Algorithms for Reasoning with graphical models

Slides Set 10:

Bounded Inference Non-iteratively; Mini-Bucket Elimination

Rina Dechter

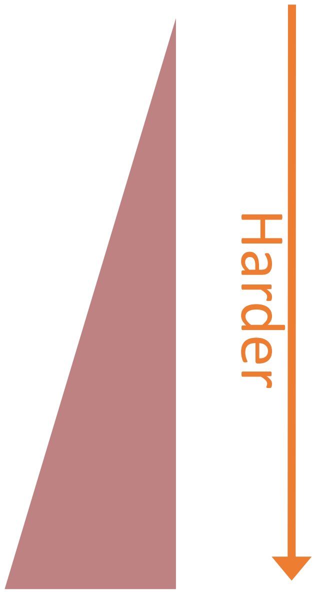
(Class Notes (8-9), Darwiche chapter 14

Outline

- Mini-bucket elimination
- Weighted Mini-bucket
- Mini-clustering
- Re-parameterization, cost-shifting
- Iterative Belief propagation
- Iterative-join-graph propagation

Types of queries

Max-Inference	$f(\mathbf{x}^*) = \max_{\mathbf{x}} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$
Sum-Inference	$Z = \sum_{\mathbf{x}} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$
Mixed-Inference	$f(\mathbf{x}_M^*) = \max_{\mathbf{x}_M} \sum_{\mathbf{x}_S} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$



- **NP-hard**: exponentially many terms
- We will focus on **approximation** algorithms
  - **Anytime**: very fast & very approximate ! Slower & more accurate

Queries

- **Probability of evidence (or partition function)**

$$P(e) = \sum_{X - \text{var}(e)} \prod_{i=1}^{n} P(x_i \mid pa_i)|_e \qquad Z = \sum_X \prod_i \psi_i(C_i)$$

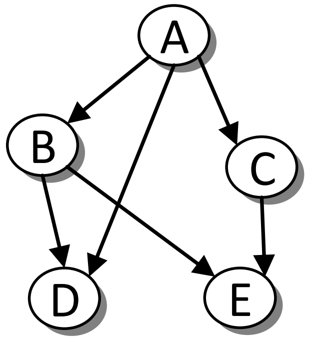
- **Posterior marginal (beliefs):**

$$P(x_i \mid e) = \frac{P(x_i, e)}{P(e)} = \frac{\sum_{X - \text{var}(e) - X_i} \prod_{j=1}^{n} P(x_j \mid pa_j)|_e}{\sum_{X - \text{var}(e)} \prod_{j=1}^{n} P(x_j \mid pa_j)|_e}$$

- **Most Probable Explanation**

$$\bar{x}^* = \arg\max_{\bar{x}} P(\bar{x}, e)$$

Bucket Elimination



Query: $P(a \mid e=0) \propto P(a, e=0)$ Elimination Order: d,e,b,c

$$P(a, e=0) = \sum_{c,b,e=0,d} P(a)P(b \mid a)P(c \mid a)P(d \mid a,b)P(e \mid b,c)$$

$$= P(a)\sum_{c} P(c \mid a)\sum_{b} P(b \mid a)\sum_{e=0} P(e \mid b,c)\sum_{d} P(d \mid a,b)$$

Original Functions

D: $P(d \mid a,b)$

E: $P(e \mid b,c)$

B: $P(b \mid a)$

C: $P(c \mid a)$

A: $P(a)$

Messages

$\boxed{f_D(a,b)} = \sum_{d} P(d \mid a,b)$

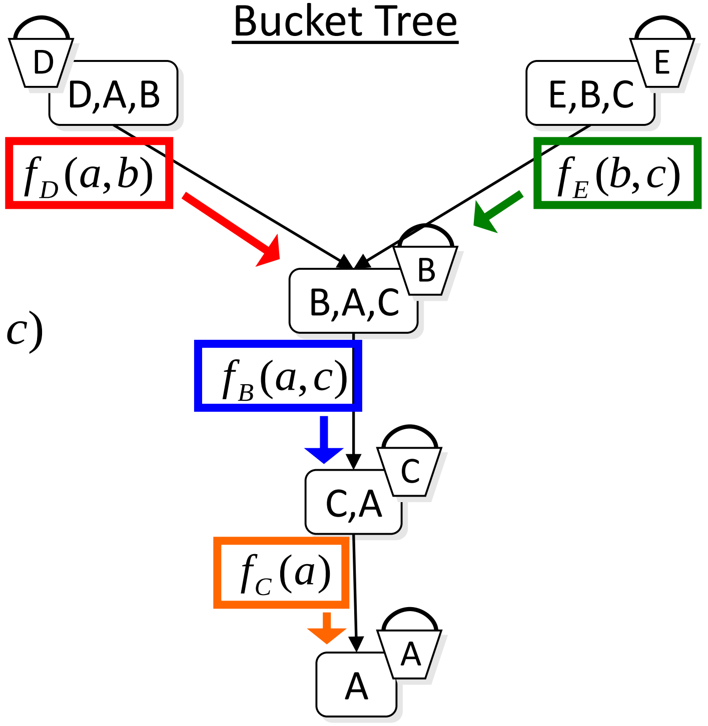
$\boxed{f_E(b,c)} = P(e=0 \mid b,c)$

$\boxed{f_B(a,c)} = \sum_{b} P(b \mid a)\, f_D(a,b)\, f_E(b,c)$

$\boxed{f_C(a)} = \sum_{c} P(c \mid a)\, f_B(a,c)$

$P(a, e=0) = p(A)\, f_C(a)$

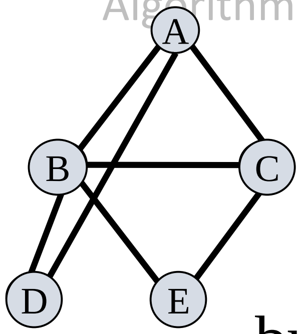
Bucket Tree



Time and space exp(w*)

Finding MPE/MAP

Algorithm BE-mpe (Dechter 1996, Bertele and



$$\text{MPE} = \max_{a,e,d,c,b} p(a)\,p(c|a)\,p(b|a)\,p(\ldots$$

$$= \max_{b}\, p(b|a) \cdot p(d|b,a) \cdot p(e|b,c)$$

$$\overbrace{}^{\max_x \prod}$$

bucket B: $p(b|a)\;\; p(d|b,a)\;\; p(e|b,c)$

bucket C: $p(c|a)\;\;\; \lambda_{B\to C}\underbrace{(a,d,c,e)}$

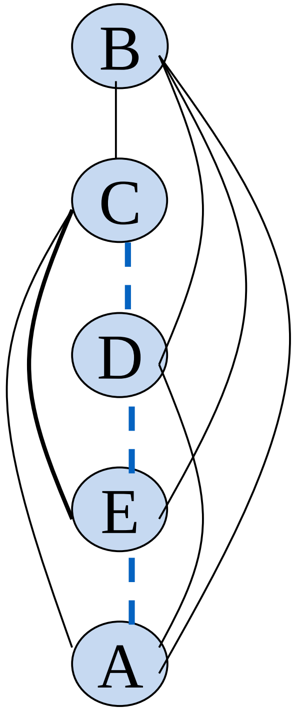
bucket D: $\lambda_{C\to D}(a,d,e)$

bucket E: $\mathbb{1}[e=0]\;\; \lambda_{D\to E}(a,e)$

bucket A: $p(a)\;\;\; \lambda_{E\to A}(a)$

OPT

W*=4
"induced width"
(max clique size)



Generating the Optimal Assignment

- Given BE messages, select optimum config in reverse order

$$\mathbf{b}^* = \arg\max_b \; p(b|a^*)\, p(d^*|b,a^*)\, p(e^*|b,c^*)$$

$$\mathbf{c}^* = \arg\max_c \; p(c|a^*)\, \lambda_{B\to C}(a^*, c, d^*, e^*)$$

$$\mathbf{d}^* = \arg\max_d \; \lambda_{C\to D}(a^*, d, e^*)$$

$$\mathbf{e}^* = \arg\max_e \; \mathbb{1}[e=0]\, \lambda_{D\to E}(a^*, e)$$

$$\mathbf{a}^* = \arg\max_a \; p(a) \cdot \lambda_{E\to A}(a)$$

Return optimal configuration $(a^*, b^*, c^*, d^*, e^*)$

B: $\underbrace{p(b|a)\; p(d|b,a)\; p(e|b,c)}$

C: $\underbrace{p(c|a) \qquad \lambda_{B\to C}(a,c,d,e)}$

D: $\underbrace{\lambda_{C\to D}(a,d,e)}$

E: $\underbrace{\mathbb{1}[e=0] \quad \lambda_{D\to E}(a,e)}$

A: $\underbrace{p(a)\; \lambda_{E\to A}(a)}$

OPT = optimal value

Approximate Inference

- Metrics of evaluation
- **Absolute error**: given e>0 and a query p= P(x|e), an estimate r has absolute error e iff |p-r|<e
- **Relative error**: the ratio r/p in [1-e,1+e].
- Dagum and Luby 1993: approximation up to a relative error is NP-hard.
- Absolute error is also NP-hard if error is less than .5

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Mini-Buckets: "Local Inference"

- Computation in a bucket is time and space exponential in the number of variables involved
- Therefore, partition functions in a bucket into "mini-buckets" on smaller number of variables

Decomposition Bounds

- Upper & lower bounds via approximate problem decomposition
- Example: MAP inference $F(x) = f_1(x) + f_2(x)$

X	F (X)
0	1.0
1	4.0
2	6.0
3	0.0

=

X	f_1(X)
0	1.0
1	2.0
2	3.0
3	4.0

×

X	f_2(X)
0	1.0
1	2.0
2	2.0
3	0.0

$$\max_x F(x) \quad = \quad \max_x \left[f_1(x) \boxed{\times} f_2(x) \right]$$

$$4.0 \quad \leq \quad \left[\max_x f_1(x) \; \boxed{\times} \; \max_x f_2(x) \right] \quad = \; 4.0 \boxed{\times} 2.0 \; = \; 8.0$$

- Relaxation: two "copies" of x, no longer required to be equal
- Bound is tight (equality) if f_1, f_2 agree on maximizing value x

Mini-Bucket Approximation

Split a bucket into mini-buckets —> bound complexity

bucket (X) =

$$\left\{\; \underbrace{f_1,\; f_2,\; \dots\; f_r,\; f_{r+1},\; \dots\; f_n}_{} \;\right\}$$

$$\lambda_X(\cdot) = \max_x \prod_{i=1}^{n} f_i(x, \dots)$$

$$\left\{\; f_1,\; \dots\; f_r \;\right\} \qquad \left\{\; f_{r+1},\; \dots\; f_n \;\right\}$$

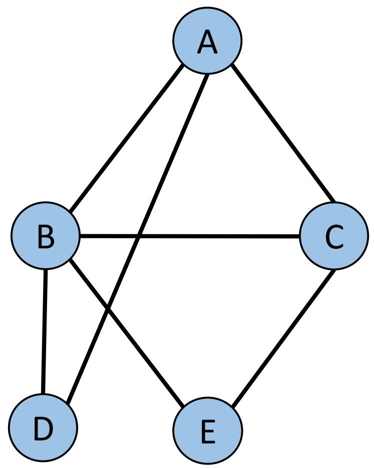
$$\lambda_{X,1}(\cdot) = \max_x \prod_{i=1}^{r} f_i(x, \dots) \qquad \lambda_{X,2}(\cdot) = \max_x \prod_{i=r+1}^{n} f_i(x, \dots)$$

$$\lambda_X(\cdot) \;\leq\; \lambda_{X,1}(\cdot)\; \lambda_{X,2}(\cdot)$$

Exponential complexity decrease: $O(e^n) \longrightarrow O(e^r) + O(e^{n-r})$

Mini-Bucket Elimination

[Dechter & Rish 2003]



mini-buckets

bucket B: $P(e|b,c)$ $P(d|a,b)\ p(b|a)$

bucket C: $P(c|a)$ $\lambda_{B\to C}(e,c)$

bucket D: $\lambda_{B\to D}(a,d)$

bucket E: $\lambda_{C\to E}(a,e)$ $e=0$

bucket A: $P(a)$ $\lambda_{E\to A}(a)$ $\lambda_{D\to A}(a)$

U = upper bound

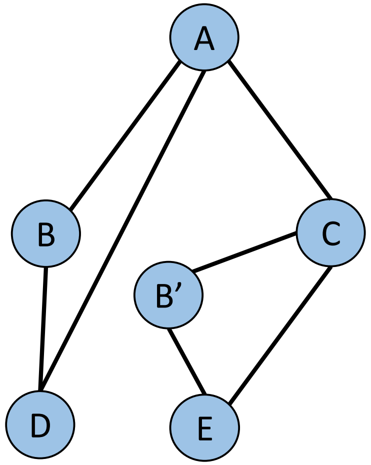
$$\lambda_{B\to D}(a,d) = max_b\ P(d|a,b)\ p(b|a)$$

$$\lambda_{B\to C}(e,c) = max_b P(e|b,c)$$

$$\lambda_{B\to D}(a,d) = max_d\ ....$$

Mini-Bucket Elimination

[Dechter & Rish 2003]



mini-buckets

bucket B: $P(e|b',c)$ $P(d|a,b)\ p(b|a)$

bucket C: $P(c|a)$ $\lambda_{B\to C}(e,c)$

bucket D: $\lambda_{B\to D}(a,d)$

bucket E: $\lambda_{C\to E}(a,e)$ $e=0$

bucket A: $P(a)$ $\lambda_{E\to A}(a)$ $\lambda_{D\to A}(a)$

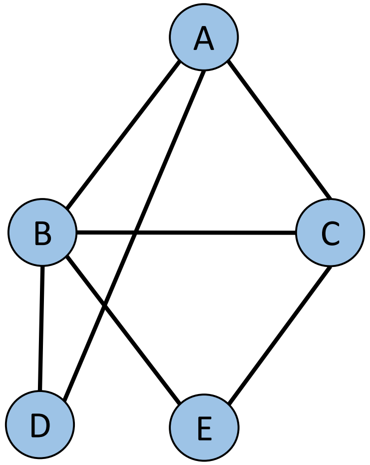
U = upper bound

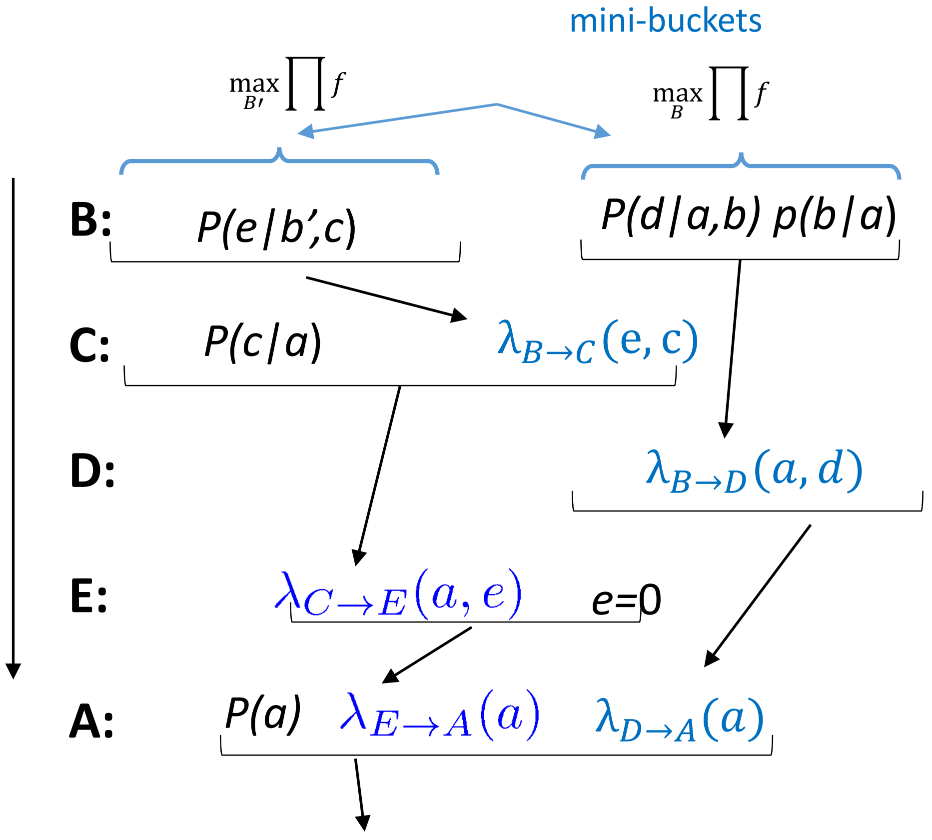
$$\lambda_{B\to D}(a,d) = max_b\ P(d|a,b)\ p(b|a)$$

$$\lambda_{B\to C}(e,c) = max_b P(e|b,c)$$

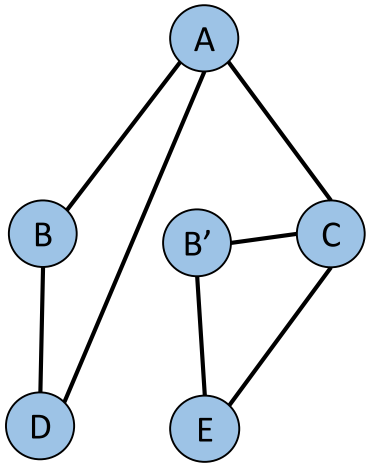
$$\lambda_{B\to D}(a,d) = max_d\ ....$$

Mini-Bucket Elimination





U = upper bound



Model relaxation:
[Kask *et al.*, 2001]
[Geffner *et al.*, 2007]
[Choi *et al.*, 2007]
[Johnson et al. 2007]

