bucket(x_6)$

$bucket(x_7)$

$bucket(x_8)$

$bucket(x_9)$

$bucket(x_{10})$

$bucket(x_{11})$

$bucket(x_{12})$

$bucket(x_{13})$

Adaptive-Consistency on the Crossword Puzzle

$R_{1,2,3,4,5} = \{(H,O,S,E,S), (L,A,S,E,R), (S,H,E,E,T), (S,N,A,I,L), (S,T,E,E,R)\}$

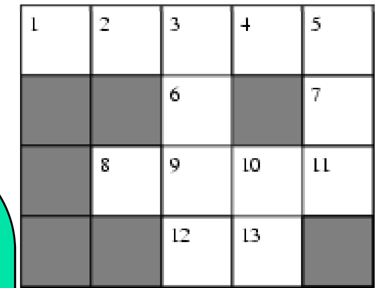
$R_{3,6,9,12} = \{(H,I,K,E), (A,R,O,N), (K,E,E,T), (E,A,R,N), (S,A,M,E)\}$

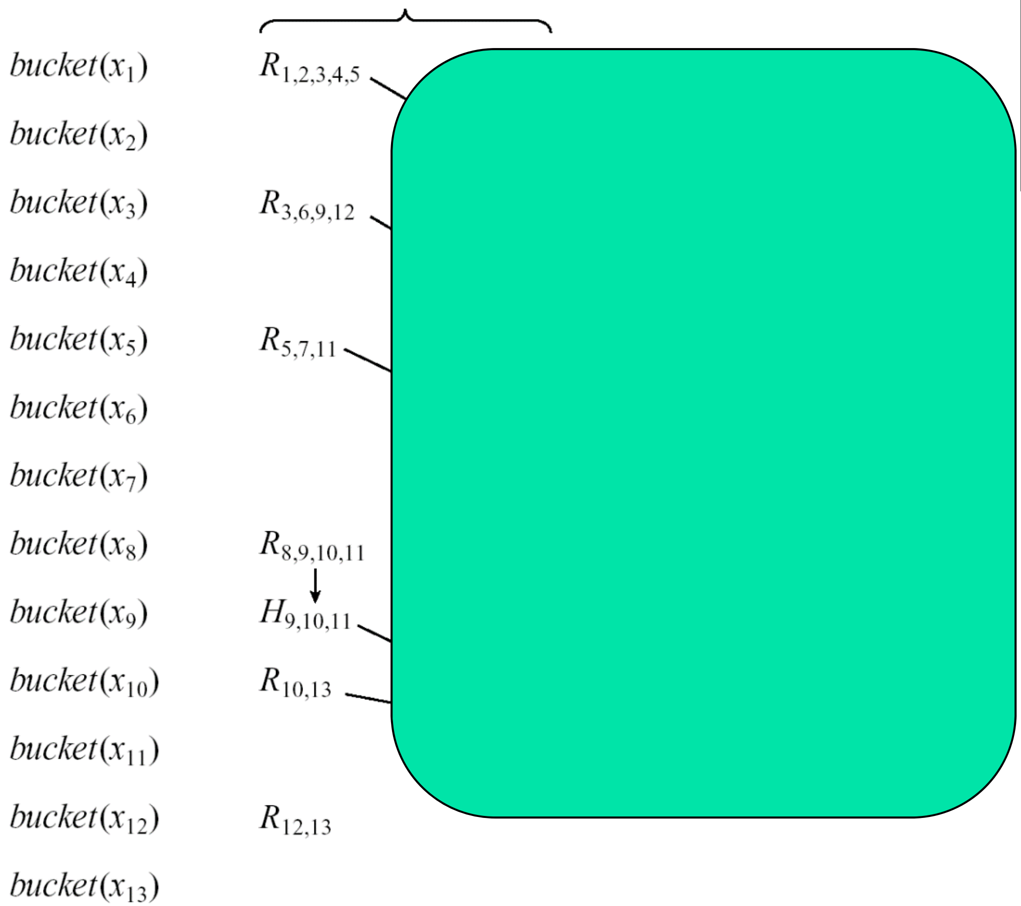
$R_{5,7,11} = \{(R,U,N), (S,U,N), (L,E,T), (Y,E,S), (E,A,T), (T,E,N)\}$

