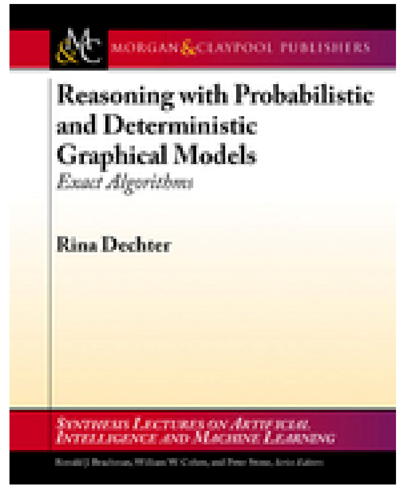
Algorithms for Reasoning with graphical models

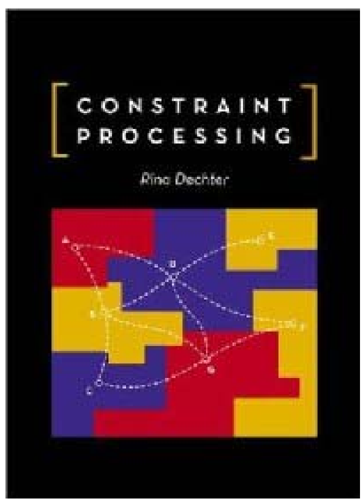
Class2: Constraint Networks

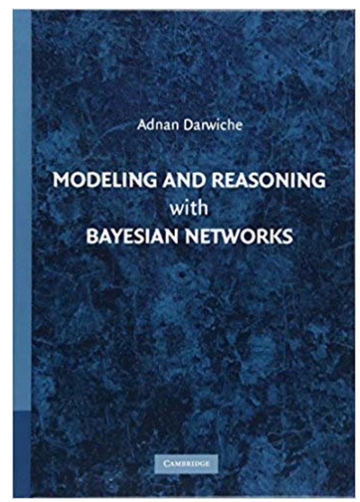
Rina Dechter

dechter1: chapters 2-3,
Dechter2: Constraint book: chapters 2 and 4

Text Books


MORGAN & CLAYPOOL PUBLISHERS
Reasoning with Probabilistic and Deterministic Graphical Models
Exact Algorithms
Rina Dechter
Synthesis Lectures on Artificial Intelligence and Machine Learning


CONSTRAINT PROCESSING
Rina Dechter


Adnan Darwiche
MODELING AND REASONING with BAYESIAN NETWORKS

Road Map

- Graphical models
- Constraint networks model
- Inference
  - Variable elimination for Constraints
  - Variable elimination for CNFs
  - Variable elimination for Linear Inequalities
  - Constraint propagation
- Search
- Probabilistic Networks

Road Map

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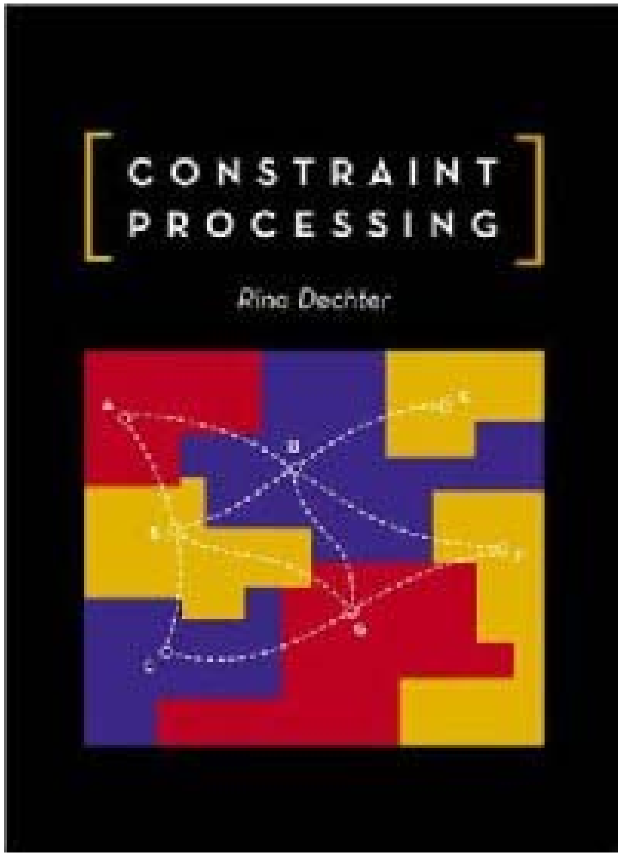
- Probabilistic Networks

Text Book (not required)

Rina Dechter,

Constraint Processing,

Morgan Kaufmann



Sudoku –
Approximation: Constraint Propagation

- ***Constraint***
- ***Propagation***

- ***Inference***

		2	4		6			
8	6	5	1			2		
	1				8	6		9
9				4		8	6	
	4	7				1	9	
	5	8		6				3
4		6	9				7	2,~~3~~ ~~4~~ ~~6~~ 2
		9			4	5	8	1
			3		2	9		

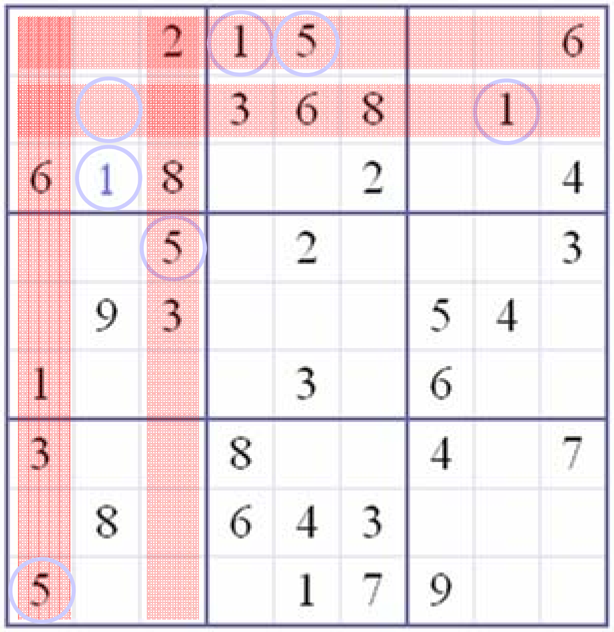
- ***Variables*: empty slots**
- ***Domains* = {1,2,3,4,5,6,7,8,9}**
- ***Constraints*:**
  - ***27 all-different***

Each row, column and major block must be alldifferent

"Well posed" if it has unique solution: 27 constraints

Sudoku

Alternative formulations:
Variables?
Domains?
Constraints?