[Dechter and Rish, 1997; 2003]

Mini-Bucket Decoding

- Assign values in reverse order using approximate messages

$b^* = \arg\max_b P(e^*|b, c^*)P(d|a^*, b)\, p(b|a^*)$

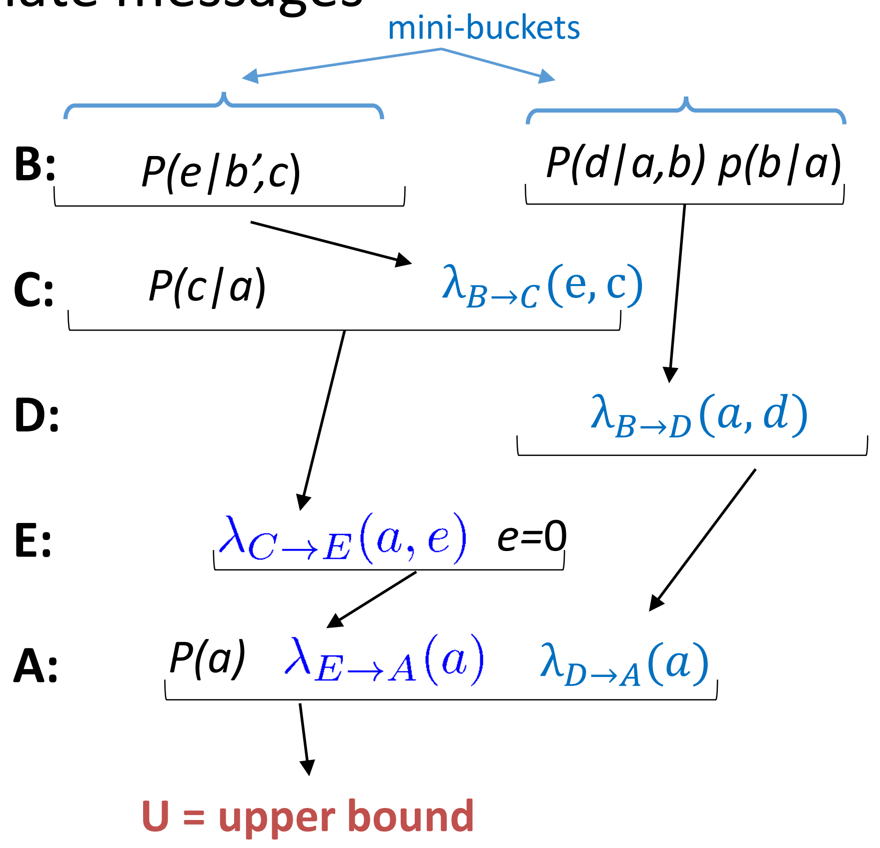
$c^* = \arg\max_e \lambda_{B\to C}(e^*, c)$

$d^* = \arg\max_d \lambda_{B\to D}(a^*, d)$

$e^* = 0$

$a^* = \arg\max_a P(a)\ \lambda_{E\to A}(a)\, \lambda_{D\to A}(a)$

Greedy configuration = lower bound

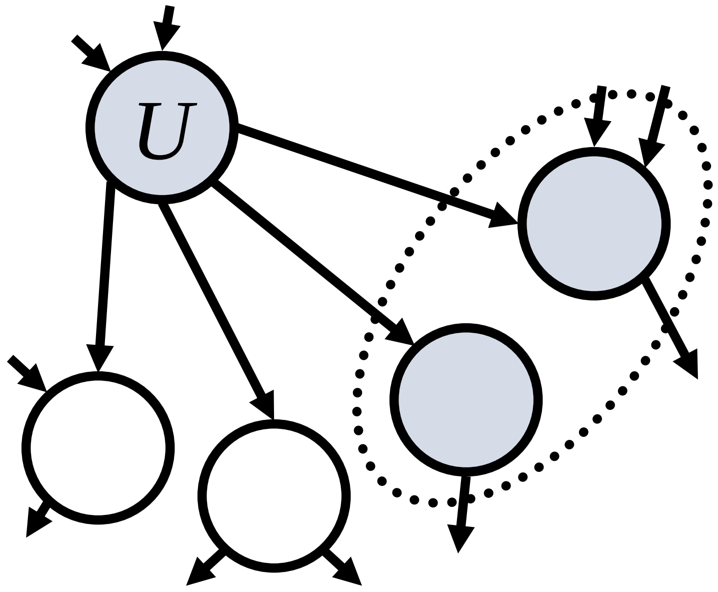


$return(a^*, e^*, d^*, c^*, b^*)$

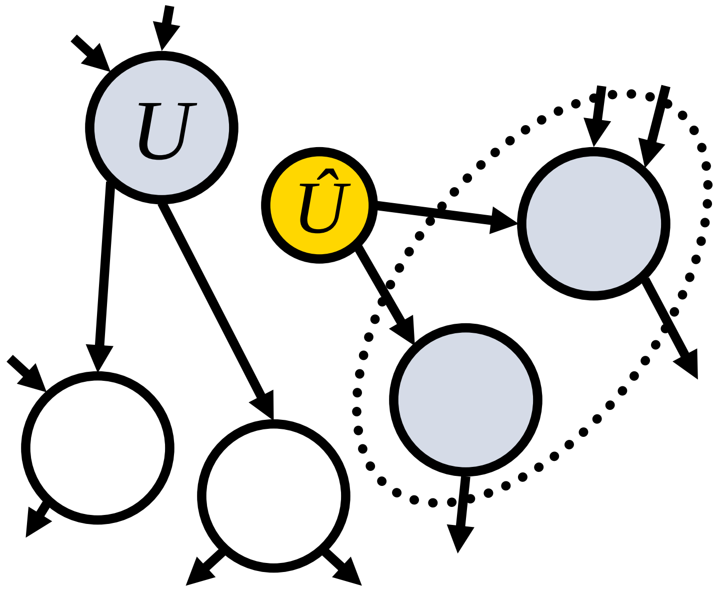
Semantics of Mini-Bucket: Splitting a Node

Variables in different buckets are renamed and duplicated
(Kask *et. al.*, 2001), (Geffner *et. al.*, 2007), (Choi, Chavira, Darwiche , 2007)

Before Splitting:
Network *N*



After Splitting:
Network *N'*



MBE-MPE(i): Algorithm MBE-mpe

- **Input**: I – max number of variables allowed in a mini-bucket
- **Output**: [lower bound (P of suboptimal solution), upper bound]

Example: MBE-mpe(3) versus **BE-mpe**

	MBE-mpe(3)		max variables in a mini-bucket		BE-mpe
B:	$P(e \mid b',c)$	$P(d \mid a,b)\ p(b \mid a)$	3:	B:	$p(b \mid a)\ p(d \mid b,a)\ p(e \mid b,c)$
C:	$P(c \mid a)$ $\lambda_{B\to C}(e,c)$		3:	C:	$p(c \mid a)\ \lambda_{B\to C}(a,d,c,e)$
D:		$\lambda_{B\to D}(a,d)$	3:	D:	$\lambda_{C\to D}(a,d,e)$
E:	$\lambda_{C\to E}(a,e)$ e=0		2:	E:	$\mathbb{1}[e=0]\ \ \lambda_{D\to E}(a,e)$
A:	$P(a)\ \ \lambda_{E\to A}(a)\ \ \lambda_{D\to A}(a)$		1:	A:	$p(a)\ \ \lambda_{E\to A}(a)$

U = Upper bound $w^* = 2$

OPT w^* =4

Mini-Bucket Decoding (for min-sum)

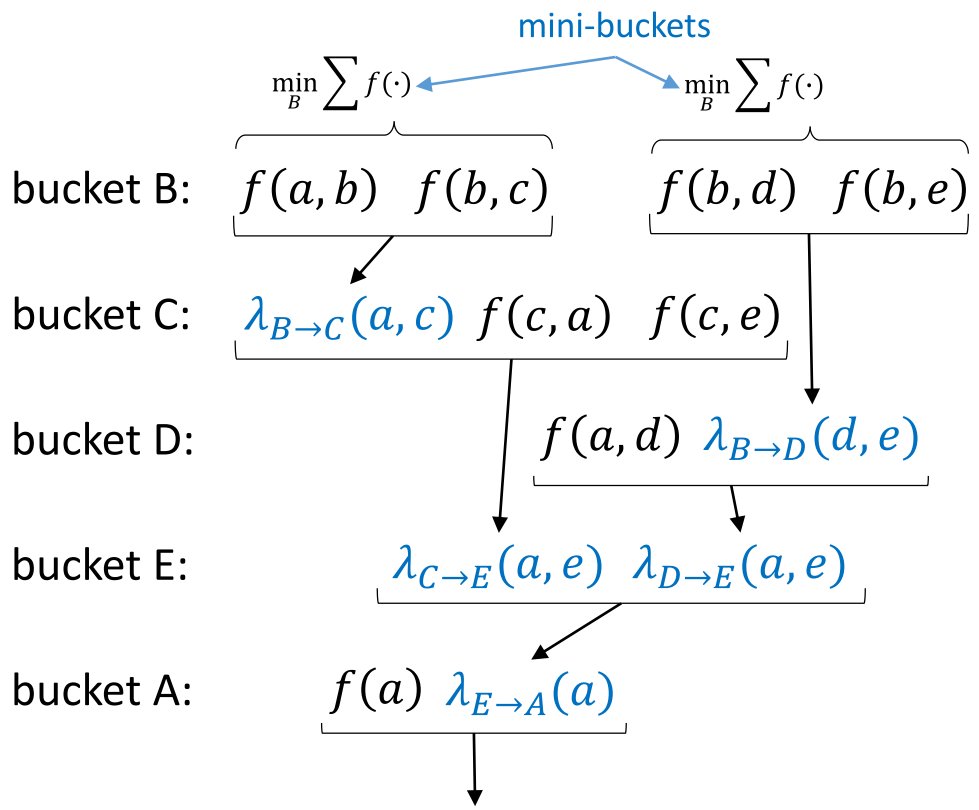
$$\hat{b} = \arg\min_b f(\hat{a}, b) + f(b, \hat{c}) + f(b, \hat{d}) + f(b, \hat{e})$$

$$\hat{c} = \arg\min_c \lambda_{B\to C}(\hat{a}, c) + f(c, \hat{a}) + f(c, \hat{e})$$

$$\hat{d} = \arg\min_d f(\hat{a}, d) + \lambda_{B\to D}(d, \hat{e})$$

$$\hat{e} = \arg\min_e \lambda_{C\to E}(\hat{a}, e) + \lambda_{D\to E}(\hat{a}, e)$$

$$\hat{a} = \arg\min_a f(a) + \lambda_{E\to A}(a)$$



Greedy configuration = upper bound

L = lower bound

[Dechter and Rish, 2003]

(i,m)-Patitionings

Definition 7.1.1 ((i,m)-partitioning) *Let H be a collection of functions $h_1, ..., h_t$ defined on scopes $S_1, ..., S_t$, respectively. We say that a function f is* subsumed *by a function h if any argument of f is also an argument of h. A partitioning of $h_1, ..., h_t$ is* canonical *if any function f subsumed by another function is placed into the bucket of one of those subsuming functions. A partitioning Q into mini-buckets is an (i, m)-partitioning if and only if (1) it is canonical, (2) at most m non-subsumed functions are included in each mini-bucket, (3) the total number of variables in a mini-bucket does not exceed i, and (4) the partitioning is* refinement-maximal, *namely, there is no other (i, m)-partitioning that it refines.*

MBE(i,m), MBE(i)

- Input: Belief network ($P_1,\ldots,P_n$)
- Output: upper and lower bounds
- Initialize: put functions in buckets along ordering
- Process each bucket from p=n to 1
  - Create (i,m)-partitions
  - Process each mini-bucket
- (For mpe): assign values in ordering d
- Return: mpe-configuration, upper and lower bounds

Algorithm MBE-mpe(i,m)

Input: A belief network $B =< X, D, G, \mathcal{P}_G, \prod >$, where $\mathcal{P} = \{P_1, ..., P_n\}$; an ordering of the variables, $d = X_1, ..., X_n$; observations $\mathbf{e}$.

Output: An upper bound U and a lower bound L on the most probable configuration given the evidence. A suboptimal solution $\bar{x}^a$ that provides the lower bound $L = P(\bar{x}^a)$.

1. **Initialize:** Generate an ordered partition of the conditional probability function, $bucket_1$, . . ., $bucket_n$, where $bucket_i$ contains all functions whose highest variable is X_i. Put each observed variable in its bucket.

2. **Backward:** For $p \leftarrow n$ downto 1, do
for all the functions $h_1, h_2, ..., h_j$ in $bucket_p$, do

- **If** (observed variable) $bucket_p$ contains $X_p = x_p$, assign $X_p = x_p$ to each function and put each in appropriate bucket.
- **else**, Generate an an (i, m)-partitioning, $Q^{'} = \{Q_1, ..., Q_r\}$ of $h_1, h_2, ..., h_t$ in $bucket_p$.
- **for each** $Q_l \in Q^{'}$ containing $h_{l_1}, ...h_{l_t}$, **do**

$$h_l \leftarrow max_{X_p} \prod_{j=1}^{t} h_{l_j} \tag{8.1}$$

Add h_l to the bucket of the largest-index in $scope(h_l)$. Put constants in $bucket_1$.

3. **Forward:**

- Compute an mpe cost by maximizing over X_1, the product in $bucket_1$. Namely $U \leftarrow max_{X_1} \Pi_{h_j \in bucket_1} h_j$.
- (Generate an approximate mpe tuple): Given $\mathbf{x}_{(1...(i-1))} = (x_1, \ldots, x_{i-1})$ choose $x_i = argmax_{X_i} \Pi_{\{h_j \in\ bucket_i\}} h_j(\mathbf{x}_{(1...(i-1))})$. $L \leftarrow P(x_1, ..., x_n)$

4. **Output** U and L and configuration: $\bar{x} = (x_1, ..., x_n)$

Figure 8.2: Algorithm *MBE-mpe(i,m)*.

Partitioning, Refinements

Clearly, as the mini-buckets get smaller, both complexity and accuracy decrease.

Definition 7.1.4 *Given two partitionings Q' and Q'' over the same set of elements, Q' is a refinement of Q'' if and only if for every set $A \in Q'$ there exists a set $B \in Q''$ such that $A \subseteq B$.*

It is easy to see that:

Proposition 7.1.5 *If Q'' is a refinement of Q' in $bucket_p$, then $h^p \leq g^p_{Q'} \leq g^p_{Q''}$.*

Properties of MBE(i)

- **Complexity**: O(r exp(i)) time and O(exp(i)) space
- Yields a lower bound and an upper bound
- **Accuracy**: determined by upper/lower (U/L) bound
- Possible use of mini-bucket approximations
  - As anytime algorithms
  - As heuristics in search
- Other tasks (similar mini-bucket approximations)
  - Belief updating, Marginal MAP, MEU, WCSP, Max-CSP

  [Dechter and Rish, 1997], [Liu and Ihler, 2011], [Liu and Ihler, 2013]

Anytime Approximation

Algorithm anytime-mpe(ϵ)
Input: Initial values of i and m, i_0 and m_0; increments i_{step} and m_{step}, and desired approximation error ϵ.
Output: U and L
1. **Initialize:** $i = i_0, m = m_0$.
2. **do**
3. run *mbe-mpe(i,m)*
4. $U \leftarrow$ upper bound of *mbe-mpe(i,m)*
5. $L \leftarrow$ lower bound of *mbe-mpe(i,m)*
6. Retain best bounds U, L, and best solution found so far
7. **if** $1 \leq U/L \leq 1+\epsilon$, return solution
8. **else** increase i and m: $i \leftarrow i + i_{step}$ and $m \leftarrow m + m_{step}$
9. **while** computational resources are available
10. **Return** the largest L
and the smallest U found so far.

MBE for Belief Updating and for Probability of Evidence or Partition Function

- Idea mini-bucket is the same:

$$\sum_X f(x) \bullet g(x) \leq \sum_X f(x) \bullet \sum_X g(x)$$

$$\sum_X f(x) \bullet g(x) \leq \sum_X f(x) \bullet \max_X g(X)$$

- So we can apply a sum in each mini-bucket, or better, one sum and the rest max, or min (for lower-bound)

- MBE-bel-max(i,m), MBE-bel-min(i,m) generating upper and lower-bound on beliefs approximates BE-bel

- MBE-map(i,m): max mini-buckets will be maximized, sum mini-buckets will be sum-max. Approximates BE-map.

Algorithm MBE-bel-max(i,m)
Input: A belief network $\mathcal{B} = \langle \mathbf{X}, \mathbf{D}, \mathbf{P}_G, \prod \rangle$, an ordering $d = (\ X_1, \ldots, X_n)$; evidence e
Output: an upper bound on $P(X_1, \bar{e})$ and an upper bound on $P(e)$.
1. **Initialize:** Partition $P = \{P_1, ..., P_n\}$ into buckets $bucket_1, \ldots, bucket_n$, where $bucket_k$ contains all CPTs $h_1, h_2, ..., h_t$ whose highest-index variable is X_k.