$R_{8,9,10,11} = R_{3,6,9,12}$

$R_{10,13} = \{(N,O), (B,E), (U,S), (I,T)\}$

$R_{12,13} = R_{10,13}$





Adaptive-Consistency on the Crossword Puzzle

$R_{1,2,3,4,5} = \{(H,O,S,E,S), (L,A,S,E,R), (S,H,E,E,T), (S,N,A,I,L), (S,T,E,E,R)\}$

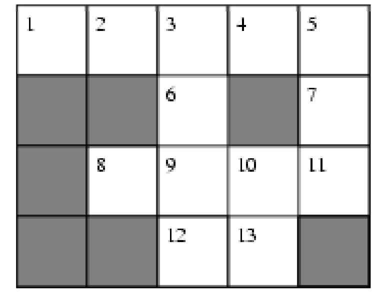
$R_{3,6,9,12} = \{(H,I,K,E), (A,R,O,N), (K,E,E,T), (E,A,R,N), (S,A,M,E)\}$

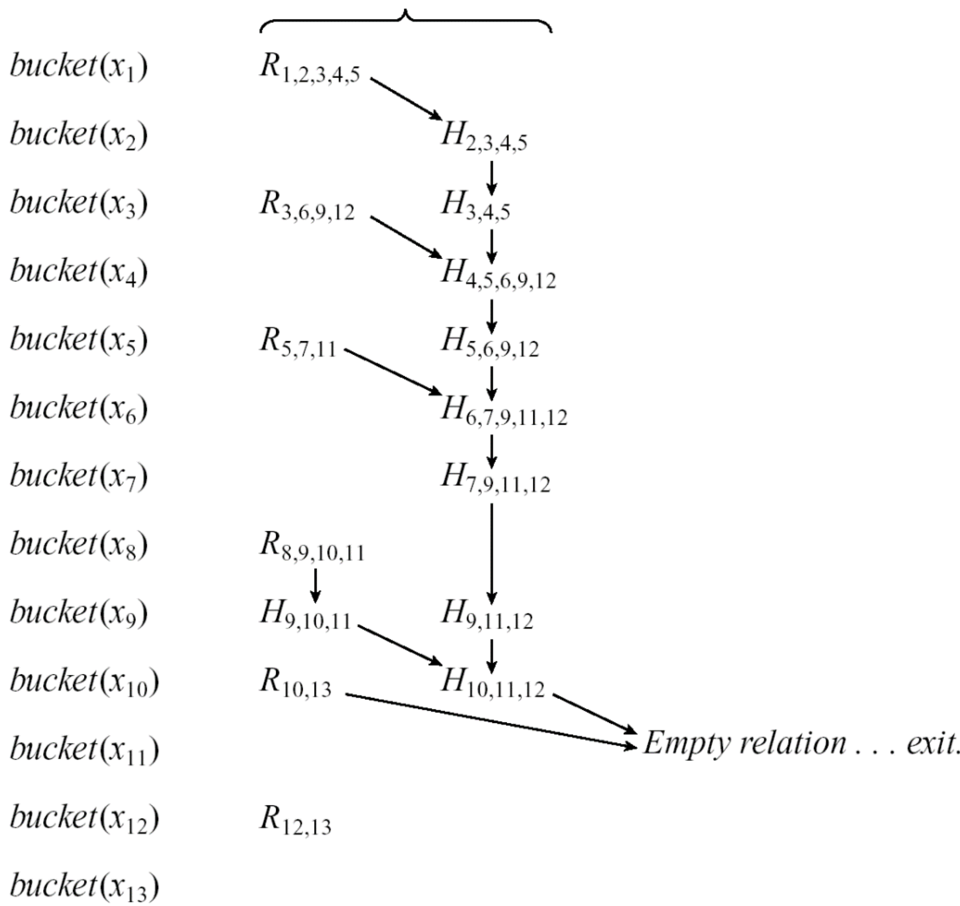
$R_{5,7,11} = \{(R,U,N), (S,U,N), (L,E,T), (Y,E,S), (E,A,T), (T,E,N)\}$

$R_{8,9,10,11} = R_{3,6,9,12}$

$R_{10,13} = \{(N,O), (B,E), (U,S), (I,T)\}$

$R_{12,13} = R_{10,13}$





Road Map

- Graphical models
- Constraint networks Model
- Inference
  - Variable elimination for Constraints
  - Variable elimination for CNFs
  - Variable elimination for Linear Inequalities
  - Constraint propagation
- Search
- Probabilistic Networks