Each row, column and major block must be alldifferent

“Well posed” if it has unique solution

Constraint Networks

A

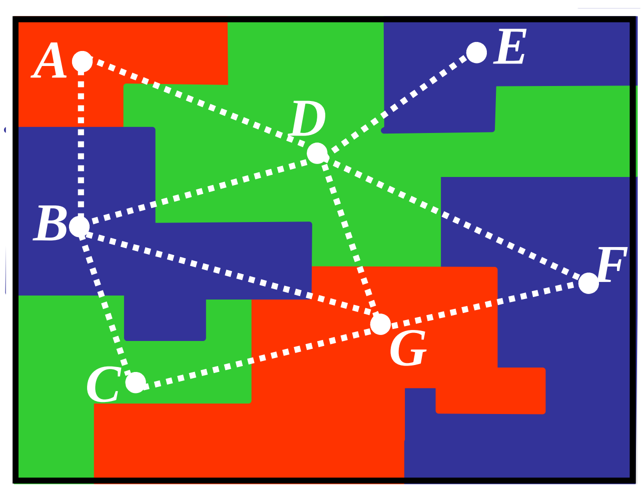
Example: map coloring

Variables - countries (A,B,C,etc.)

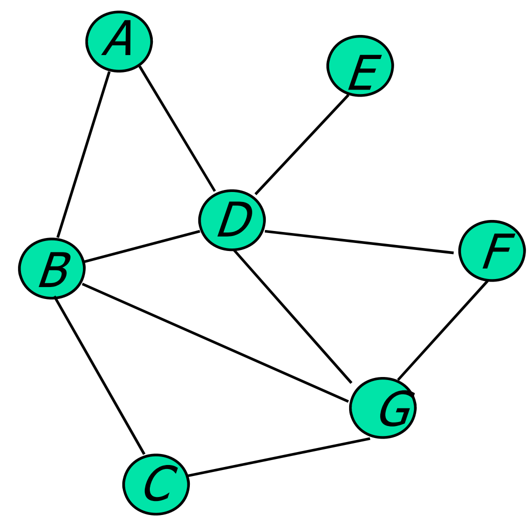
Values - colors (red, green, blue)

Constraints: **A ≠ B, A ≠ D, D ≠ E,** *etc.*

A	B
red	green
red	yellow
green	red
green	yellow
yellow	green
yellow	red



Constraint graph



Constraint Satisfaction Tasks

Example: map coloring

Variables - countries (A,B,C,etc.)

Values - colors (e.g., red, green, yellow)

Constraints:

$A \neq B$, $A \neq D$, $D \neq E$, *etc.*

A	B	C	D	E…
red	green	red	green	blue
red	blue	green	green	blue
…	…	…	…	green
…	…	…	…	red
red	blue	red	green	red

Are the constraints consistent?

Find a solution, find all solutions

Count all solutions

Find a good (optimal) solution

Constraint Network

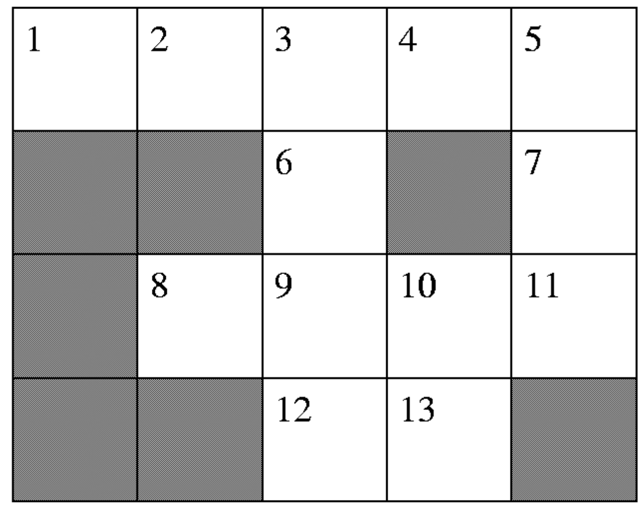
- A constraint network is: *R=(X,D,C)*
  - **X** **variables** $X = \{X_1, \ldots, X_n\}$
  - **D** **domain** $D = \{D_1, \ldots, D_n\}, D_i = \{v_1, \ldots v_k\}$
  - **C** **constraints** $C = \{C_1, \ldots C_t\}$

    $C_i = (S_i, R_i)$
  - ***R*** **expresses allowed tuples over** **scopes**
- **A solution** is an assignment to all variables that satisfies all constraints (join of all relations).
- **Tasks:** consistency?, one or all solutions, counting, optimization

Crossword Puzzle

Formulation?

- Variables: $x_1, \ldots, x_{13}$
- Domains: letters
- Constraints: words from



{HOSES, LASER, SHEET, SNAIL, STEER, ALSO, EARN, HIKE, IRON, SAME, EAT, LET, RUN, SUN, TEN, YES, BE, IT, NO, US}

Crossword Puzzle

$$R_{1,2,3,4,5} = \{(H,O,S,E,S),(L,A,S,E,R),(S,H,E,E,T),\\ (S,N,A,I,L),(S,T,E,E,R)\}$$

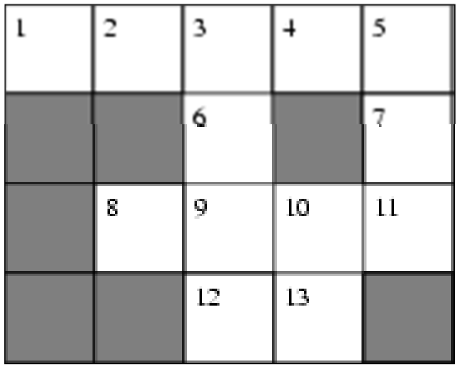
$$R_{3,6,9,12} = \{(H,I,K,E),(A,R,O,N),(K,E,E,T),(E,A,R,N),\\ (S,A,M,E)\}$$