2. **Backward:** for $k = n$ to 2 do

- **If** X_p is observed $(X_k = a)$, assign $X_k \leftarrow a$ in each h_j and put the result in the highest-variable bucket of its scope (put constants in $bucket_1$).
- **Else** for $h_1, h_2, ..., h_t$ in $bucket_k$ Generate an (i,m)-partitioning, $Q^{'} = \{Q_1, ..., Q_r\}$. **For each** $Q_l \in Q^{'}$, containing $h_{l_1}, ...h_{l_t}$, do

$$h_l \leftarrow \sum_{X_k} \Pi_{j=1}^{t} h_{1_j},, \ if \ l = 1$$

$$h_l \leftarrow max_{X_k} \Pi_{j=1}^{t} h_{l_j}, \ if \ k \neq 1$$

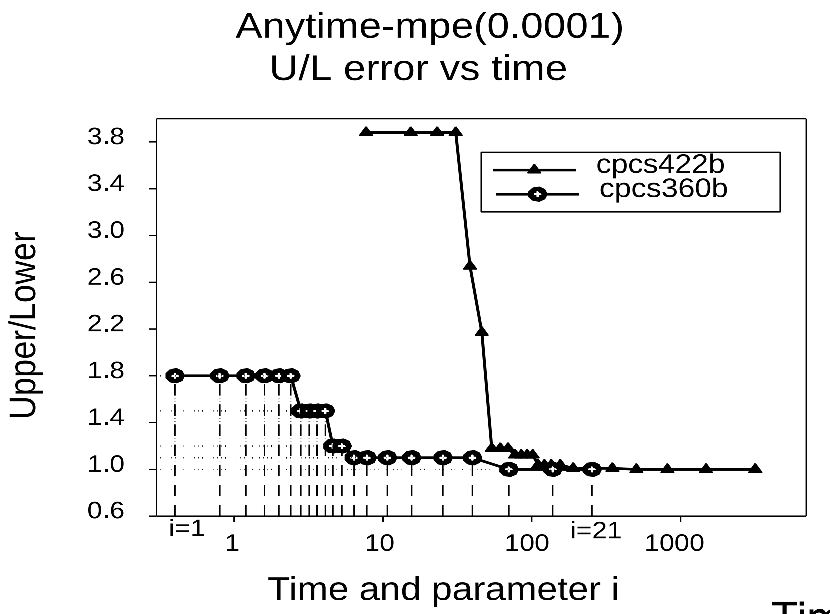
Add h_l to the bucket of the highest-index variable in its scope $\bigcup_{j=1}^{t} scope(h_{l_j}) - \{X_k\}$. (put constant functions in $bucket_1$).

3. **Return** $P'(\bar{x_1}, e) \leftarrow$ the product of functions in the bucket of X_1, which is an upper bound on $P(x_1, \bar{e})$.
$P'(e) \leftarrow \sum_{x_1} P'(\bar{x_1}, e)$, which is an upper bound on probability of evidence.

Figure 8.5: Algorithm *MBE-bel-max(i,m)*.

CPCS Networks – Medical Diagnosis (noisy-OR model)

Test case: no evidence



Algorithm	Time (sec) cpcs360	cpcs422
elim-mpe	115.8	1697.6
anytime-mpe(ε), $\varepsilon = 10^{-4}$	70.3	505.2
anytime-mpe(ε), $\varepsilon = 10^{-1}$	70.3	110.5

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Decomposition for Sum

$$F(x) = f_1(x) \cdot f_2(x)$$

$$\sum_x f_1(x) \cdot f_2(x) \quad \leq \quad \left[\sum_x f_1(x)^{\frac{1}{w_1}}\right]^{w_1} \cdot \left[\sum_x f_2(x)^{\frac{1}{w_2}}\right]^{w_2}$$

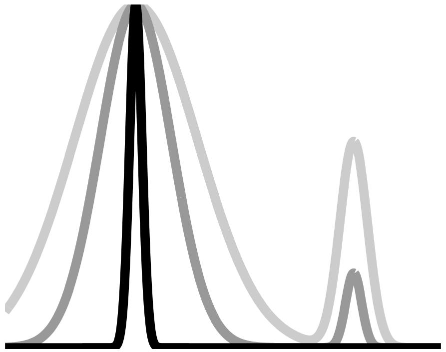
- Generalize technique to sum via Holder's inequality:

$$w_1 + w_2 = 1$$

$$\sum_{x_1}^{w_1} f(x_1) = \left[\sum_{x_1} f(x_1)^{\frac{1}{w_1}}\right]^{w_1}$$

- Define the weighted (or powered) sum:

$$\sum_{x_1}^{w_1}\sum_{x_2}^{w_2} f(x_1, x_2) \neq \sum_{x_2}^{w_2}\sum_{x_1}^{w_1} f(x_1, x_2)$$



- "Temperature" interpolates between sum & max:

$$\lim_{w \to 0^+} \sum_x^{w} f(x) = \max_x f(x)$$

- Different weights do not commute:

The Power Sum and Holder Inequality

The power sum is defined as follows:

$$\sum_x^w f(x) = (\sum_x f(x)^{\frac{1}{w}})^w \tag{1.2}$$

where w is a non-negative weight. The power sum reduce to a standard summation when $w = 1$ and approaches max when $w \to 0^+$.

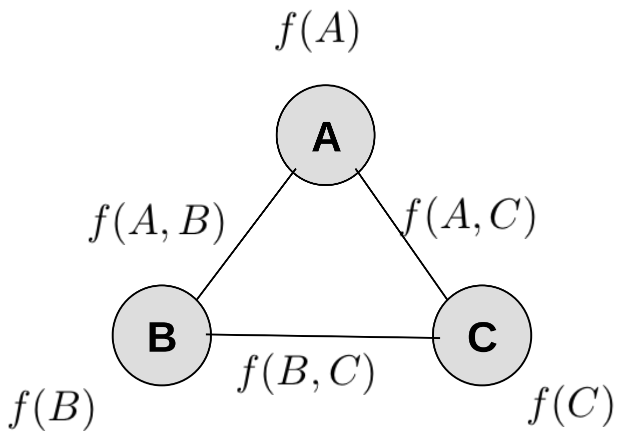
[Holder inequality] Let $f_i(x)$, $i = 1..r$ be a set of functions and $w_1, ..., w_r$ b a set of of non-zero weights, s.t., $w = \sum_{i=1}^r w_i$ then,

$$\sum_x^w \prod_{i=1}^r f_i(x) \leq \prod_{i=1}^r \sum_x^{w_i} f_i(x)$$

Working Example

- Model:
  - Markov network
- Task:
  - Partition function

$$Z = \sum_{A,B,C} f(A)f(B)f(C)f(A,B)f(A,C)f(B,C)$$



(Qiang Liu slides)

Mini-Bucket (Basic Principles)

- Upper bound

$$\sum_i a_i b_i \leq (\sum_i a_i) \max_i(b_i)$$

- Lower bound

$$\sum_i a_i b_i \geq (\sum_i a_i) \min_i(b_i)$$

(Qiang Liu slides)

I am using a_i b_i to represent the general constant.

Holder Inequality

$$\sum_i a_i b_i \leq \left(\sum_i a_i^{1/w_1}\right)^{w_1} \left(\sum_i b_i^{1/w_2}\right)^{w_2}$$

- Where $a_i > 0, b_i > 0$ and $w_1 + w_2 = 1$ $w_1 > 0, w_2 > 0$

- When $\frac{a_i^{1/w_1}}{\sum a_i^{1/w_1}} = \frac{b_i^{1/w_2}}{\sum b_i^{1/w_2}}$ the equality is achieved.

(Qiang Liu slides)

G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities, Cambridge Univ. Press, London and New York, 1934.*

Reverse Holder Inequality

- If $w_1 + w_2 = 1$, but $w_1 < 0, w_2 > 1$

the direction of the inequality reverses.

$$\sum_i a_i b_i \geq \left(\sum_i a_i^{1/w_1}\right)^{w_1} \left(\sum_i b_i^{1/w_2}\right)^{w_2}$$

(Qiang Liu slides)

G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities, Cambridge Univ. Press, London and New York, 1934.*

Weighted Mini-Bucket

(for summation)

Exact bucket elimination:

$$\lambda_B(a,c,d,e) = \sum_b [f(a,b)\cdot f(b,c)\cdot f(b,d)\cdot f(b,e)]$$

$$\leq \left[\sum_b^{w_1} f(a,b)f(b,c)\right]\cdot\left[\sum_b^{w_2} f(b,d)f(b,e)\right]$$

$$= \lambda_{B\to C}\,(a,c)\cdot\ \lambda_{B\to D}\,(d,e)$$

(mini-buckets)

where $\sum_x^{w} f(x) = \left[\sum_x f(x)^{\frac{1}{w}}\right]^w$

is the weighted or "power" sum operator

$$\sum_x^{w} f_1(x)f_2(x) \leq \left[\sum_x^{w_1} f_1(x)\right]\left[\sum_x^{w_2} f_2(x)\right]$$

where $w_1 + w_2 = w$ and $w_1 > 0, w_2 > 0$

(lower bound if $w_1 > 0, w_2 < 0$)

mini-buckets $\sum_x^{w}\prod f(x)$

bucket B: $\underbrace{f(a,b)\ \ f(b,c)}$ $\underbrace{f(b,d)\ \ f(b,e)}$

bucket C: $\lambda_{B\to C}(a,c)\ \ f(a,c)\ \ f(c,e)$

bucket D: $f(a,d)\ \ \lambda_{B\to D}(d,e)$

bucket E: $\lambda_{C\to E}(a,e)\ \ \lambda_{D\to E}(a,e)$

bucket A: $f(a)\ \ \lambda_{E\to A}(a)$

U = upper bound

[Liu and Ihler, 2011]

Algorithm Weighted WMBE(i,m), $(w_1, ...w_n)$

Input: A belief network $\mathcal{B} = \langle \mathbf{X}, \mathbf{D}, \mathbf{P}_G, \prod \rangle$, an ordering $d = (\ X_1, \ldots, X_n)$; evidence e

Output: an upper bound on $\sum_{\mathbf{X}}^{w} \prod_{i=1}^{n} P_i$

1. **Initialize:** Partition $P = \{P_1, ..., P_n\}$ into buckets $bucket_1$, ..., $bucket_n$, where $bucket_k$ contains all CPTs $h_1, h_2, ..., h_t$ whose highest-index variable is X_k.

2. **Backward:** for $k = n$ to 1 do

   - **If** X_p is observed $(X_k = a)$, assign $X_k \leftarrow a$ in each h_j and put the result in the highest-variable bucket of its scope (put constants in $bucket_1$).

   - **Else** for $h_1, h_2, ..., h_t$ in $bucket_k$ Generate an (i, m)-partitioning, $Q' = \{Q_1, ..., Q_r\}$. Select a set of weights $w_1, ...w_r$ s.t $\sum_l w_l = w$.
     For each $Q_l \in Q'$, containing $h_{l_1}, ...h_{l_t}$, do

$$h_l \leftarrow \sum_{X_k}^{w_l} \prod_{j=1}^{t} h_{l_j} = (\sum_{X_k} \prod_{j=1}^{t} (h_{l_j})^{w_l})^{\frac{1}{w_l}}$$

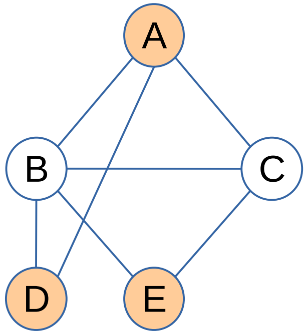
     Add h_l to the bucket of the highest-index variable in its scope (and put constant functions in $bucket_1$).

3. **Return** $U) \leftarrow$ the weighted product of functions in the bucket of X_1, which is an upper bound on $P(x_1, \mathrm{e})$.

Weighted-mini-bucket for Marginal Map

Bucket Elimination for MMAP

Bucket Elimination



$$\mathbf{X}_M = \{A, D, E\}$$
$$\mathbf{X}_S = \{B, C\}$$

$$\max_{\mathbf{X}_M} \sum_{\mathbf{X}_S} P(\mathbf{X})$$

constrained elimination order

SUM

B: $\underbrace{f(A,B)\, f(B,C)\, f(B,D)\, f(B,E)}$

Σ_B

C: $\underbrace{\lambda^B(A,C,D,E)\, f(A,C)\, f(C,E)}$

Σ_C

MAX

D: $\underbrace{\lambda^C(A,D,E)\, f(A,D)}$

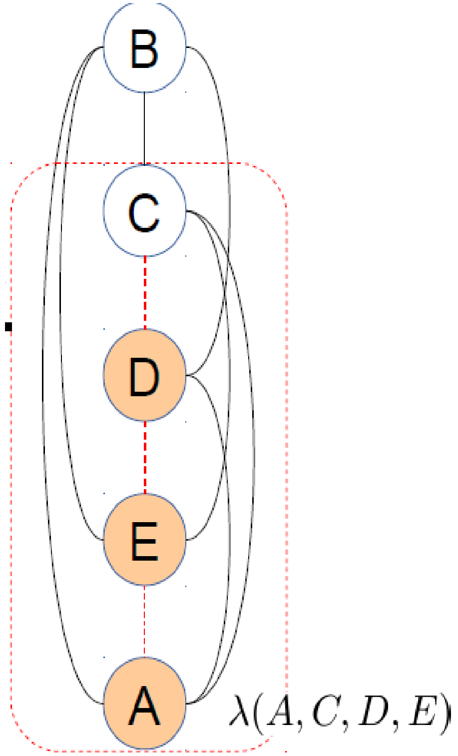
max_D

E: $\underbrace{\lambda^D(A,E)}$

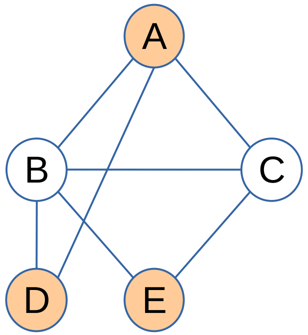
max_E

A: $\underbrace{\lambda^E(A)}$

MAP* is the marginal MAP value



MB and WMB for Marginal MAP



$\mathbf{X}_M = \{A, D, E\}$

$\mathbf{X}_S = \{B, C\}$

$$\max_{\mathbf{X}_M} \sum_{\mathbf{X}_S} P(\mathbf{X})$$

$$\lambda_{B\to C}(a,c) = \sum_b^{w_1} f(a,b) f(b,c)$$

$$\lambda_{B\to D}(d,e) = \sum_b^{w_2} f(b,d) f(b,e)$$

$$(w_1 + w_2 = 1)$$

$$\vdots$$

$$\lambda_{E\to A}(a) = \max_e \lambda_{C\to E}(a,e)\lambda_{D\to E}(a,e)$$

$$U = \max_a f(a)\lambda_{E\to A}(a)$$

Marginal MAP

mini-buckets

Σ_B	bucket B:	$f(a,b)\ \ f(b,c)$	$f(b,d)\ \ f(b,e)$
Σ_C	bucket C:	$\lambda_{B\to C}(a,c)\ \ f(a,c)\ \ f(c,e)$	
$\max_D$	bucket D:	$f(a,d)\ \ \lambda_{B\to D}(d,e)$	
$\max_E$	bucket E:	$\lambda_{C\to E}(a,e)\ \ \lambda_{D\to E}(a,e)$	
$\max_A$	bucket A:	$f(a)\ \ \lambda_{E\to A}(a)$	

U = upper bound

Can optimize over cost-shifting and weights
(single pass "MM" or iterative message passing)

[Liu and Ihler, 2011; 2013]
[Dechter and Rish, 2003]

MBE-map

Process max buckets
With max mini-buckets
And sum buckets with weighted
Mini-buckets

Algorithm MBE-map(i,m)

Input: A Bayesian network $\mathcal{B} = \langle \mathbf{X}, \mathbf{D}, \mathbf{P}_G, \prod \rangle$, $P = \{P_1, ..., P_n\}$; a subset of hypothesis variables $A = \{A_1, ..., A_k\}$; an ordering of the variables, d, in which the A's are first in the ordering; observations $\mathbf{e}$.

Output: An upper bound on the map and a and a suboptimal solution $A = \bar{a}_k^a$.

1. **Initialize:** Partition $P = \{P_1, ..., P_n\}$ into $bucket_1, \ldots, bucket_n$, where $bucket_i$ contains all functions whose highest variable is X_i.

2. **Backwards** For $p \leftarrow n$ downto 1, do

for all the functions $h_1, h_2, ..., h_j$ in $bucket_p$ do

- **If** (observed variable) $bucket_p$ contains the observation $X_p = x_p$, assign $X_p = x_p$ to each h_i and and put each in appropriate bucket.
- **Else** for $h_1, h_2, ..., h_j$ in $bucket_p$ generate an (i,m)-partitioning, $Q' = \{Q_1, ..., Q_r\}$.
- **If** $X_P \notin A$ assign $w_p = 1$, otherwise $w_p = 0$. Select weights for the mini-buckets in X_p bucket: $w_{p_1}, ..., w_{p_r}$. s.t $\sum_i w_{p_i} = w_p$.
  foreach $Q_l \in Q'$, containing $h_{l_1}, ...h_{l_t}$, do

$$h_l \leftarrow \sum_{X_k}^{w_{pl}} \prod_{j=1}^{t} h_{l_j} = \left(\sum_{X_k} \left(\prod_{j=1}^{t} h_{l_j}\right)^{w_{pl}}\right)^{\frac{1}{w_{pl}}}$$

  Add h_l to the bucket of the highest-index variable in its scope.

3. **Forward:** for $p = 1$ to k, given $A_1 = a_1^a, ..., A_{p-1} = a_{p-1}^a$, assign a value a_p^a to A_p that maximizes the product of all functions in $bucket_p$. conditioned on earlier assignments.