Gausian and Boolean Propagation, Resolution

- Linear inequalities

$$x + y + z \leq 15, z \geq 13 \Rightarrow$$

$$x \leq 2, y \leq 2$$

- Boolean constraint propagation, unit resolution

$$(A \vee B \vee \neg C), (\neg B) \Rightarrow$$

$$(A \vee \neg C)$$

Extended Composition

Definition 3.2.1 (extended composition) *The extended composition of relation* $R_{S_1}, \ldots, R_{S_m}$ *relative to a subset of variables* $A \subseteq \bigcup_{i=1}^{m} S_i$*, denoted* $EC_A(R_{S_1}, \ldots, R_{S_m})$*, is defined by*

$$EC_A(R_{S_1}, \ldots, R_{S_m}) = \pi_A(\bowtie_{i=1}^{m} R_{S_i})$$

Example 3.2.2 Consider the two clauses $\alpha = (P \vee \neg Q \vee \neg O)$ and $\beta = (Q \vee \neg W)$. Now let the relation $R_{PQO} = \{000, 100, 010, 001, 110, 101, 111\}$ be the models of α and the relation $R_{QW} = \{00, 10, 11\}$ be the models of β. Resolving these two clauses over Q generates the resolvent clause $\gamma = res(\alpha, \beta) = (P \vee \neg O \vee \neg W)$. The models of γ are $\{(000, 100, 010, 001, 110, 101, 111\}$. It is easy to see that $EC_{PQW}(R_{PQO}, R_{QW}) = \pi_{RQW}(R_{PQO} \bowtie R_{Qw})$ yields the models of γ. □

Lemma 3.2.3 *The* resolution *operation over two clauses,* $(\alpha \vee Q)$ *and* $(\beta \vee \neg Q)$*, results in a clause* $(\alpha \vee \beta)$ *for which* $models(\alpha \vee \beta) = EC_{Q'}(models(\alpha \vee Q), models(\beta \vee \neg Q))$*, where* Q' *is the union of scopes of both clauses excluding* Q*.* □

The Effect of Resolution on Its Graph

(~C) (AVBVC) (~AvBvE)(~B,C,E)

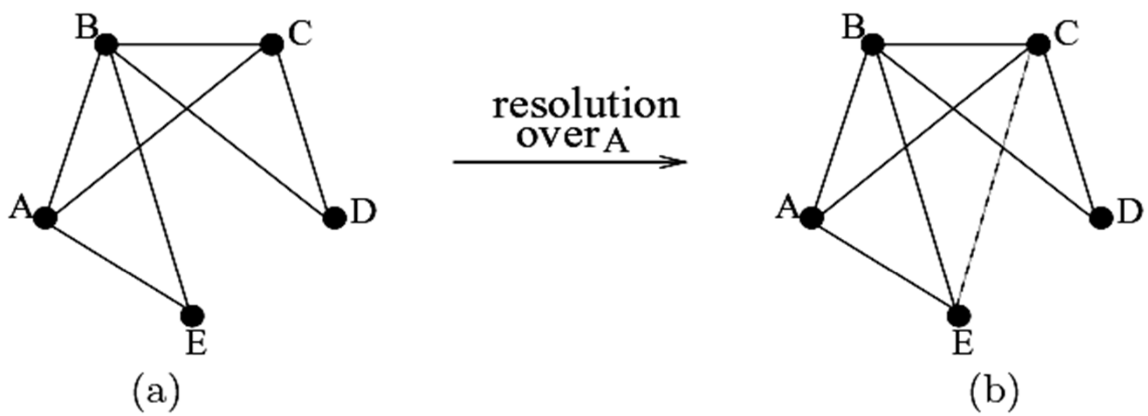


Figure 4.19: (a) The interaction graph of theory $\varphi_1 = \{(\neg C), (A \vee B \vee C), (\neg A \vee B \vee E), (\neg B \vee C \vee D)\}$, and (b) the effect of resolution over A on that graph.

Directional Resolution ⇔ Adaptive Consistency

(~C) (AVBVC) (~AvBvE)(~B,C,E)