$$R_{5,7,11} = \{(R,U,N),(S,U,N),(L,E,T),(Y,E,S),(E,A,T),(T,E,N)\}$$

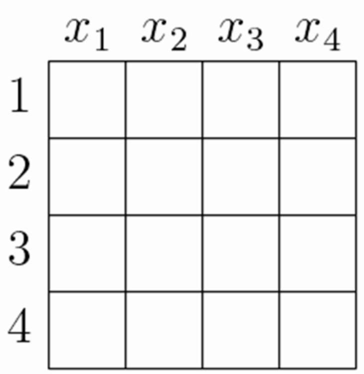
$$R_{8,9,10,11} = R_{3,6,9,12}$$

$$R_{10,13} = \{(N,O),(B,E),(U,S),(I,T)\}$$

$$R_{12,13} = R_{10,13}$$



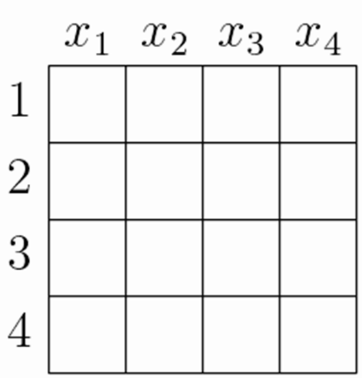
The Queen Problem



(a)

(b)

The Queen Problem



(a)

$$
\begin{aligned}
R_{12} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{13} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{14} &= \{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (2,4),\ (3,1),\ (3,2),\ (3,4) \\
&\quad\ (4,2),\ (4,3)\} \\
R_{23} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{24} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{34} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}
\end{aligned}
$$

(b)

The network has four variables, all with domains $D_i = \{1, 2, 3, 4\}$.
(a) The labeled chess board. (b) The constraints between variables.

Varieties of Constraints

Unary constraints involve a single variable,

e.g., SA ≠ green

Binary constraints involve pairs of variables,

e.g., SA ≠ WA

Higher-order constraints involve 3 or more variables,

e.g., cryptarithmetic column constraints

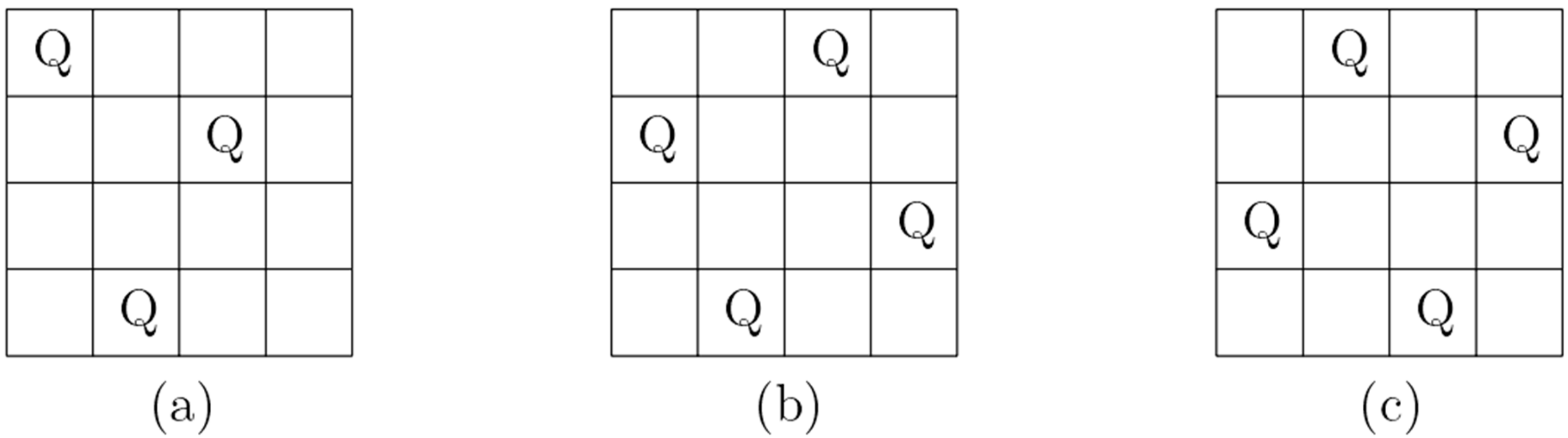
Constraint's Representations

- Relation: allowed tuples

X	Y	Z
1	3	2
2	1	3

- Algebraic expression: $X + Y^2 \leq 10, X \neq Y$
- Propositional formula: $(a \vee b) \rightarrow \neg c$
- Semantics: by a relation

Partial Solutions

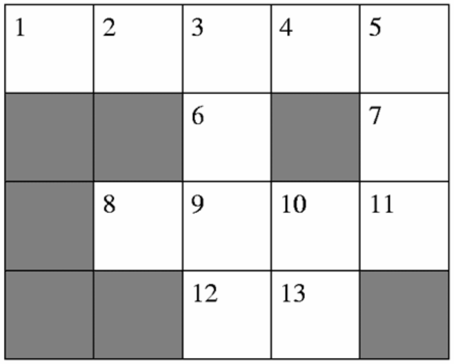


Not all partial consistent instantiations are part of a solution: (a) A partial consistent instantiation that is not part of a solution. (b) The placement of the queens corresponding to the solution (2, 4, 1, 3). (c) The placement of the queens corresponding to the solution (3, 1, 4, 2).