4. **Return** An upper bound $U = max_{a_1} \prod_{h_i \in bucket_1} h_i$ on the map value, computed in the first bucket, and the assignment $\bar{a}_k^a = (a_1^a, ..., a_k^a)$.

Figure 8.7: Algorithm *MBE-map(i,m)*.

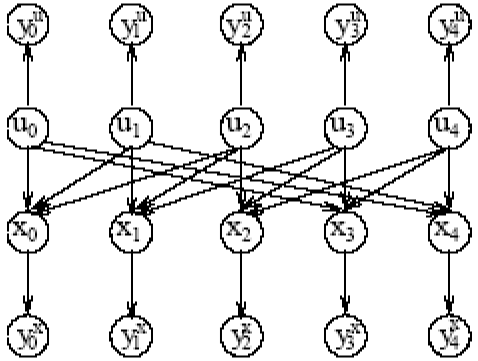


Figure 7.7: Belief network for a linear block code.

Example 7.3.1 We will next demonstrate the mini-bucket approximation for MAP on an example of *probabilistic decoding* (see Chapter 2) Consider a belief network which describes the decoding of a *linear block code*, shown in Figure 7.7. In this network, U_i are *information bits* and X_j are *code bits*, which are functionally dependent on U_i. The vector (U, X), called the channel input, is transmitted through a noisy channel which adds Gaussian noise and results in the channel output vector $Y = (Y^u, Y^x)$. The decoding task is to assess the most likely values for the U's given the observed values $Y = (\bar{y}^u, \bar{y}^x)$, which is the MAP task where U is the set of hypothesis variables, and $Y = (\bar{y}^u, \bar{y}^x)$ is the evidence. After processing the observed buckets we get the following bucket configuration (lower case y's are observed values):

$$\left.\begin{aligned}
&bucket(X_0) = P(y_0^x|X_0), P(X_0|U_0, U_1, U_2),\\
&bucket(X_1) = P(y_1^x|X_1), P(X_1|U_1, U_2, U_3),\\
&bucket(X_2) = P(y_2^x|X_2), P(X_2|U_2, U_3, U_4),\\
&bucket(X_3) = P(y_3^x|X_3), P(X_3|U_3, U_4, U_0),\\
&bucket(X_4) = P(y_4^x|X_4), P(X_4|U_4, U_0, U_1),\\
&bucket(U_0) = P(U_0), P(y_0^u|U_0),\\
&bucket(U_1) = P(U_1), P(y_1^u|U_1),\\
&bucket(U_2) = P(U_2), P(y_2^u|U_2),\\
&bucket(U_3) = P(U_3), P(y_3^u|U_3),\\
&bucket(U_4) = P(U_4), P(y_4^u|U_4).
\end{aligned}\right\}\ \text{Initial partitioning}$$

Processing by *mbe-map(4,1)* of the first top five buckets by summation and the rest by maximization, results in the following mini-bucket partitionings and function generation:

$bucket(X_0) = \{P(y_0^x|X_0), P(X_0|U_0, U_1, U_2)\},$
$bucket(X_1) = \{P(y_1^x|X_1), P(X_1|U_1, U_2, U_3)\},$
$bucket(X_2) = \{P(y_2^x|X_2), P(X_2|U_2, U_3, U_4)\},$
$bucket(X_3) = \{P(y_3^x|X_3), P(X_3|U_3, U_4, U_0)\},$
$bucket(X_4) = \{P(y_4^x|X_4), P(X_4|U_4, U_0, U_1)\},$
$bucket(U_0) = \boxed{\{P(U_0), P(y_0^u|U_0), h^{X_0}(U_0, U_1, U_2)\}}, \boxed{\{h^{X_3}(U_3, U_4, U_0)\}}, \boxed{\{h^{X_4}(U_4, U_0, U_1)\}},$
$bucket(U_1) = \{P(U_1), P(y_1^u|U_1), h^{X_1}(U_1, U_2, U_3), h^{U_0}(U_1, U_2)\}, \{h^{U_0}(U_4, U_1)\},$
$bucket(U_2) = \{P(U_2), P(y_2^u|U_2), h^{X_2}(U_2, U_3, U_4), h^{U_1}(U_2, U_3)\},$
$bucket(U_3) = \{P(U_3), P(y_3^u|U_3), h^{U_0}(U_3, U_4), h^{U_1}(U_3, U_4), h^{U_2}(U_3, U_4)\},$
$bucket(U_4) = \{P(U_4), P(y_4^u|U_4), h^{U_1}(U_4), h^{U_3}(U_4)\}.$

The first five buckets are not partitioned at all and are processed as full buckets, since in this case a full bucket is a (4,1)-partitioning. This processing generates five new functions, three are placed in bucket U_0, one in bucket U_1 and one in bucket U_2. Then bucket U_0 is partitioned into three mini-buckets processed by maximization, creating two functions placed in bucket U_1 and one function placed in bucket U_3. Bucket U_1 is partitioned into two mini-buckets, generating functions placed in bucket U_2 and bucket U_3. Subsequent buckets are processed as full buckets. Note that the scope of recorded functions is bounded by 3.

In the bucket of U_4 we get an upper bound U satisfying $U \geq MAP = P(U, \bar{y}^u, \bar{y}^x)$ where $\bar{y}^u$ and $,\bar{y}^x$ are the observed outputs for the U's and the X's bits transmitted. In order to bound $P(U|\bar{e})$, where $\bar{e} = (\bar{y}^u, \bar{y}^x)$, we need $P(\bar{e})$ which is not available. Yet, again, in most cases we are interested in the ratio $P(U = \bar{u}_1|\bar{e})/P(U = \bar{u}_2|\bar{e})$ for competing hypotheses $U = \bar{u}_1$ and $U = \bar{u}_2$ rather than in the absolute values. Since $P(U|\bar{e}) = P(U, \bar{e})/P(\bar{e})$ and the probability of the evidence is just a constant factor independent of U, the ratio is equal to $P(U_1, \bar{e})/P(U_2, \bar{e})$. $\square$

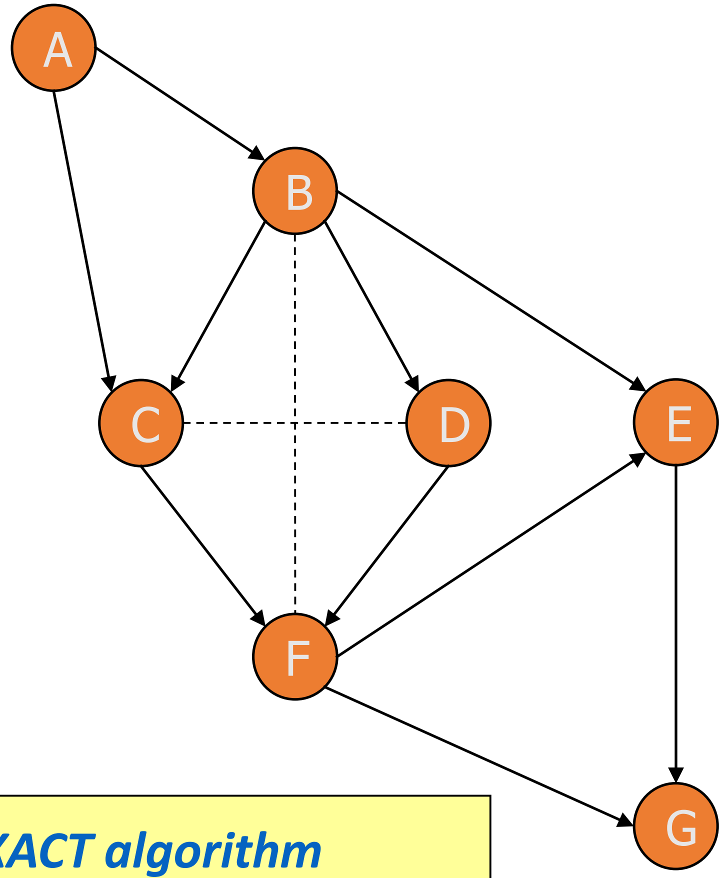
Complexity and Tractability of MBE(i,m)

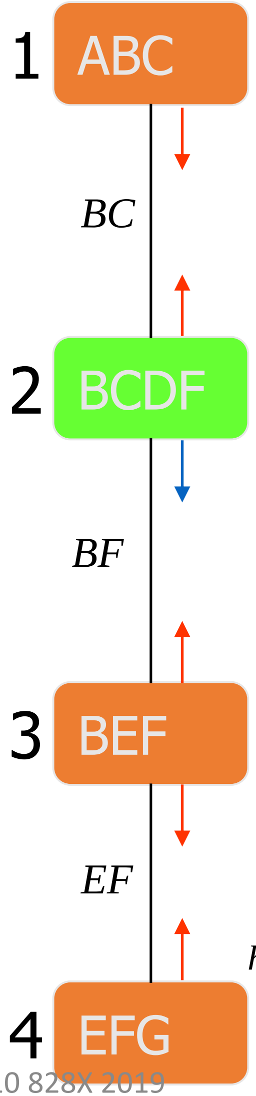
Theorem 8.10 *Algorithm* WMB(i,m) *takes* $O(r \cdot k^i)$ *time and space, where,* k *bounds the domain size and* r *is the number of input functions*[1]. *For* $m = 1$ *the algorithm is time and space linear and is bounded by* $O(r \cdot exp(|S|))$, *where* $|S|$ *is the maximum scope of any input function,* $|S| \leq i \leq n$.

Outline

- Mini-bucket elimination
- Weighted Mini-bucket
- Mini-clustering
- Re-parameterization, cost-shifting
- Iterative Belief propagation
- Iterative-join-graph propagation

Join-Tree Clustering (Cluster-Tree Elimination)





$$h_{(1,2)}(b,c) = \sum_{a} p(a) \cdot p(b \mid a) \cdot p(c \mid a,b)$$

$$h_{(2,1)}(b,c) = \sum_{d,f} p(d \mid b) \cdot p(f \mid c,d) \cdot h_{(3,2)}(b,f)$$

$$h_{(2,3)}(b,f) = \sum_{c,d} p(d \mid b) \cdot p(f \mid c,d) \cdot h_{(1,2)}(b,c)$$

$$h_{(3,2)}(b,f) = \sum_{e} p(e \mid b,f) \cdot h_{(4,3)}(e,f)$$

$$h_{(3,4)}(e,f) = \sum_{b} p(e \mid b,f) \cdot h_{(2,3)}(b,f)$$

$$h_{(4,3)}(e,f) = p(G = g_e \mid e,f)$$

EXACT algorithm

Time and space:
exp(cluster size)= exp(treewidth)

Mini-Clustering

Split a cluster into mini-clusters => bound complexity

$$\underbrace{\{h_1, \ldots, h_r, h_{r+1}, \ldots, h_n\}}$$

We can replace the sum with power sum
For weights that sum to 1 in each mini-bucket

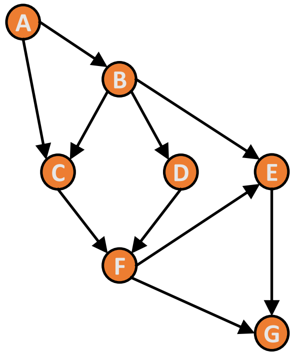
APPROXIMATE algorithm

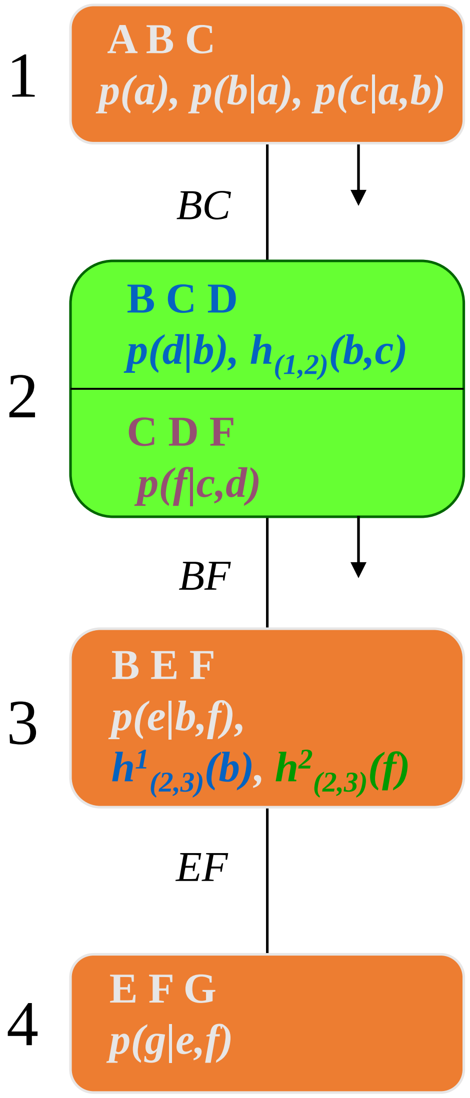
$$\underbrace{\{h_1, \ldots, h_r\}} \qquad \underbrace{\{h_{r+1}, \ldots, h_n\}}$$

$$\sum_{elim} \prod_{i=1}^{n} h_i \le \left(\sum_{elim} \prod_{i=1}^{r} h_i\right) \cdot \left(\sum_{elim} \prod_{i=r+1}^{n} h_i\right)$$

Exponential complexity decrease $O(e^n) \rightarrow O(e^{\mathrm{var}(r)}) + O(e^{\mathrm{var}(n-r)})$

Mini-Clustering, i-bound=3





$$h^1_{(1,2)}(b,c) = \sum_a p(a) \cdot p(b \mid a) \cdot p(c \mid a,b)$$

$$h^1_{(2,3)}(b) = \sum_{c,d} p(d \mid b) \cdot h^1_{(1,2)}(b,c)$$

$$h^2_{(2,3)}(f) = \max_{c,d} p(f \mid c,d)$$

APPROXIMATE algorithm

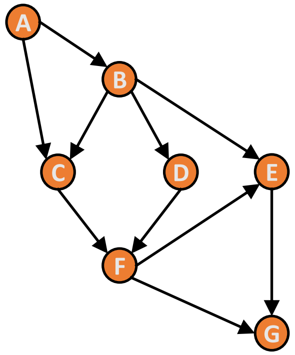
Time and space:

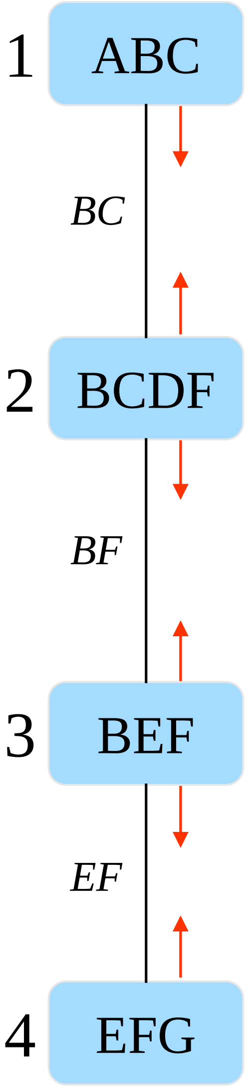
exp(i-bound)

$\downarrow$

Number of variables in a mini-cluster

Mini-Clustering - Example





$H_{(1,2)}$ $\quad h^1_{(1,2)}(b,c) := \sum_a p(a) \cdot p(b \mid a) \cdot p(c \mid a,b)$

$H_{(2,1)}$ $\quad h^1_{(2,1)}(b) := \sum_{d,f} p(d \mid b) \cdot h^1_{(3,2)}(b,f)$

$\quad h^2_{(2,1)}(c) := \max_{d,f} p(f \mid c,d)$

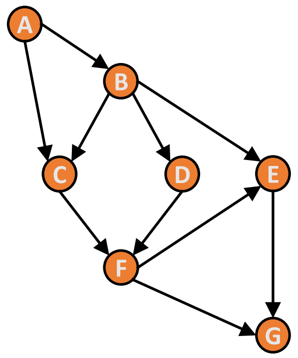
$H_{(2,3)}$ $\quad h^1_{(2,3)}(b) := \sum_{c,d} p(d \mid b) \cdot h^1_{(1,2)}(b,c)$

$\quad h^2_{(2,3)}(f) := \max_{c,d} p(f \mid c,d)$

$H_{(3,2)}$ $\quad h^1_{(3,2)}(b,f) := \sum_e p(e \mid b,f) \cdot h^1_{(4,3)}(e,f)$

$H_{(3,4)}$ $\quad h^1_{(3,4)}(e,f) := \sum_b p(e \mid b,f) \cdot h^1_{(2,3)}(b) \cdot h^2_{(2,3)}(f)$

$H_{(4,3)}$ $\quad h^1_{(4,3)}(e,f) := p(G = g_e \mid e,f)$



Cluster Tree Elimination vs. Mini-Clustering

CTE

1 **ABC**

$h_{(1,2)}(b,c)$

BC

$h_{(2,1)}(b,c)$

2 **BCDF**

$h_{(2,3)}(b,f)$

BF

$h_{(3,2)}(b,f)$

3 **BEF**

$h_{(3,4)}(e,f)$

EF

$h_{(4,3)}(e,f)$

4 **EFG**

MC

1 **ABC**

$H_{(1,2)}$ $\boxed{h^1_{(1,2)}(b,c)}$

BC

$H_{(2,1)}$ $\boxed{h^1_{(2,1)}(b) \quad h^2_{(2,1)}(c)}$

2 **BCDF**

$H_{(2,3)}$ $\boxed{h^1_{(2,3)}(b) \quad h^2_{(2,3)}(f)}$

BF

$H_{(3,2)}$ $\boxed{h^1_{(3,2)}(b,f)}$

3 **BEF**

$H_{(3,4)}$ $\boxed{h^1_{(3,4)}(e,f)}$

EF

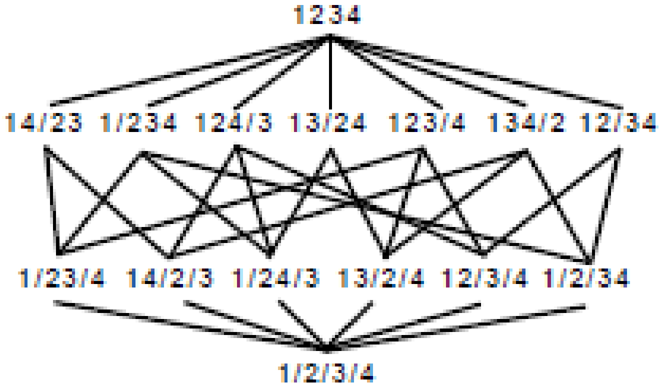
$H_{(4,3)}$ $\boxed{h^1_{(4,3)}(e,f)}$

4 **EFG**

Heuristics for partitioning

(Dechter and Rish, 2003, Rollon and Dechter 2010)

Scope-based Partitioning Heuristic (SCP) aims at minimizing the number of mini-buckets in the partition by including in each minibucket as many functions as respecting the *i* bound is satisfied



Partitioning lattice of bucket $\{f_1, f_2, f_3, f_4\}$.