Bucket A

Bucket B

Bucket C

Bucket D

Bucket E

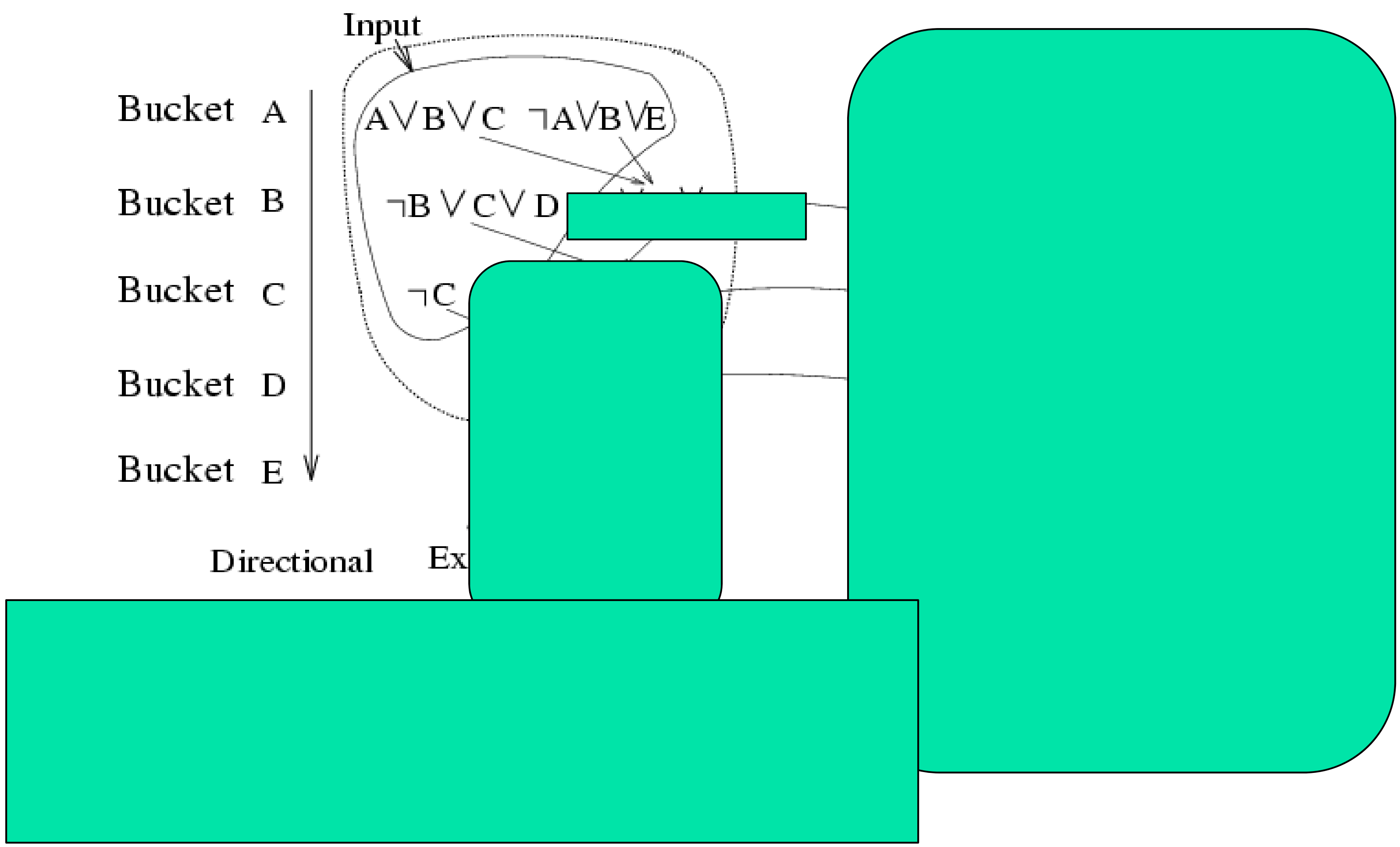
Direct

$| bucket_i | = O(\exp(w\ ))$

$\text{DR time and space} : O(n \exp(w^*))$