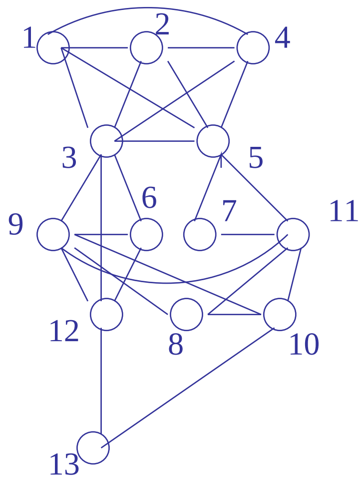
Constraint Graphs:

Primal, dual and hypergraphs

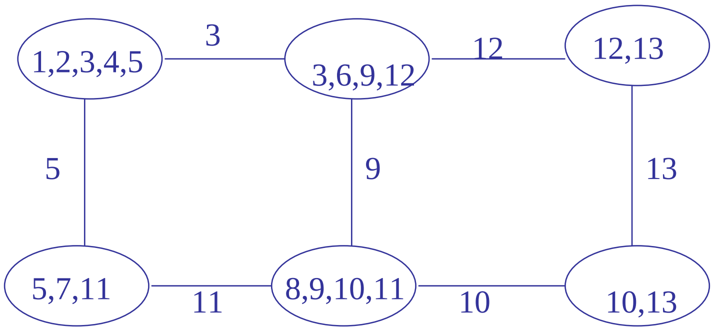
CSP: When defining variables as squares:



A (primal) constraint graph: a node per variable arcs connect constrained variables.
A dual constraint graph: a node per constraint's scope, an arc connect nodes sharing variables =hypergraph



(a)

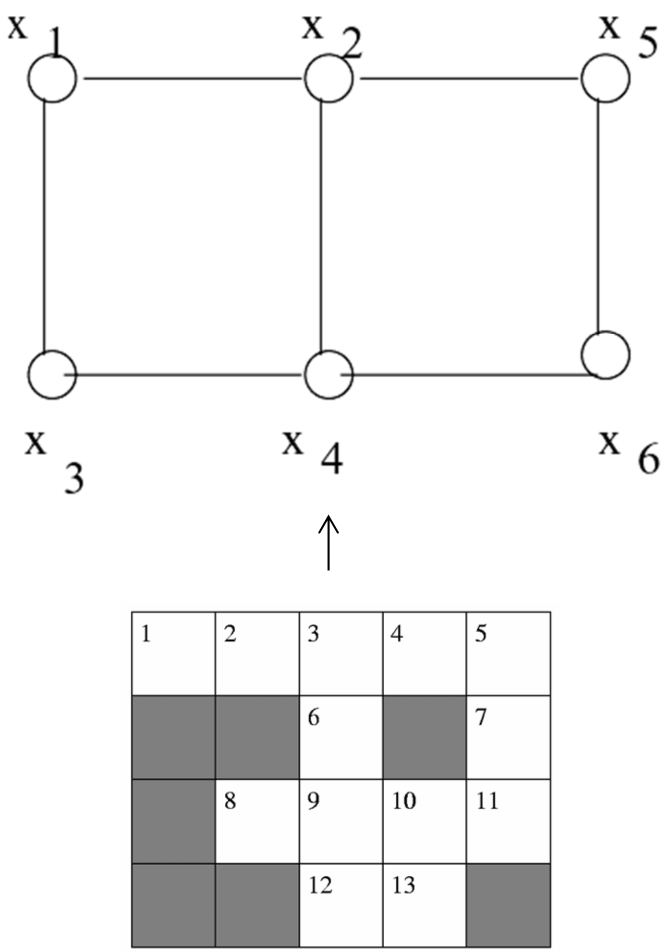


(b)

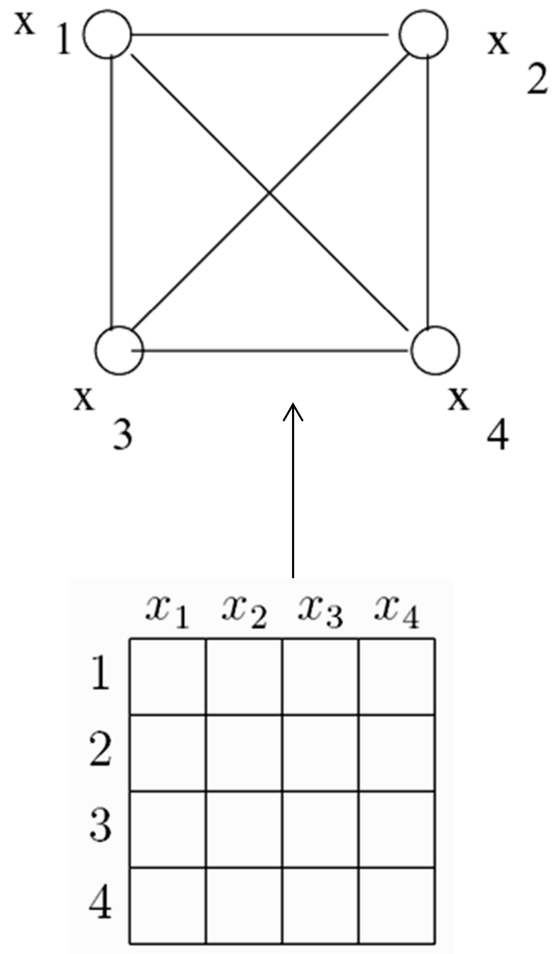
Primal graph?

Dual graph?

Constraint Graphs (primal)



When variables are words



Queen problem

Graph Concepts

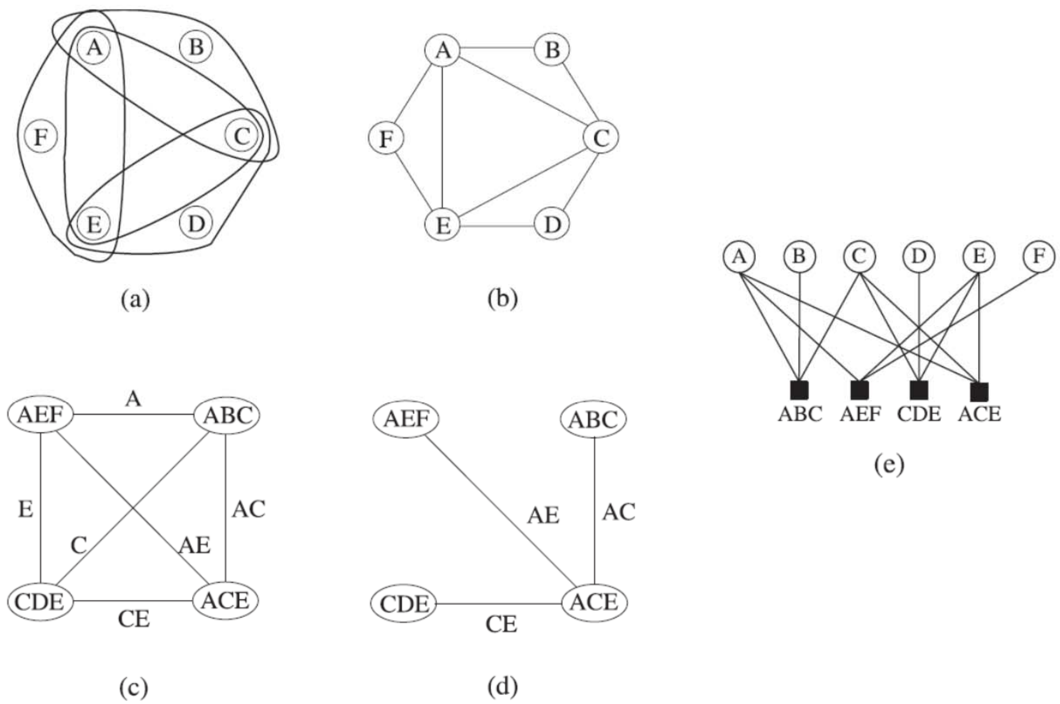
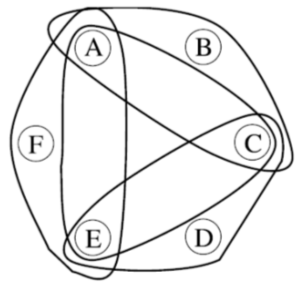