- *Log relative error*:

$$RE(f,h) = \sum_t (\log(f(t)) - \log(h(t)))$$

- *Max log relative error*:

$$MRE(f,h) = \max_t \{\log(f(t)) - \log(h(t))\}$$

Use greedy heuristic derived from a distance function to decide which functions go into a single mini-bucket

Greedy Scope-based Partitioning

Procedure **Greedy Partitioning**

Input: $\{h_1, \ldots, h_k\}$, *i-bound*;
Output: A partitioning $mb(1), \ldots, mb(p)$ such that every $mb(i)$ contains at most *i-bound* variables;
1. Sort functions by the size of their scopes. Let $\{h_1, \ldots, h_k\}$ be the sorted array of functions, with h_1 having the largest scope.
2. **for** $i = 1$ to k
    if h_i can be placed in existing mini-buckets without making the scope
      greater than the *i-bound*, place it in the one with the most functions.
    else create a new mini-bucket an place h_i in it.
  endfor

Heuristic for Partitioning

Scope-based Partitioning Heuristic. The *scope-based* partition heuristic (SCP) aims at minimizing the number of mini-buckets in the partition by including in each mini-bucket as many functions as possible as long as the *i* bound is satisfied. First, single function mini-buckets are decreasingly ordered according to their arity from left to right. Then, each mini-bucket is absorbed into the left-most mini-bucket with whom it can be merged.

The time complexity of Partition(B, i) , where B is the bucket to be partitioned, and |B|,the number of functions in the bucket, using the SCP heuristic is $O(|B| \log(|B|) + |B|^2)$.

The scope-based
heuristic is is quite fast, its shortcoming is that it does not consider the actual information in the functions.

Greedy Partition as a function of a distance function h

function `GreedyPartition`(B,i, h)
1. **Initialize** Q as the bottom partition of B;
2. **While** $\exists Q' \in ch(Q)$ which is a i-partition
$Q \leftarrow \arg\min_{Q'}\{h(Q \rightarrow Q')\}$ among child i-partitions of Q;
3.Return Q;

Figure 8.13: Greedy partitioning

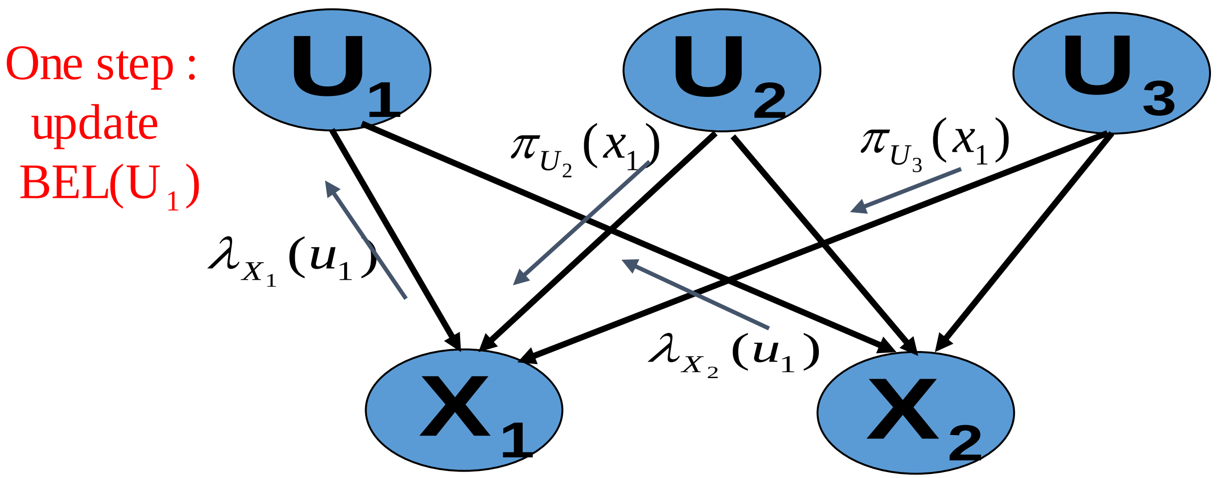
Proposition 8.6.5 *The time complexity of* `GreedyPartition` *is* $O(|B| \times T)$ *where* $O(T)$ *is the time complexity of selecting the min child partition according to* h.

Comparing Mini-clustering against Belief Propagation.

What is belief propagation

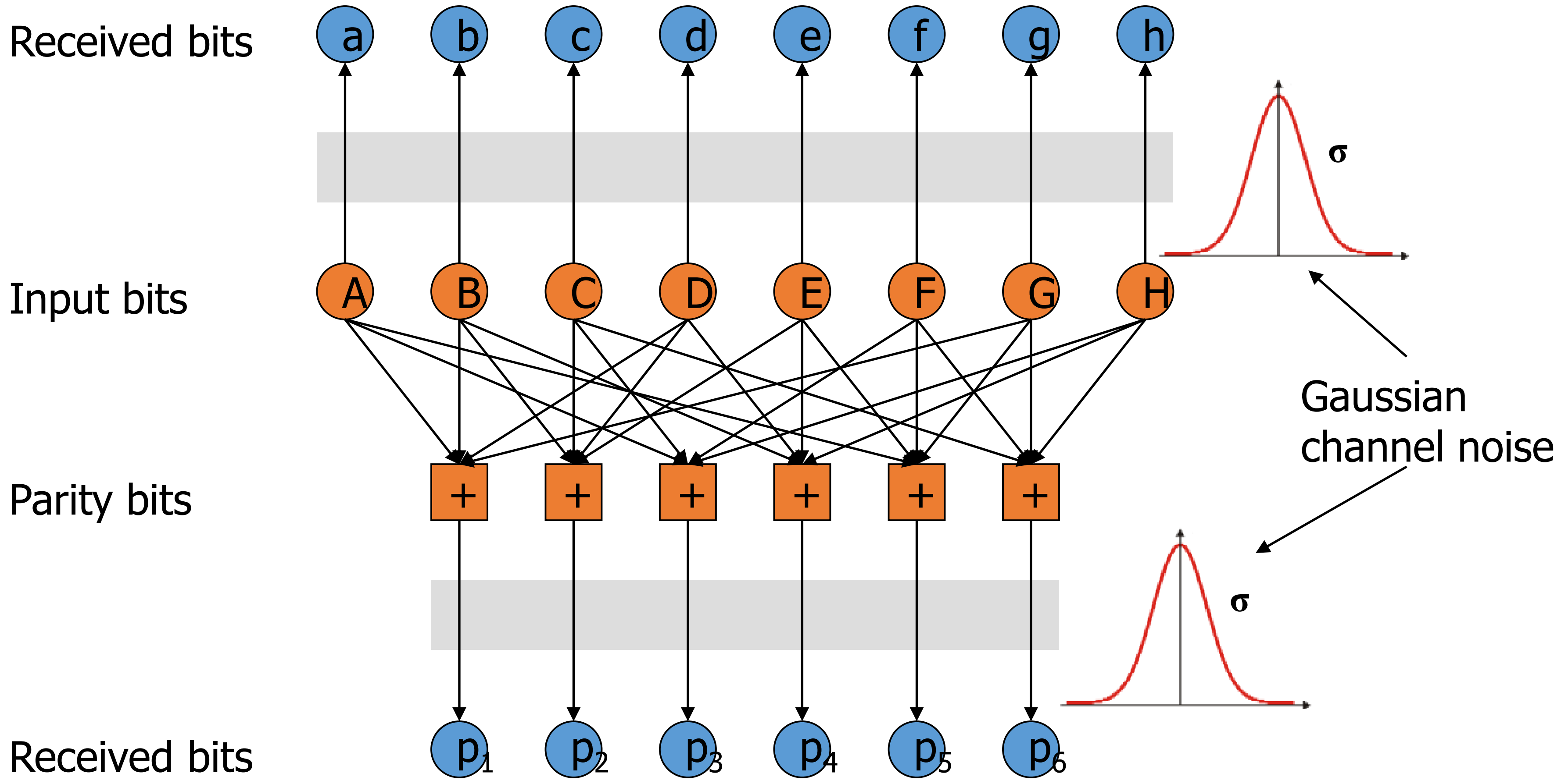
Iterative Belief Proapagation

- Belief propagation is exact for poly-trees
- IBP - applying BP iteratively to cyclic networks



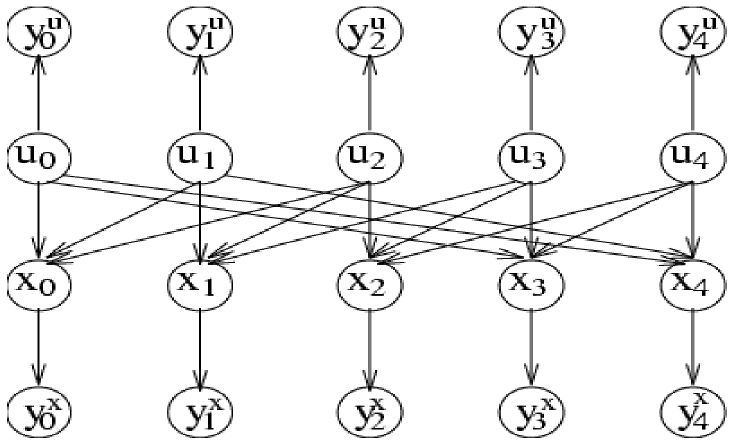
- No guarantees for convergence
- Works well for many coding networks

Linear Block Codes



Probabilistic decoding

Error-correcting linear block code

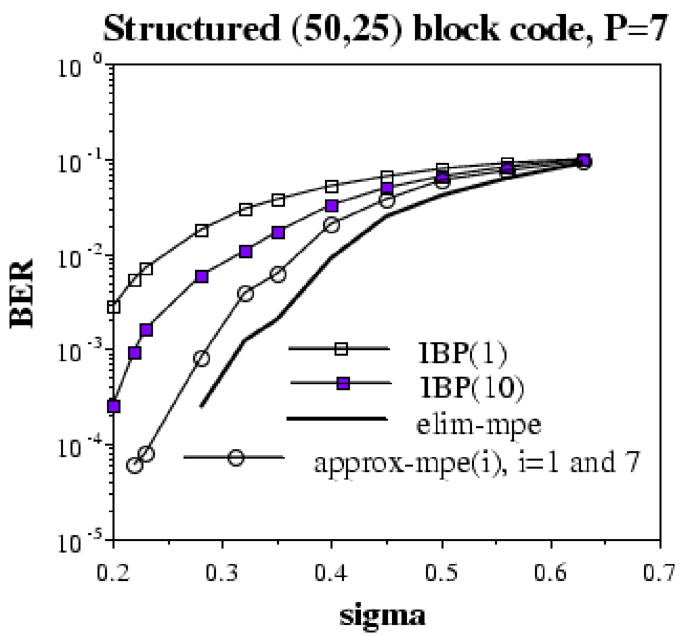


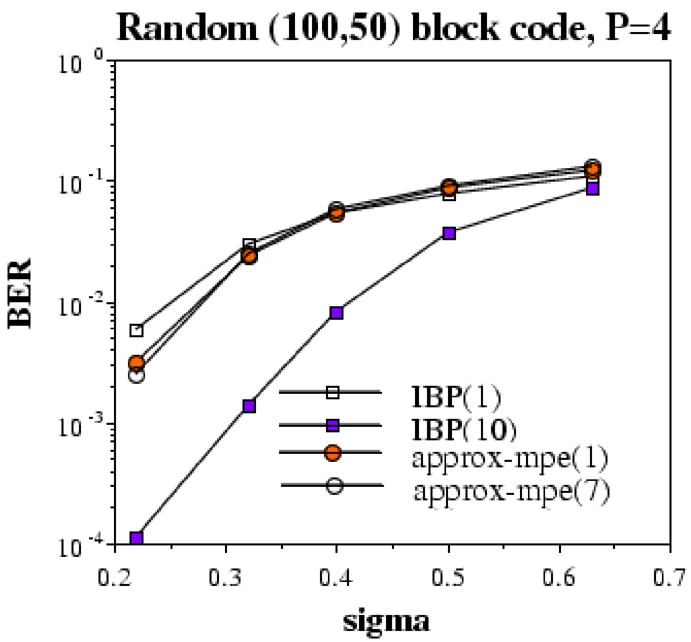
State-of-the-art:
approximate algorithm – iterative belief propagation (IBP)
(Pearl's poly-tree algorithm applied to loopy networks)

MBE-mpe vs. IBP

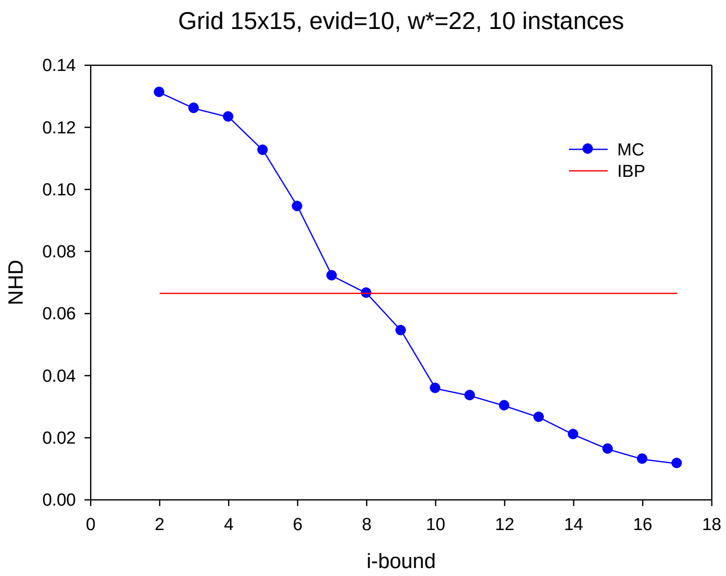
MBE-mpe is better on low w* codes
IBP (or BP) is better on randomly generated (high w*) codes.