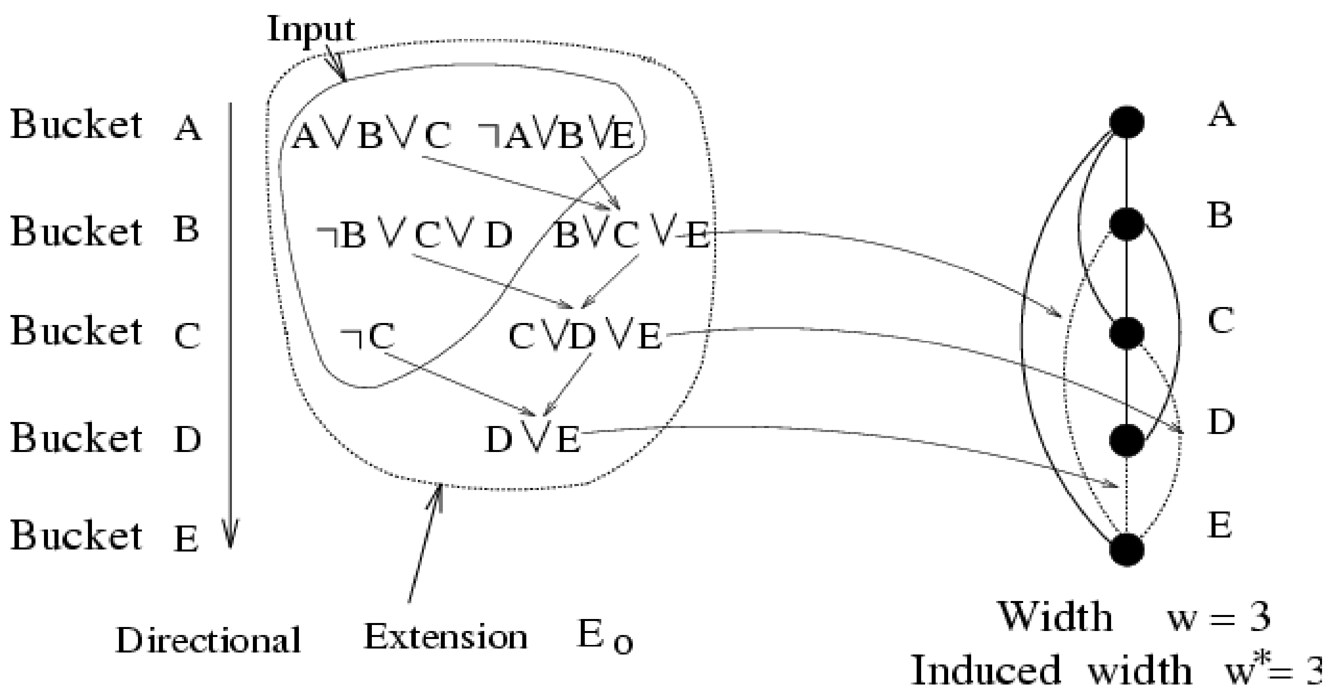
Directional Resolution ⇔ Adaptive Consistency

(~C) (AVBVC) (~AvBvE)(~B,C,E)



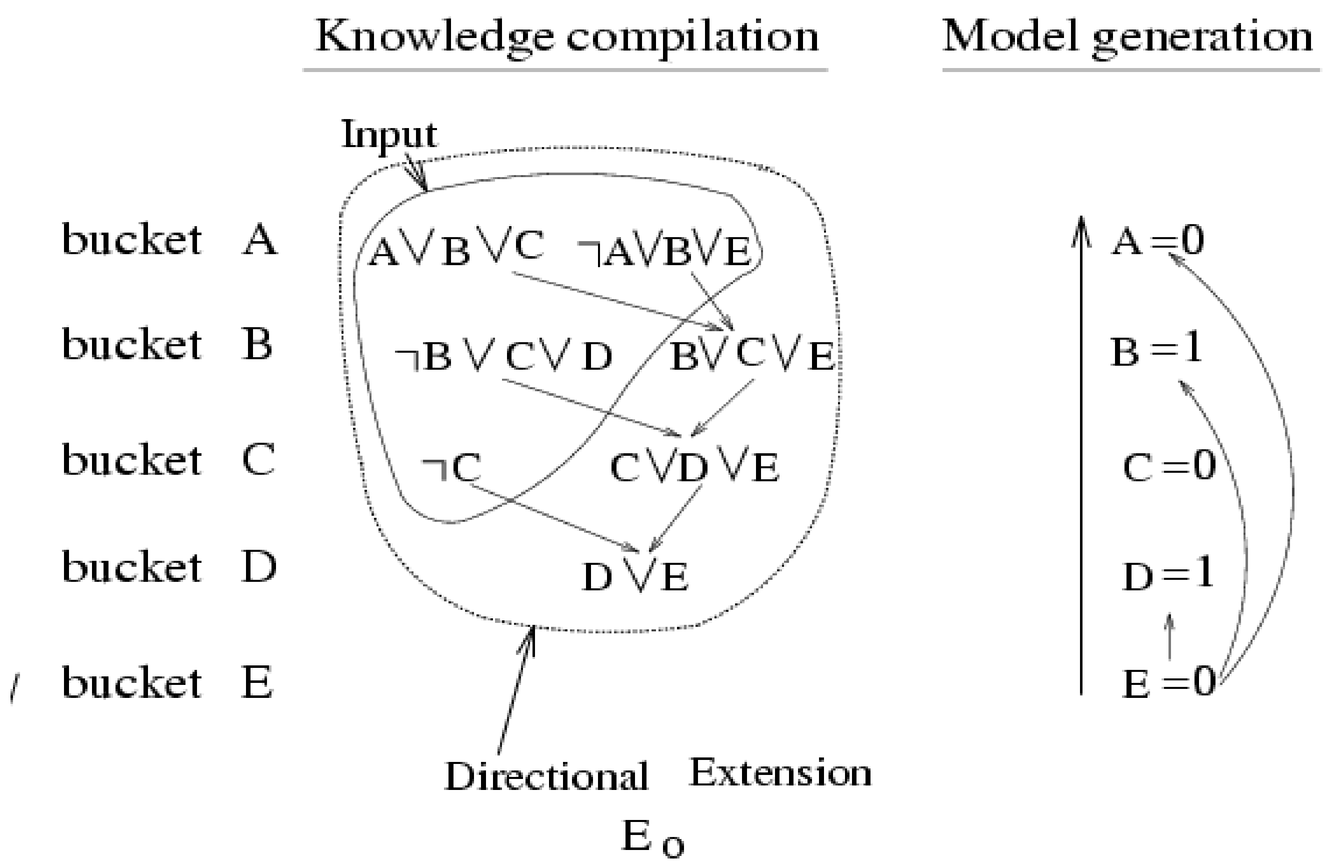
Directional Resolution ⇔ Adaptive Consistency