Figure 2.1: (a) Hyper; (b) primal; (c) dual; (d) join-tree of a graphical model having scopes ABC, AEF, CDE and ACE; and (e) the factor graph.

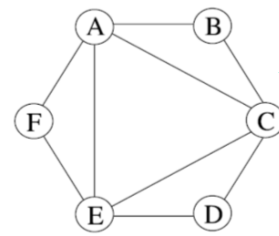
Graph Concepts Reviews:

Hyper Graphs and Dual Graphs

A hypergraph

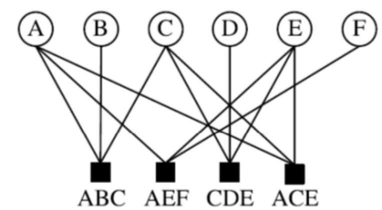


(a)



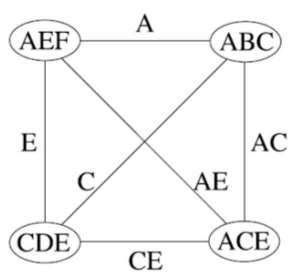
(b)

Primal graphs



(e)

Dual graph

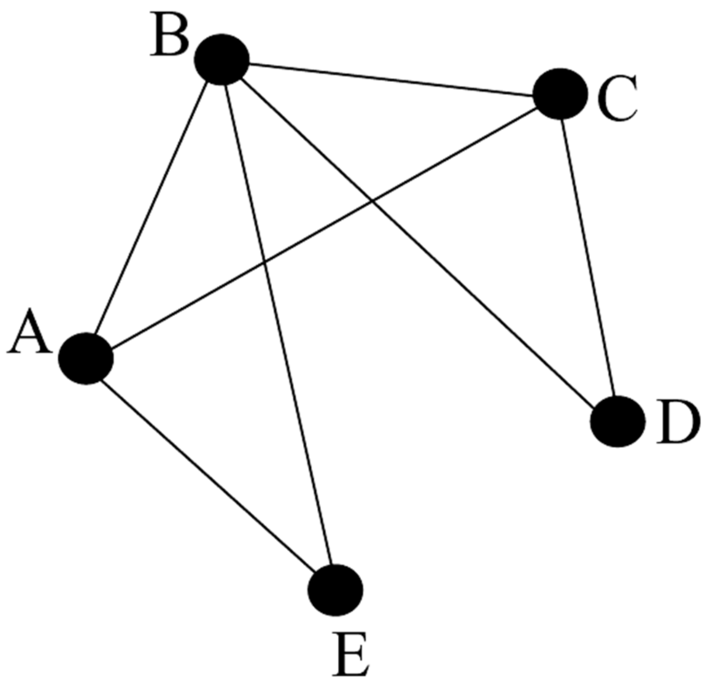


(c)

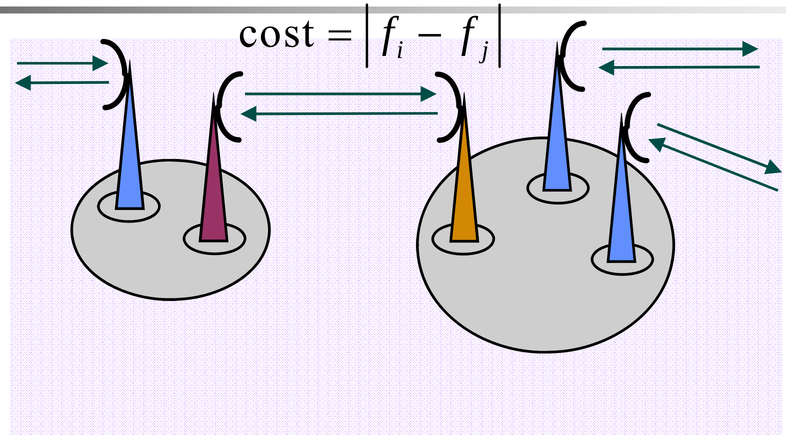
Factor graphs

Propositional Satisfiability

$\varphi = \{(\neg C), (A \vee B \vee C), (\neg A \vee B \vee E), (\neg B \vee C \vee D)\}$.



Example: Radio Link Assignment



Given a telecommunication network (where each communication link has various antenas) , assign a frequency to each antenna in such a way that all antennas may operate together without noticeable interference.