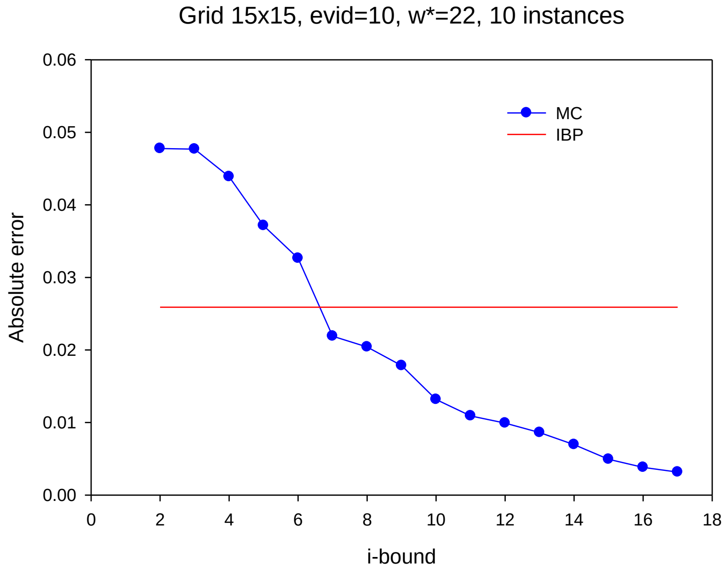
Bit error rate (BER) as a function of noise (sigma):

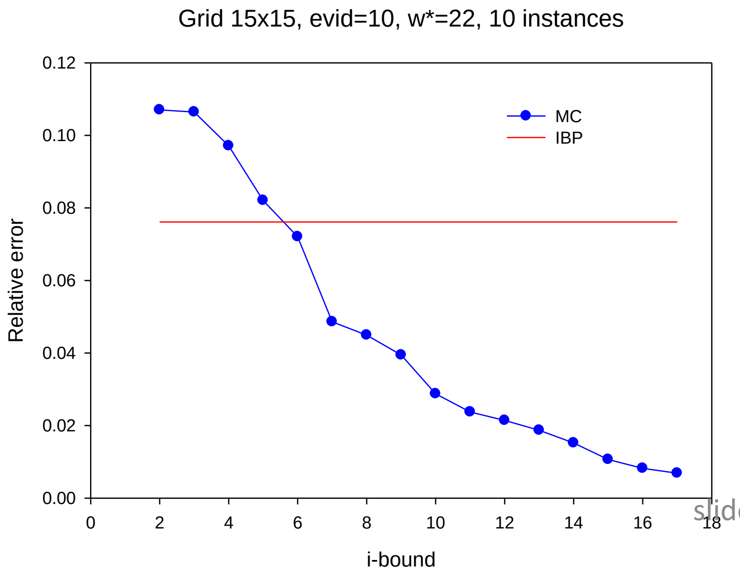


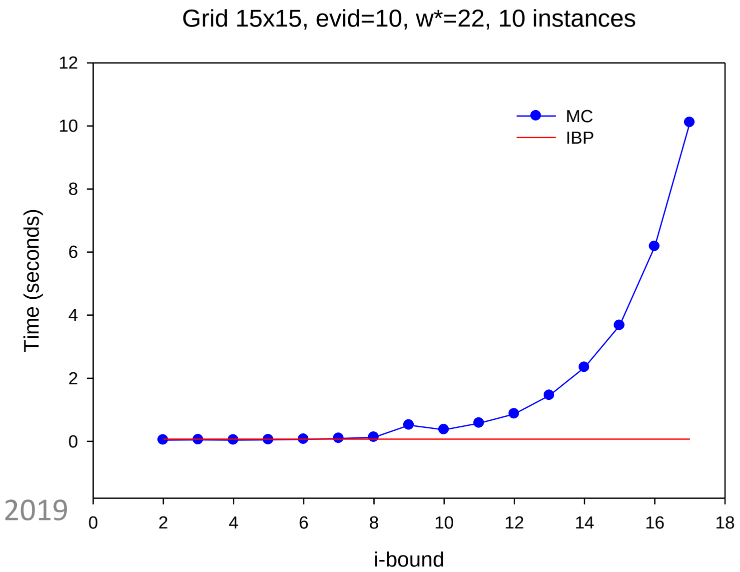


Grid 15x15 - 10 evidence







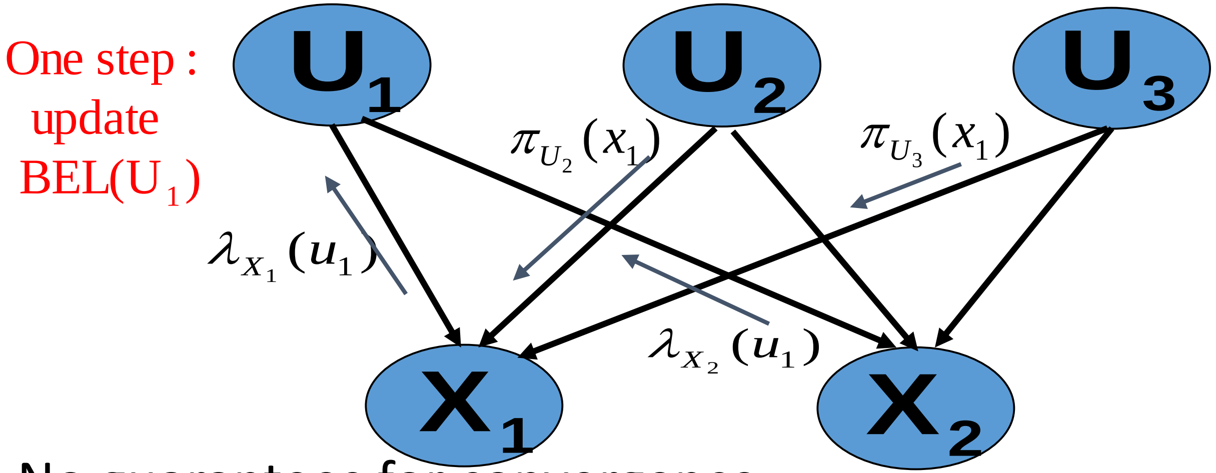


Outline

- Mini-bucket elimination
- Weighted Mini-bucket
- Mini-clustering
- Iterative Belief propagation
- Iterative-join-graph propagation
- Re-parameterization, cost-shifting

Iterative Belief Proapagation

- Belief propagation is exact for poly-trees
- IBP - applying BP iteratively to cyclic networks



- No guarantees for convergence
- Works well for many coding networks
- Lets combine iterative-nature with anytime--IJGP

Iterative Join Graph Propagation

- Loopy Belief Propagation
  - Cyclic graphs
  - Iterative
  - Converges fast in practice (no guarantees though)
  - Very good approximations (e.g., turbo decoding, LDPC codes, SAT – survey propagation)

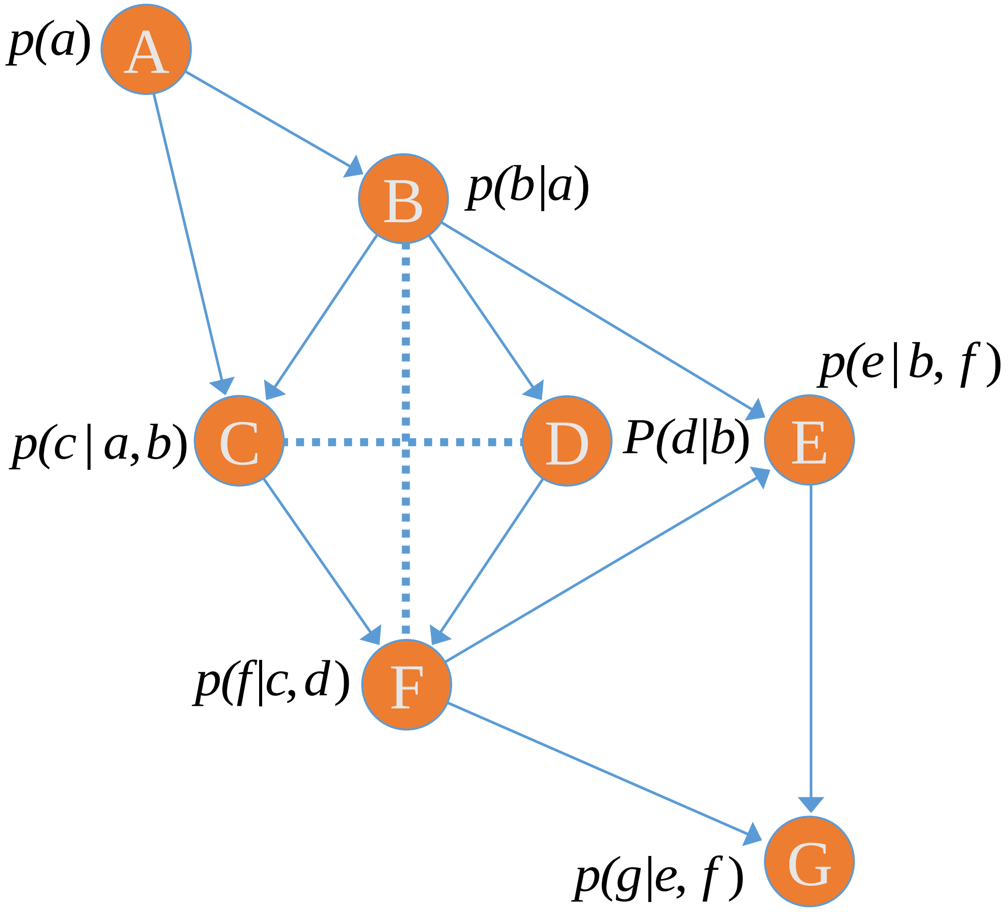
- Mini-Clustering(i)
  - Tree decompositions
  - Only two sets of messages (inward, outward)
  - Anytime behavior – can improve with more time by increasing the i-bound

- We want to combine:
  - Iterative virtues of Loopy BP
  - Anytime behavior of Mini-Clustering(i)

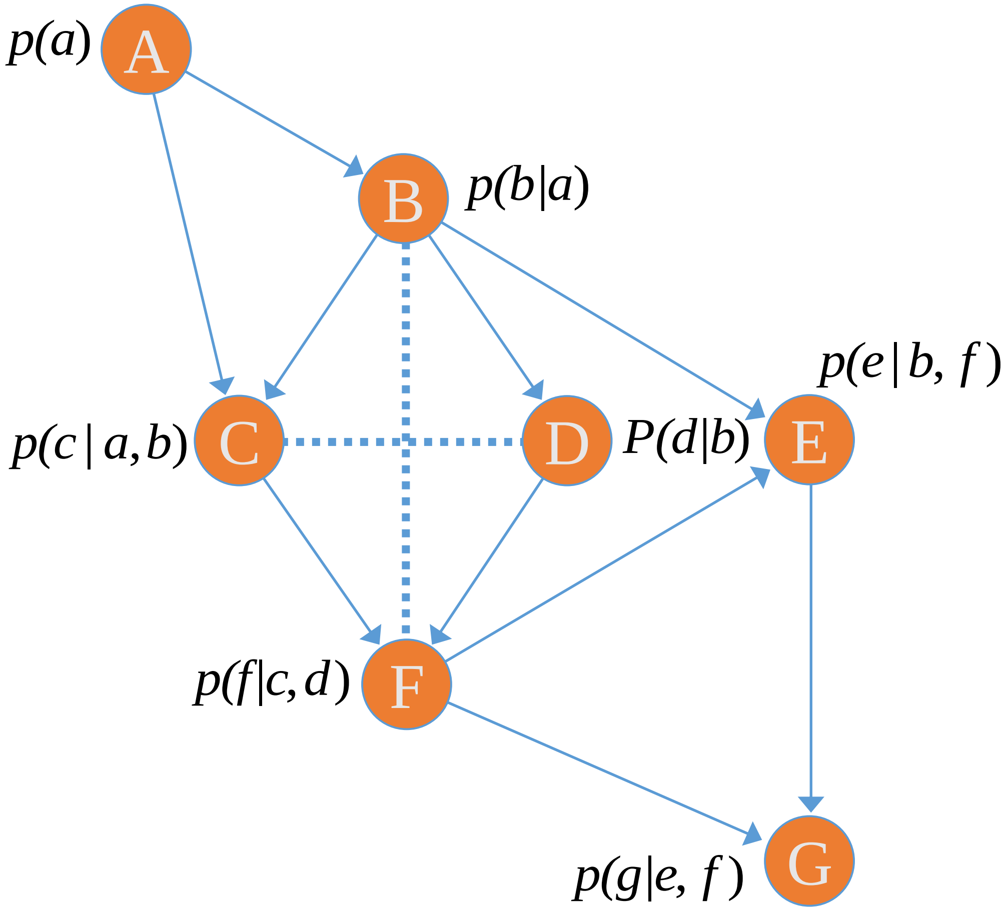
IJGP - The basic idea

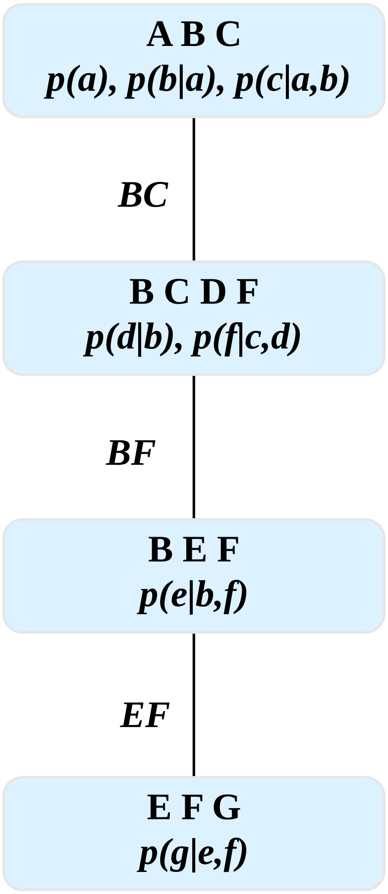
- Apply Cluster Tree Elimination to any *join-graph*
- We commit to graphs that are *I-maps*
- Avoid cycles as long as I-mapness is not violated
- Result: use *minimal arc-labeled* join-graphs

Tree Decomposition for Belief Updating

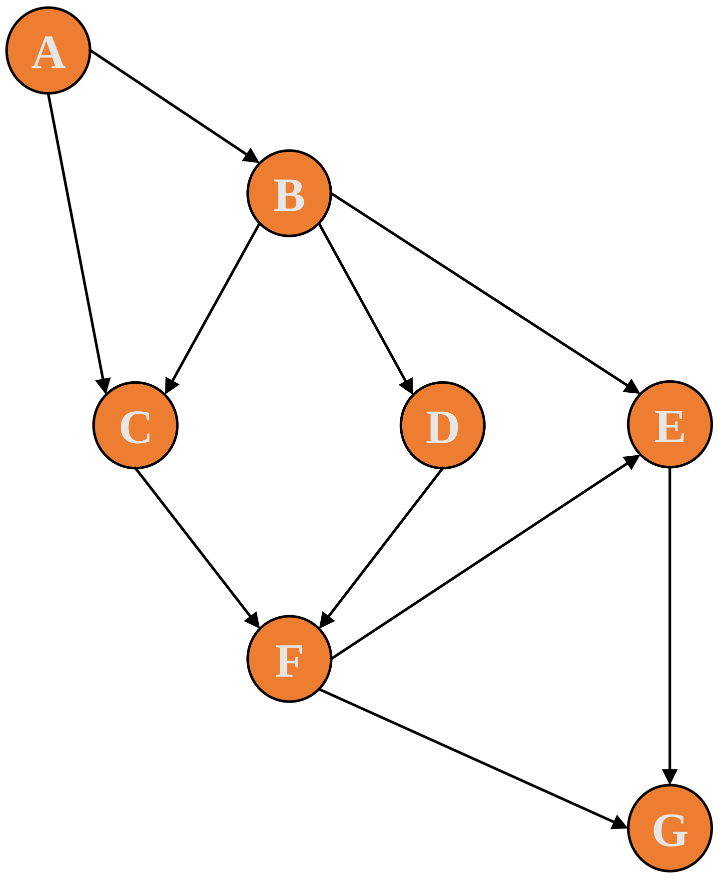


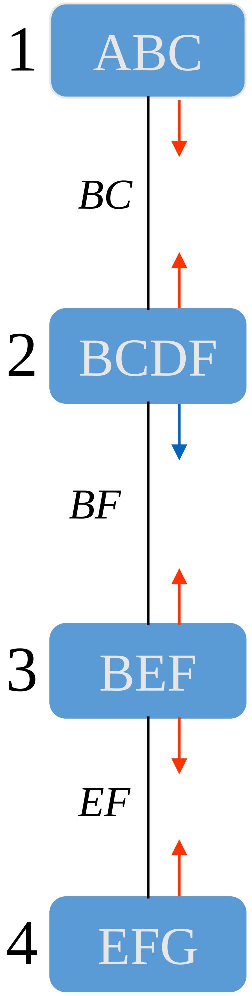
Tree Decomposition for belief updating





CTE: Cluster Tree Elimination





$$h_{(1,2)}(b,c) = \sum_{a} p(a) \cdot p(b \mid a) \cdot p(c \mid a,b)$$

$$h_{(2,1)}(b,c) = \sum_{d,f} p(d \mid b) \cdot p(f \mid c,d) \cdot h_{(3,2)}(b,f)$$

$$h_{(2,3)}(b,f) = \sum_{c,d} p(d \mid b) \cdot p(f \mid c,d) \cdot h_{(1,2)}(b,c)$$

$$h_{(3,2)}(b,f) = \sum_{e} p(e \mid b,f) \cdot h_{(4,3)}(e,f)$$

$$h_{(3,4)}(e,f) = \sum_{b} p(e \mid b,f) \cdot h_{(2,3)}(b,f)$$

$$h_{(4,3)}(e,f) = p(G = g_e \mid e,f)$$

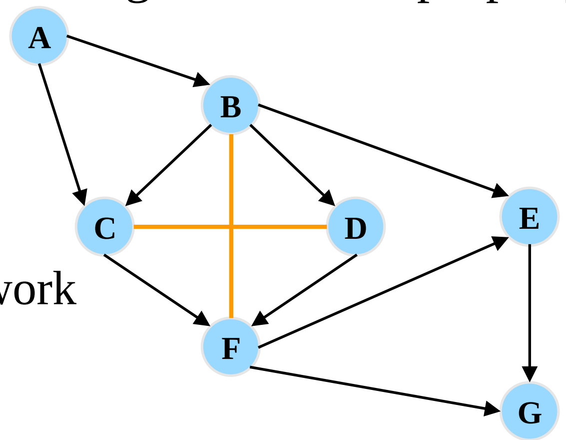
Time: *O (exp(w+1))*
Space: *O (exp(sep))*

For each cluster P(X|e) is computed, also P(e)

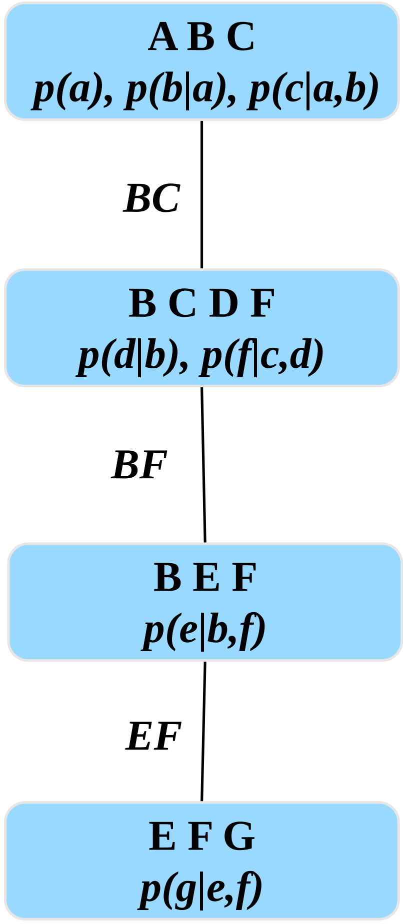
Example

A *tree decomposition* for a belief network $BN = \langle X, D, G, P \rangle$ is a triple $\langle T, \chi, \psi \rangle$, where $T = (V,E)$ is a tree and χ and ψ are labeling functions, associating with each vertex $v \in V$ two sets, $\chi(v) \subseteq X$ and $\psi(v) \subseteq P$ satisfying :