(~C) (AVBVC) (~AvBvE)(~B,C,E)



$$|bucket_i| = O(\exp(w^*))$$
$$\text{DR time and space}: O(n \exp(w^*))$$

Directional Resolution ⇔ Adaptive Consistency



Directional Resolution

DIRECTIONAL-RESOLUTION

Input: A *CNF* theory φ, an ordering $d = Q_1, \ldots, Q_n$ of its variables.

Output: A decision of whether φ is satisfiable. If it is, a theory $E_d(\varphi)$, equivalent to φ, else an empty directional extension.

1. **Initialize:** generate an ordered partition of clauses into buckets. $bucket_1, \ldots, bucket_n$, where $bucket_i$ contains all clauses whose highest literal is Q_i.
2. **for** $i \leftarrow n$ **downto** 1 process $bucket_i$:
3.   **if** there is a unit clause **then** (the instantiation step)
           apply unit-resolution in $bucket_i$ and place the resolvents in their right buckets.
           if the empty clause was generated, theory is not satisfiable.
4.   **else** resolve each pair $\{(\alpha \vee Q_i), (\beta \vee \neg Q_i)\} \subseteq bucket_i$.
           if $\gamma = \alpha \vee \beta$ is empty, return $E_d(\varphi) = \{\}$, theory is not satisfiable
           else determine the index of γ and add it to the appropriate bucket.
5. **return** $E_d(\varphi) \leftarrow \bigcup_i bucket_i$

History

- 1960 – resolution-based Davis-Putnam algorithm
- 1962 – resolution step replaced by conditioning (Davis, Logemann and Loveland, 1962) to avoid memory explosion, resulting into a backtracking search algorithm known as Davis-Putnam (DP), or DPLL procedure.
- The dependency on induced-width was not known in 1960.
- 1994 – Directional Resolution (DR), a rediscovery of the original Davis-Putnam, identification of tractable classes (Dechter and Rish, 1994).

Properties of DR

Lemma 3.2.6 *Given a theory φ and an ordering $d = (Q_1, ..., Q_n)$, if Q_i has at most k parents in the induced graph along d, then the bucket of Q_i in $E_d(\varphi)$ contains no more than 3^{k+1} clauses.*

Proof: Given a clause α in the bucket of Q_i, there are three possibilities for each parent P of Q_i: either P appears in α, $\neg P$ appears in α, or neither of them appears in α. Since Q_i also appears in α, either positively or negatively, the number of possible clauses in a bucket is no more than $2 \cdot 3^k < 3^{k+1}$. ■

Theorem 3.2.7 *(complexity of DR)*

Given a theory φ and an ordering of its variables d, the time complexity of algorithm DR along d is $O(n \cdot 9^{w_d^*})$, and $E_d(\varphi)$ contains at most $n \cdot 3^{w_d^*+1}$ clauses, where w_d^* is the induced width of φ's interaction graph along d. □

Algorithms for Reasoning with graphical models

Class4

Rina Dechter