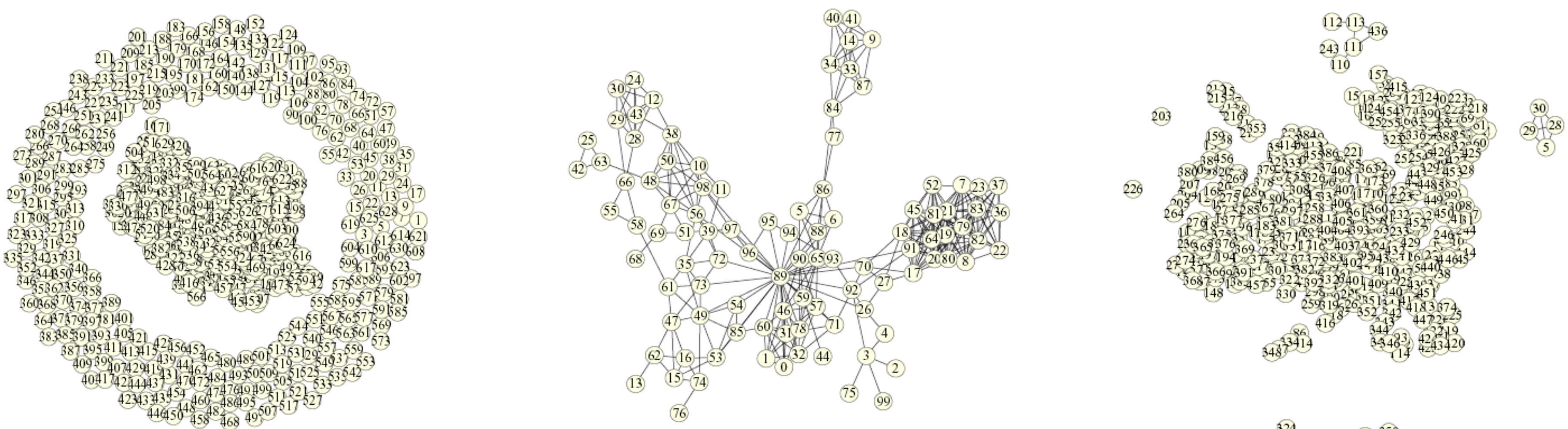
Encoding?

Variables: one for each antenna

Domains: the set of available frequencies

Constraints: the ones referring to the antennas in the same communication link

Constraint graphs of 3 instances of the Radio frequency assignment problem in CELAR's benchmark



Operations With Relations

- Intersection
- Union
- Difference
- Selection
- Projection
- Join
- Composition

Local Functions

Combination $f \bowtie g$

- Join :

x_1	x_2
a	a
b	b

$\bowtie$

x_2	x_3
a	a
a	b
b	a

$=$

x_1	x_2	x_3
a	a	a
a	a	b
b	b	a

$f \wedge g$

- Logical AND:

x_1	x_2	f
a	a	true
a	b	false
b	a	false
b	b	true

$\wedge$

x_2	x_3	g
a	a	true
a	b	true
b	a	true
b	b	false

$=$

x_1	x_2	x_3	h
a	a	a	true
a	a	b	true
a	b	a	false
a	b	b	false
b	a	a	false
b	a	b	false
b	b	a	true
b	b	b	false

Global View of the Problem

C_1

x_1	x_2
a	a
b	b

$\bowtie$

C_2

x_2	x_3
a	a
a	b
b	a

=

Global View

x_1	x_2	x_3
a	a	a
a	a	b
b	b	a

TASK

Does the problem a solution?

The problem has a solution if the global view is not empty

x_1	x_2	x_3	h
a	a	a	true
a	a	b	true
a	b	a	false
a	b	b	false
b	a	a	false
b	a	b	false
b	b	a	true
b	b	b	false

The problem has a solution if there is some true tuple in the global view, the universal relation

Example of Selection, Projection and Join

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s

(a) Relation R

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

(b) Relation R'

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

(c) Relation R''

x_1	x_2	x_3
b	b	c
c	b	c

(a) $\sigma_{x_3=c}(R')$

x_2	x_3
b	c
n	n

(b) $\pi_{\{x_2,x_3\}}(R')$

x_1	x_2	x_3	x_4
b	b	c	2
b	b	c	3
c	b	c	2
c	b	c	3

(c) $R' \bowtie R''$

Global View of the Problem

C_1	
x_1	x_2
a	a
b	b

⋈

C_2	
x_2	x_3
a	a
a	b
b	a

=

Global View		
x_1	x_2	x_3
a	a	a
a	a	b
b	b	a

TASK

What about counting?

x_1	x_2	x_3	h
a	a	a	true
a	a	b	true
a	b	a	false
a	b	b	false
b	a	a	false
b	a	b	false
b	b	a	true
b	b	b	false

Number of true tuples

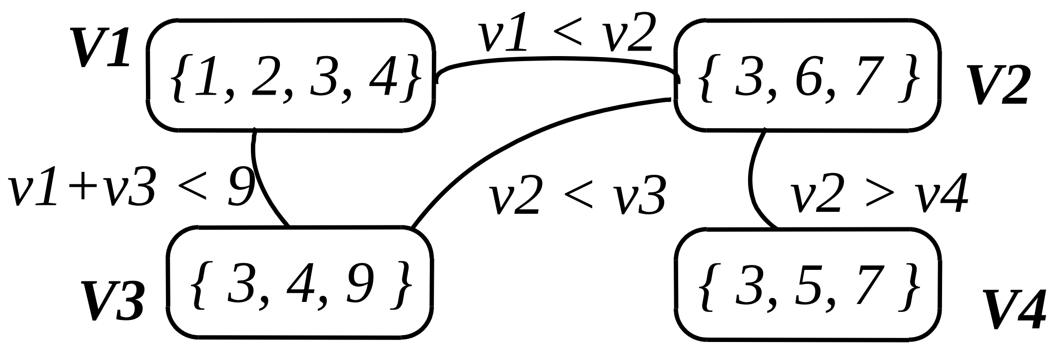
→ *true is 1*
false is 0
logical AND?

x_1	x_2	x_3	h
a	a	a	1
a	a	b	1
a	b	a	0
a	b	b	0
b	a	a	0
b	a	b	0
b	b	a	1
b	b	b	0

Sum over all the tuples

Examples

Numeric constraints



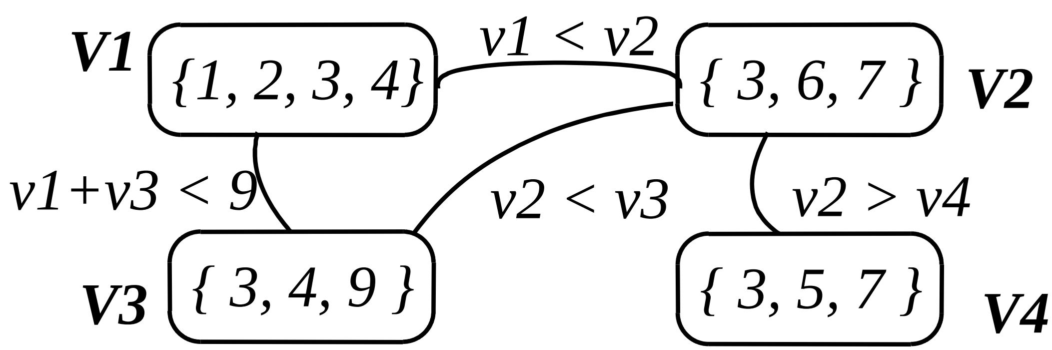
Can we specify numeric constraints as relations?

Numeric Constraints

- *Given* ***P*** *= (****V****,* ***D****,* ***C*** *), where*

$$\mathbf{V} = \{V_1, V_2, \ldots, V_n\}$$
$$\mathbf{D} = \{D_{V1}, D_{V2}, \ldots, D_{Vn}\}$$
$$\mathbf{C} = \{C_1, C_2, \ldots, C_l\}$$

Example I:



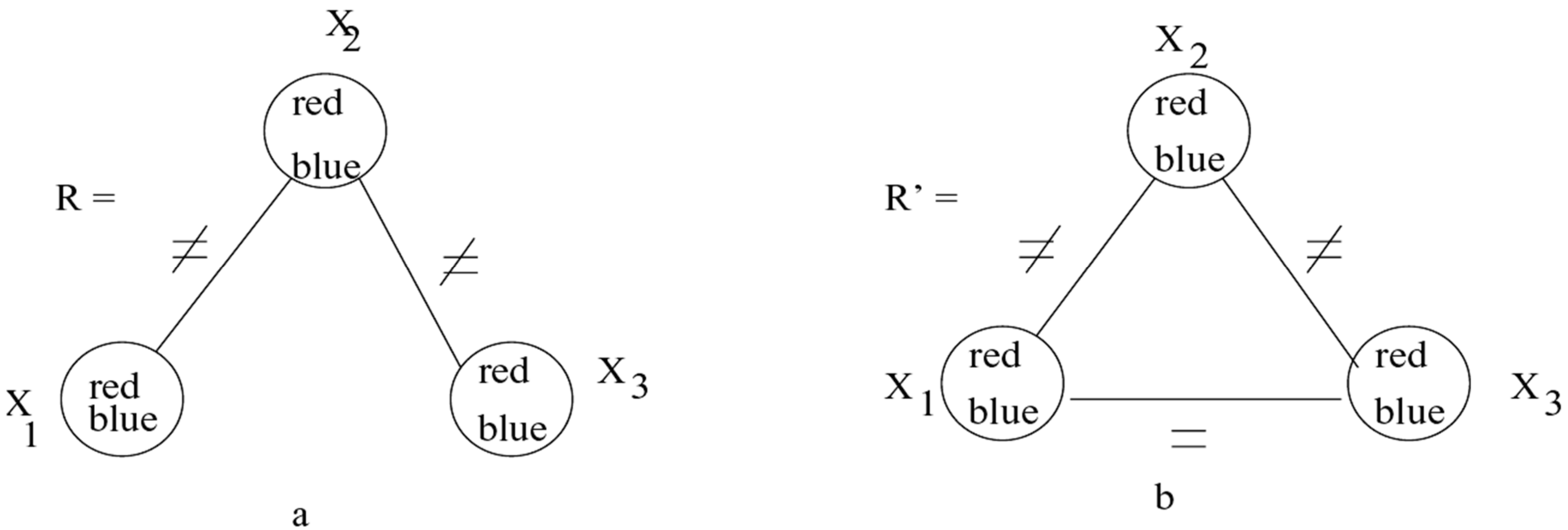
- *Define* ***C*** *?*

The minimal network,
An extreme case of re-parameterization

Binary Constraint Networks

Properties of Binary Constraint Networks

A graph ℜ to be colored by two colors,
an equivalent representation ℜ' having a newly inferred constraint between x1 and x3.



Equivalence and deduction with constraints (composition)

Equivalence, Redundancy, Composition

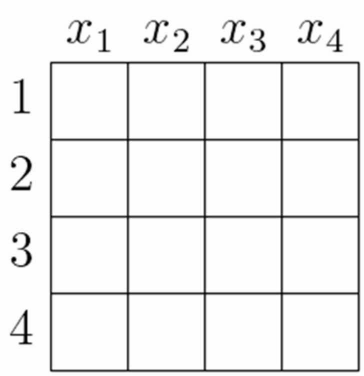
- Equivalence: Two constraint networks are equivalent if they have the same set of solutions.

- Composition in relational operation

$$R_{xz} = \pi_{xz}(R_{xy} \bowtie R_{yz})$$

The N-queens Constraint Network

The network has four variables, all with domains Di = {1, 2, 3, 4}.
(a) The labeled chess board. (b) The constraints between variables.



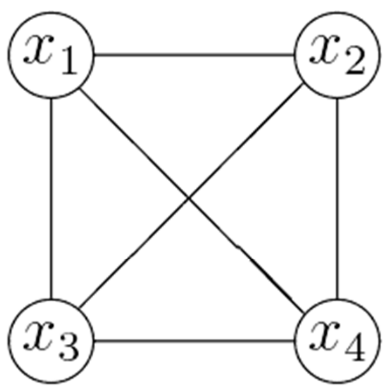
(a)

$$
\begin{aligned}
R_{12} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}\\
R_{13} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\}\\
R_{14} &= \{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (2,4),\ (3,1),\ (3,2),\ (3,4)\\
&\qquad (4,2),\ (4,3)\}\\
R_{23} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}\\
R_{24} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\}\\
R_{34} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}
\end{aligned}
$$

(b)

Solutions are: (2,4,1,3) (3,1,4,2)

The 4-queens constraint network



(a)

$$
\begin{aligned}
M_{12} &= \{(2,4),\ (3,1)\} \\
M_{13} &= \{(2,1),\ (3,4)\} \\
M_{14} &= \{(2,3),\ (3,2)\} \\
M_{23} &= \{(1,4),\ (4,1)\} \\
M_{24} &= \{(1,2),\ (4,3)\} \\
M_{34} &= \{(1,3),\ (4,2)\}
\end{aligned}
$$

The minimal network

(b)

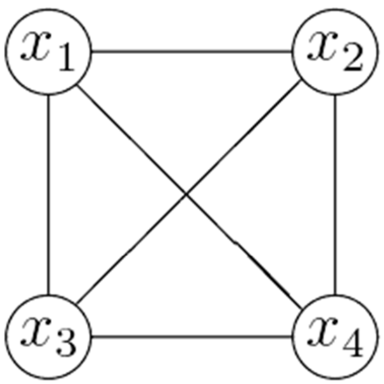
$$
\begin{aligned}
D_1 &= \{2,3\} \\
D_2 &= \{1,4\} \\
D_3 &= \{1,4\} \\
D_4 &= \{2,3\}
\end{aligned}
$$

The minimal domains

(c)

Solutions are: (2,4,1,3) (3,1,4,2)

The 4-queen problem



The constraint graph

(a)

$$M_{12} = \{(2,4),\ (3,1)\}$$
$$M_{13} = \{(2,1),\ (3,4)\}$$
$$M_{14} = \{(2,3),\ (3,2)\}$$
$$M_{23} = \{(1,4),\ (4,1)\}$$
$$M_{24} = \{(1,2),\ (4,3)\}$$
$$M_{34} = \{(1,3),\ (4,2)\}$$

The minimal constraints

(b)

$$D_1 = \{1,3\}$$
$$D_2 = \{1,4\}$$
$$D_3 = \{1,4\}$$
$$D_4 = \{1,3\}$$

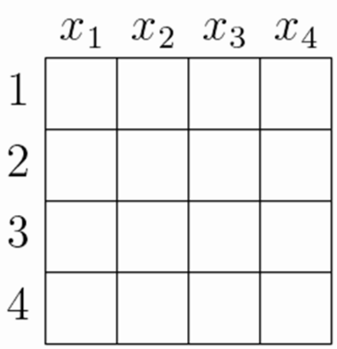
The minimal domains

(c)

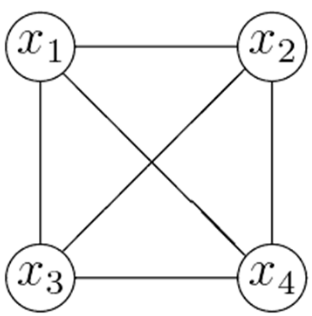
Solutions are: (2,4,1,3) (3,1,4,2)

The 4-queens problem

Solutions are: (2,4,1,3) (3,1,4,2)



$$
\begin{aligned}
R_{12} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{13} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{14} &= \{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (2,4),\ (3,1),\ (3,2),\ (3,4) \\
&\quad (4,2),\ (4,3)\} \\
R_{23} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{24} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{34} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}
\end{aligned}
$$

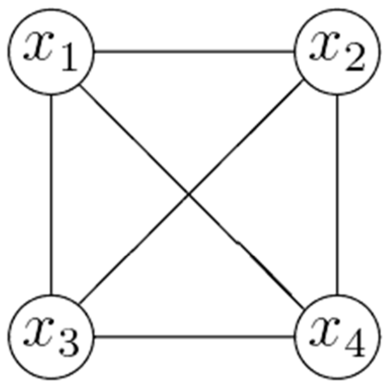


$$
\begin{aligned}
M_{12} &= \{(2,4),\ (3,1)\} \\
M_{13} &= \{(2,1),\ (3,4)\} \\
M_{14} &= \{(2,3),\ (3,2)\} \\
M_{23} &= \{(1,4),\ (4,1)\} \\
M_{24} &= \{(1,2),\ (4,3)\} \\
M_{34} &= \{(1,3),\ (4,2)\}
\end{aligned}
$$

$$
\begin{aligned}
D_1 &= \{1,3\} \\
D_2 &= \{1,4\} \\
D_3 &= \{1,4\} \\
D_4 &= \{1,3\}
\end{aligned}
$$

(a) (b) (c)

Figure 2.11: The 4-queens constraint network: (a) The constraint graph. (b) The minimal binary constraints. (c) The minimal unary constraints (the domains).



(a)

$$
\begin{aligned}
M_{12} &= \{(2,4),\ (3,1)\} \\
M_{13} &= \{(2,1),\ (3,4)\} \\
M_{14} &= \{(2,3),\ (3,2)\} \\
M_{23} &= \{(1,4),\ (4,1)\} \\
M_{24} &= \{(1,2),\ (4,3)\} \\
M_{34} &= \{(1,3),\ (4,2)\}
\end{aligned}
$$

(b)

$$
\begin{aligned}
D_1 &= \{1,3\} \\
D_2 &= \{1,4\} \\
D_3 &= \{1,4\} \\
D_4 &= \{1,3\}
\end{aligned}
$$

(c)

Solutions are: (2,4,1,3) (3,1,4,2)

The Minimal Network

- The minimal network is perfectly explicit for binary and unary constraints:
  - Every pair of values permitted by the minimal constraint is in a solution.

The Projection Networks

- The projection network of a relation is obtained by projecting it onto each pair of its variables (yielding a binary network).
- *Relation = {(1,1,2)(1,2,2)(1,2,1)}*
  - *What is the projection network?*
- What is the relationship between a relation and its projection network?
- {(1,1,2)(1,2,2)(2,1,3)(2,2,2)} are the solutions of its projection network?

Example: Sudoku

What is the minimal network?
The projection network?

Constraint propagation

		2	4		6			
8	6	5	1			2		
	1				8	6		9
9				4		8	6	
	4	7				1	9	
	5	8		6				3
4		6	9				7	2 ~~3~~ ~~4~~ ~~6~~
		9			4	5	8	1
			3		2	9		

- ***Variables**: 81 slots*
- ***Domains** = {1,2,3,4,5,6,7,8,9}*
- ***Constraints**:*
  - ***27 not-equal***

Each row, column and major block must be alldifferent

"Well posed" if it has unique solution: 27 constraints

Algorithms for Reasoning with graphical models

Class3

Rina Dechter