1. For each function $p_i \in P$ there is exactly one vertex such that $p_i \in \psi(v)$ and $scope(p_i) \subseteq \chi(v)$
2. For each variable $X_i \in X$ the set $\{v \in V | X_i \in \chi(v)\}$ forms a connected subtree (running intersection property)



Belief network

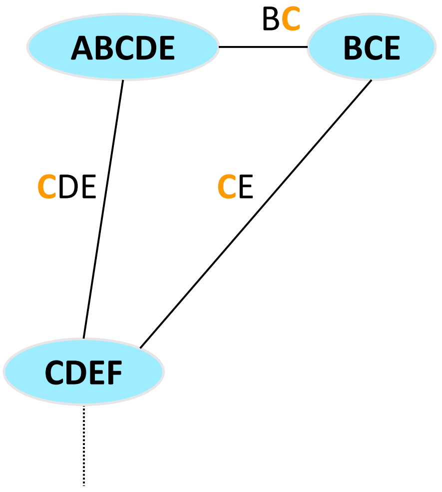


Tree decomposition

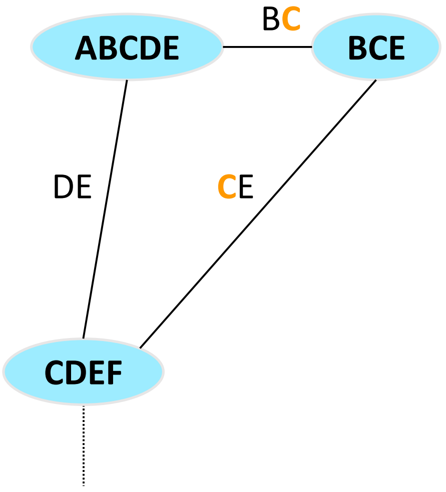
IJGP - The basic idea

- Apply Cluster Tree Elimination to any *join-graph*
- We commit to graphs that are *I-maps*
- Avoid cycles as long as I-mapness is not violated
- Result: use *minimal arc-labeled* join-graphs

Minimal Arc-Labeled Decomposition



a) Fragment of an
arc-labeled join-graph



a) Shrinking labels to make it a
minimal arc-labeled join-graph

- Use a DFS algorithm to eliminate cycles relative to each variable

Minimal arc-labeled join-graph

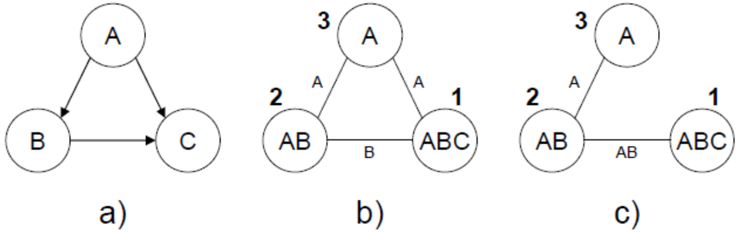


Figure 1.17: a) A belief network; b) A dual join-graph with singleton labels; c) A dual join-graph which is a join-tree

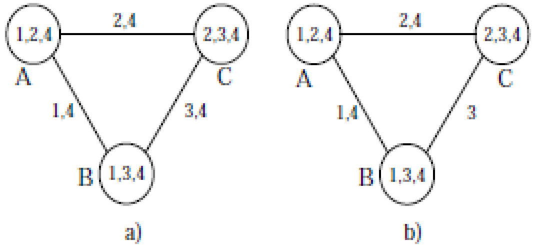
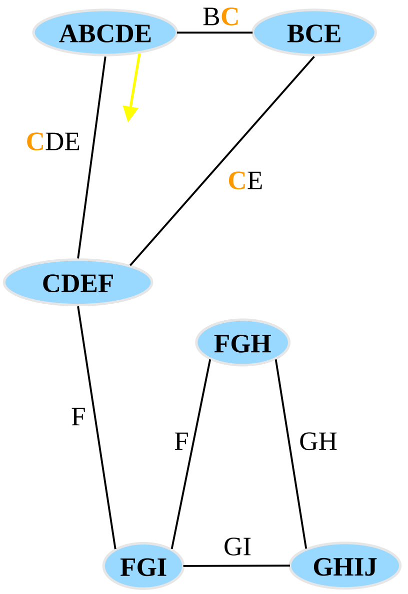
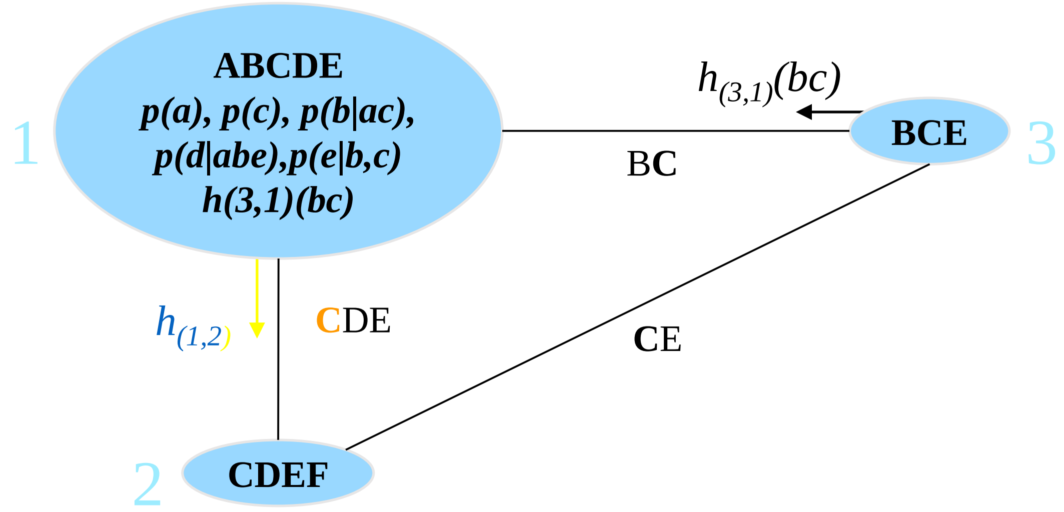


Figure 1.15: An arc-labeled decomposition

Message propagation





Minimal arc-labeled:

sep(1,2)={D,E}

elim(1,2)={A,B,C}

$$h_{(1,2)}(de) = \sum_{a,b,c} p(a)p(c)p(b|ac)p(d|abe)p(e|bc)h_{(3,1)}(bc)$$

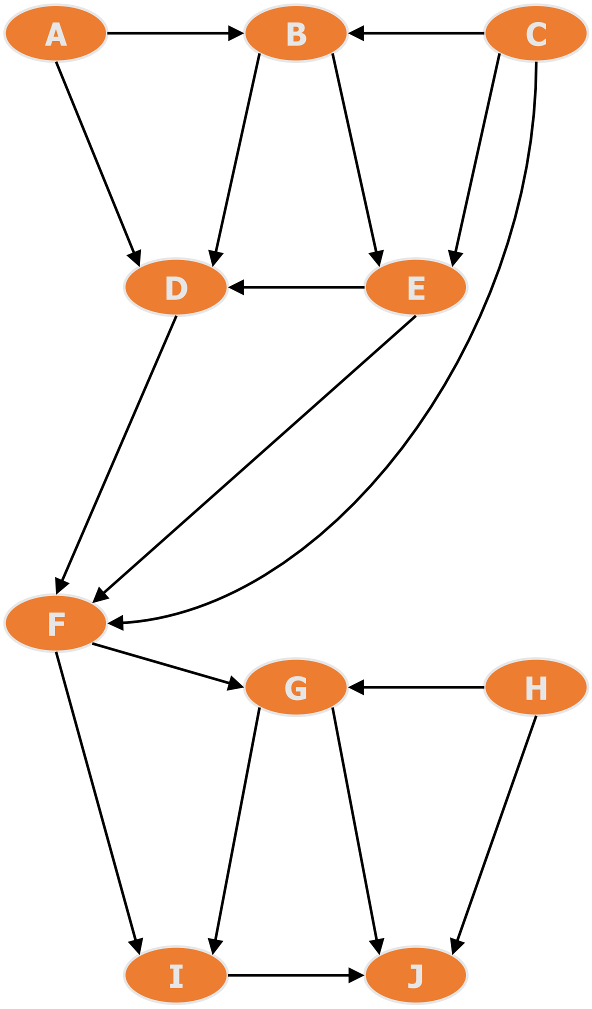
Non-minimal arc-labeled:

sep(1,2)={C,D,E}

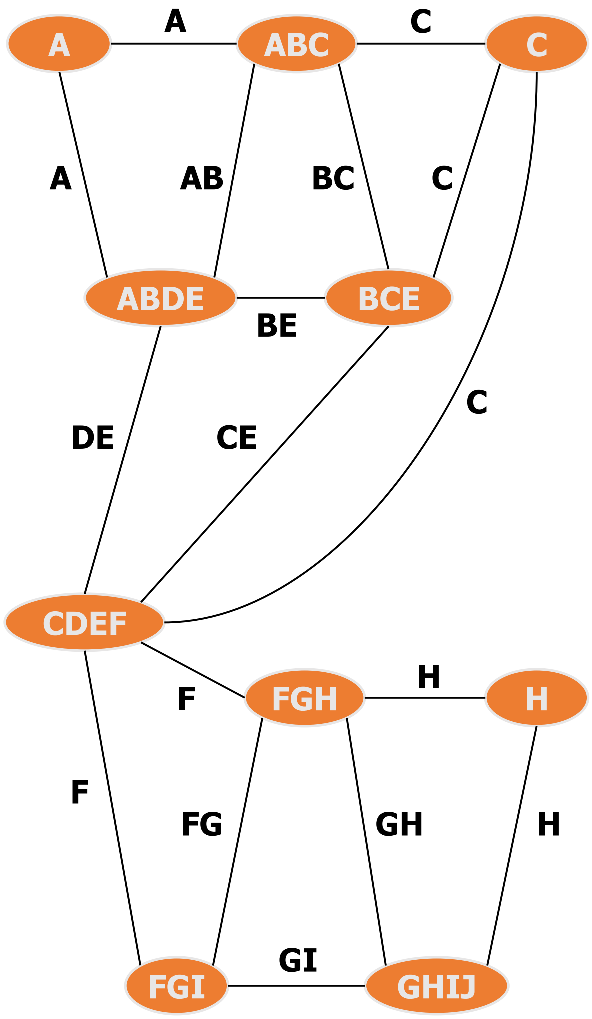
elim(1,2)={A,B}

$$h_{(1,2)}(cde) = \sum_{a,b} p(a)p(c)p(b|ac)p(d|abe)p(e|bc)h_{(3,1)}(bc)$$

IJGP - Example

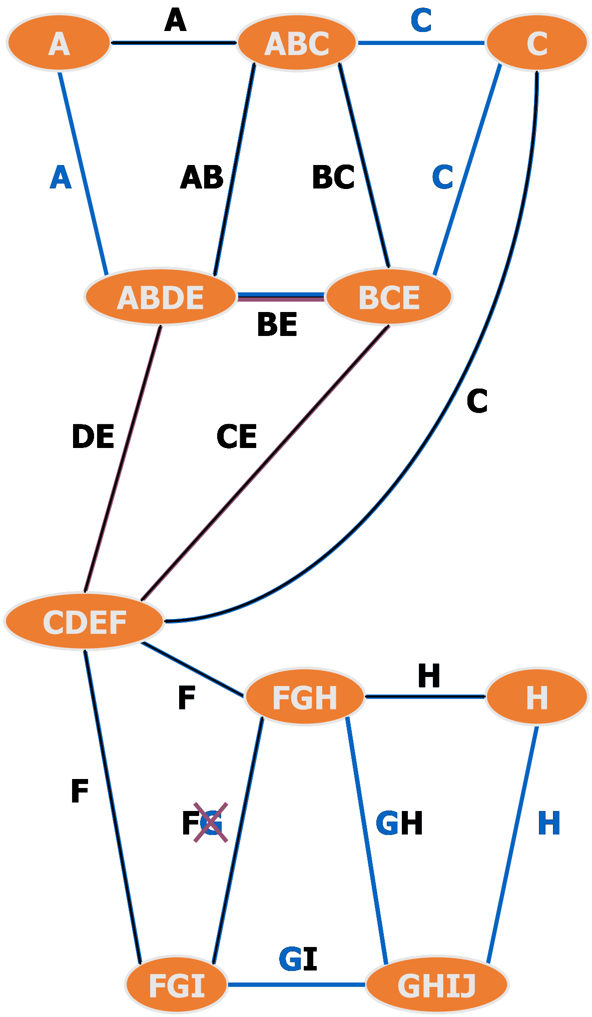


Belief network



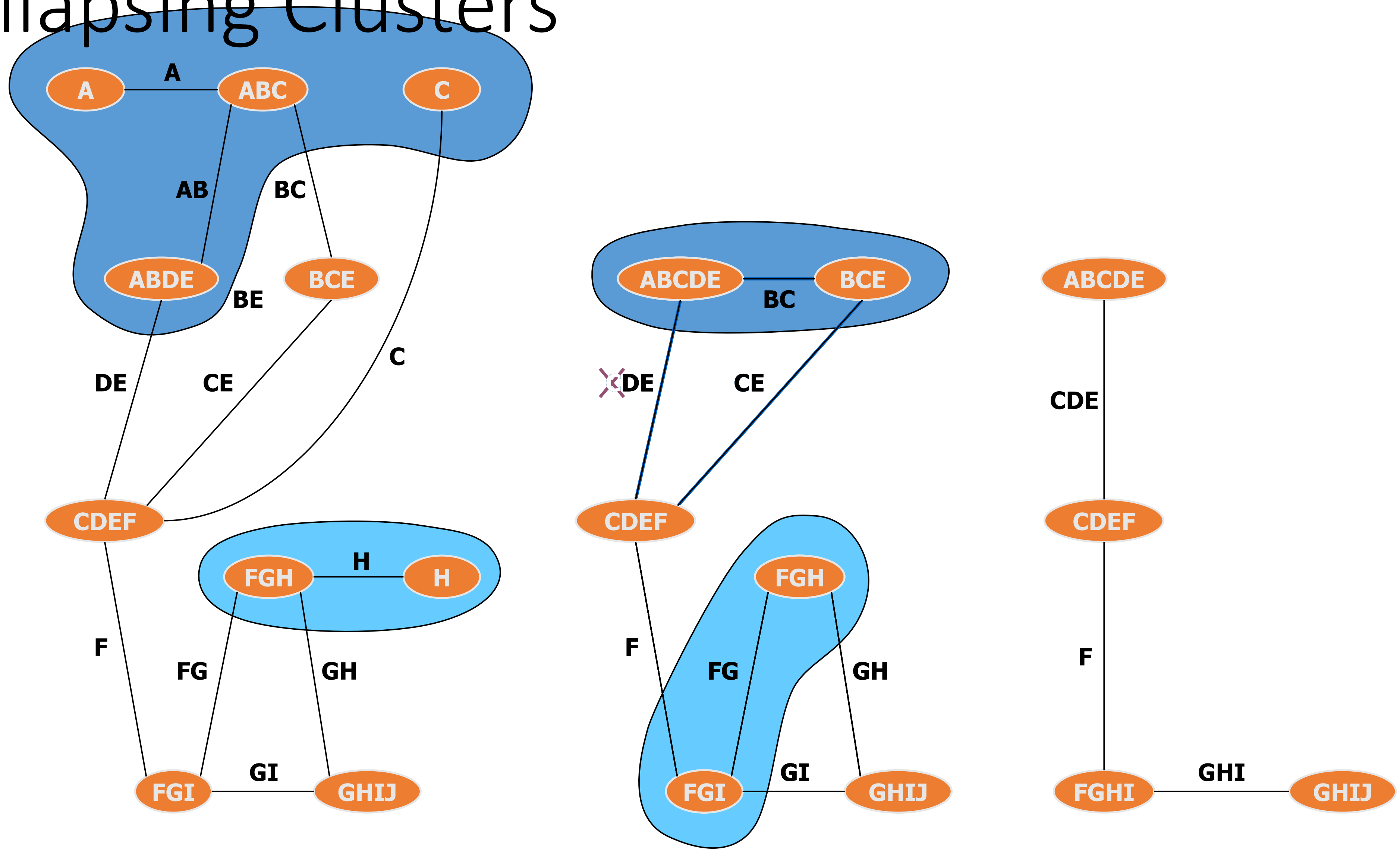
Loopy BP graph

Arc-Minimal Join-Graph

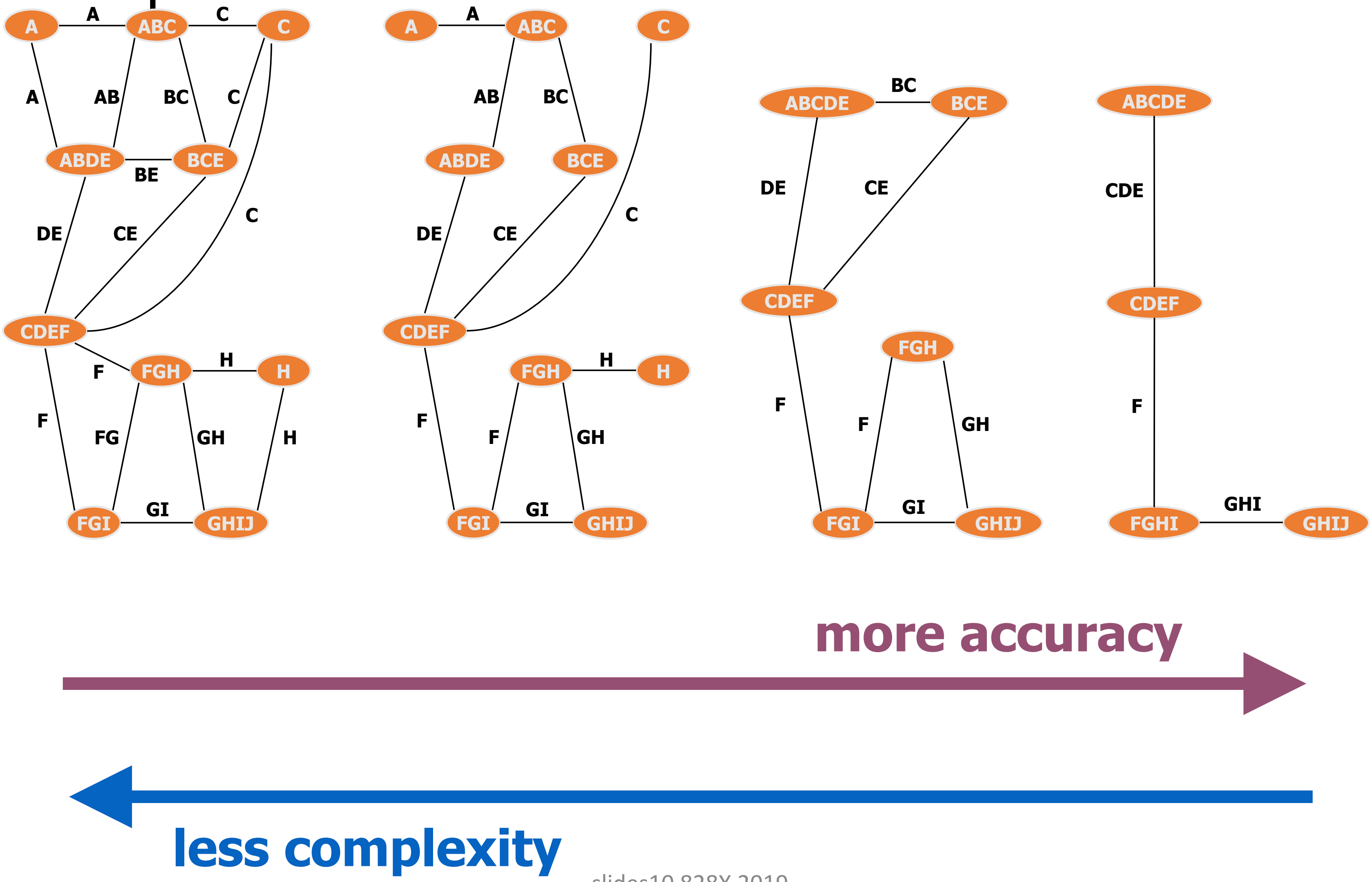


Arcs labeled with any single variable should form a TREE

Collapsing Clusters



Join-Graphs



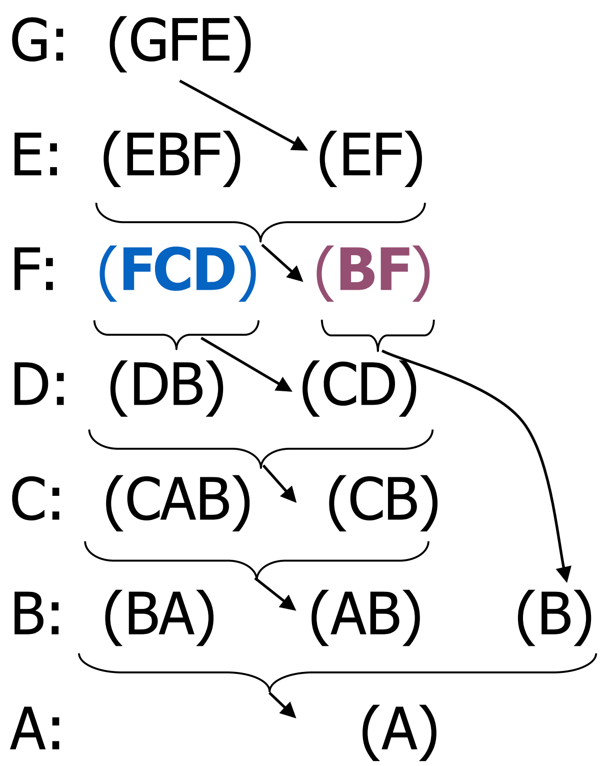
A
ABC
C
ABDE
BCE
CDEF
FGH
H
FGI
GHIJ
AB
BC
BE
DE
CE
F
FG
GH
GI
ABCDE
CDE
FGHI
GHI
more accuracy
less complexity

Bounded decompositions

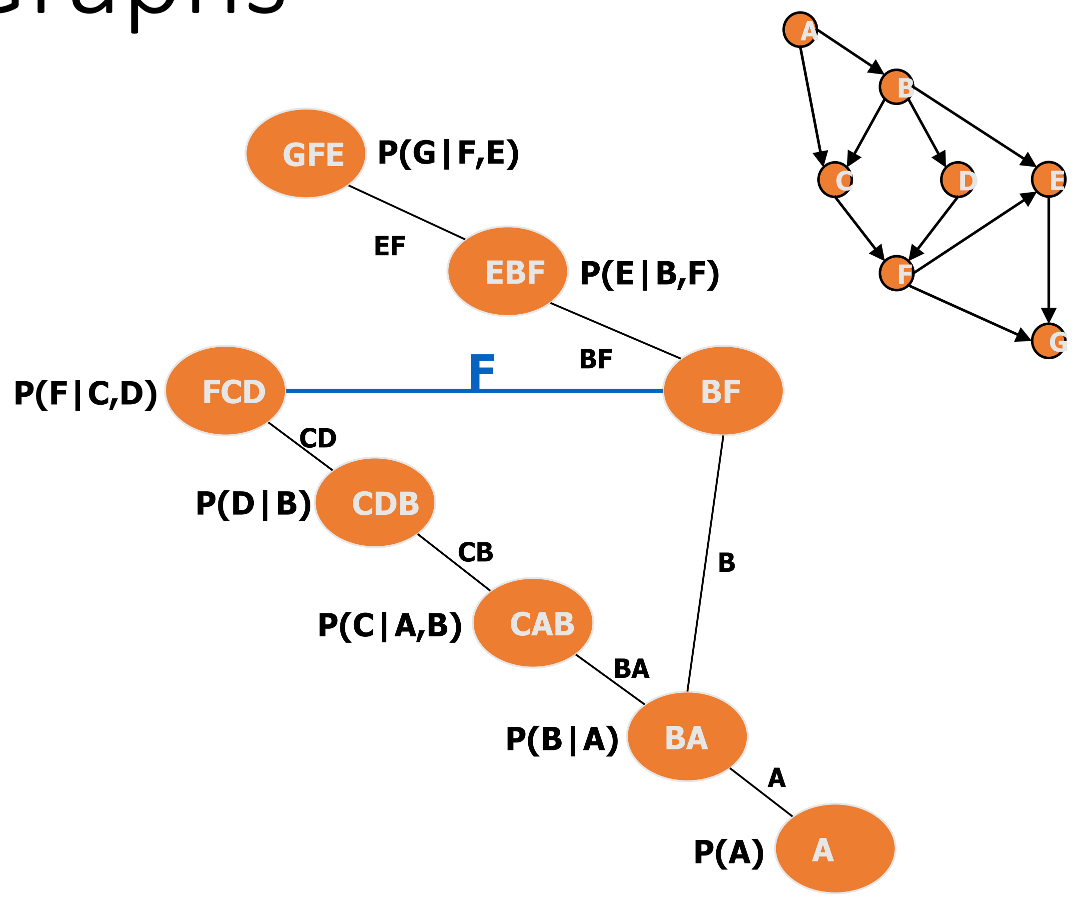
- We want arc-labeled decompositions such that:
  - the cluster size (internal width) is bounded by *i* (the accuracy parameter)

- Possible approaches to build decompositions:
  - partition-based algorithms - inspired by the mini-bucket decomposition
  - grouping-based algorithms

Constructing Join-Graphs



a) schematic mini-bucket(i), i=3



b) arc-labeled join-graph decomposition

IJGP properties

- IJGP(i) applies BP to min arc-labeled join-graph, whose cluster size is bounded by i
- On join-trees IJGP finds exact beliefs
- IJGP is a Generalized Belief Propagation algorithm (Yedidia, Freeman, Weiss 2001)
- Complexity of one iteration:
  - time: $O(deg \bullet (n+N) \bullet d^{i+1})$
  - space: $O(N \bullet d^{\theta})$

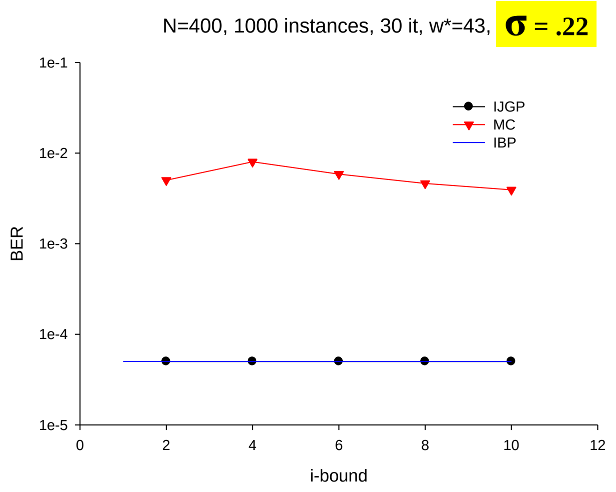
Empirical evaluation

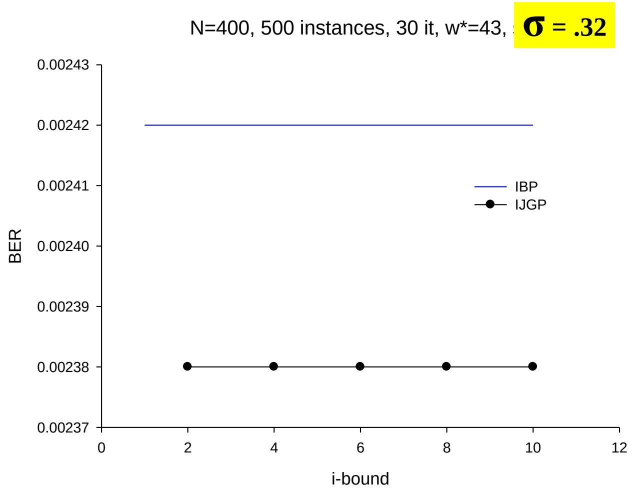
- Algorithms:
  - Exact
  - IBP
  - MC
  - IJGP

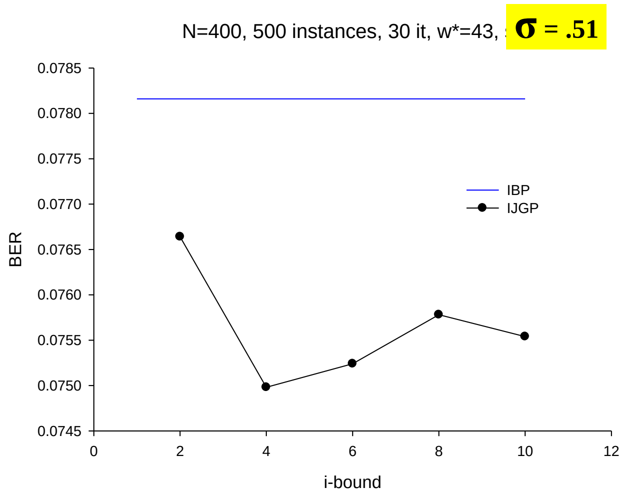
- Measures:
  - Absolute error
  - Relative error
  - Kulbach-Leibler (KL) distance
  - Bit Error Rate
  - Time

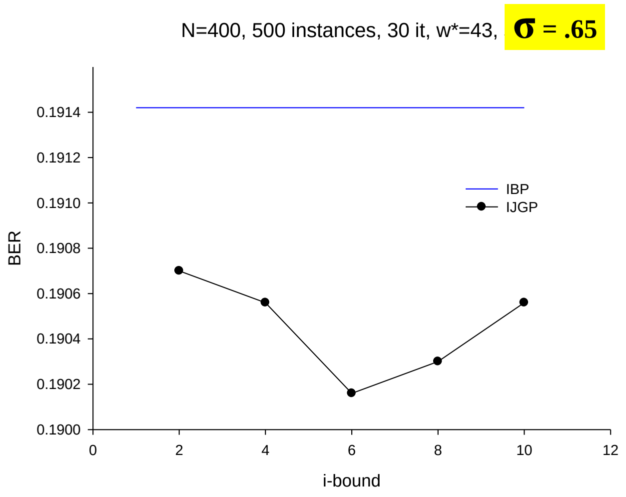
- Networks (all variables are binary):
  - Random networks
  - Grid networks (MxM)
  - CPCS 54, 360, 422
  - Coding networks

Coding Networks – Bit Error Rate

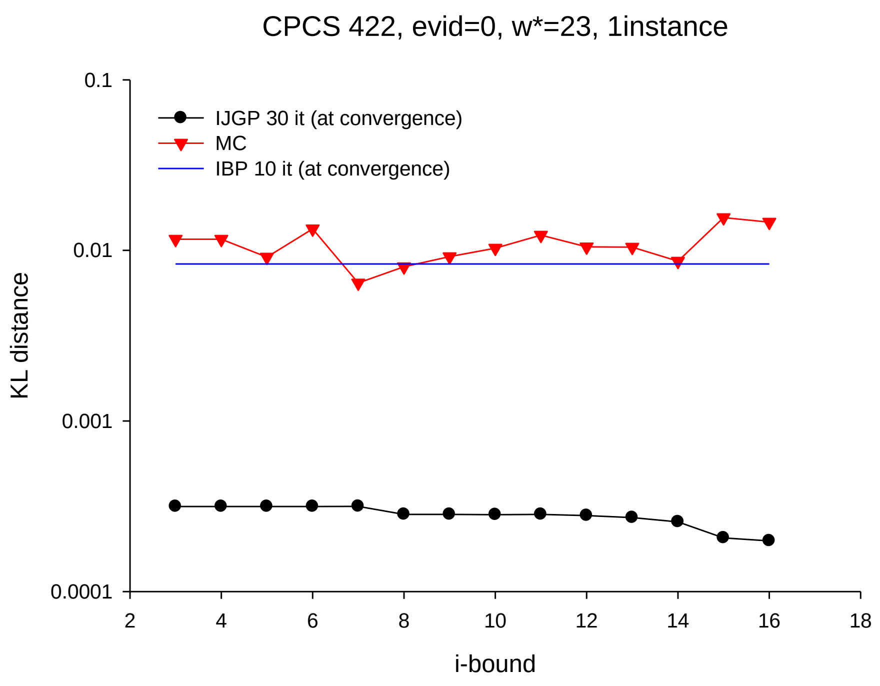




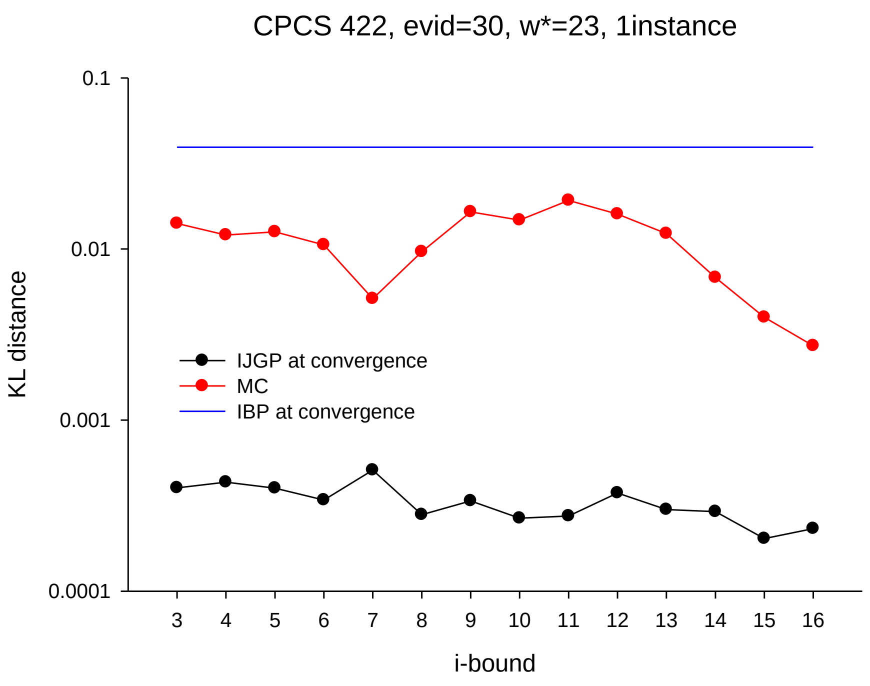




CPCS 422 – KL Distance

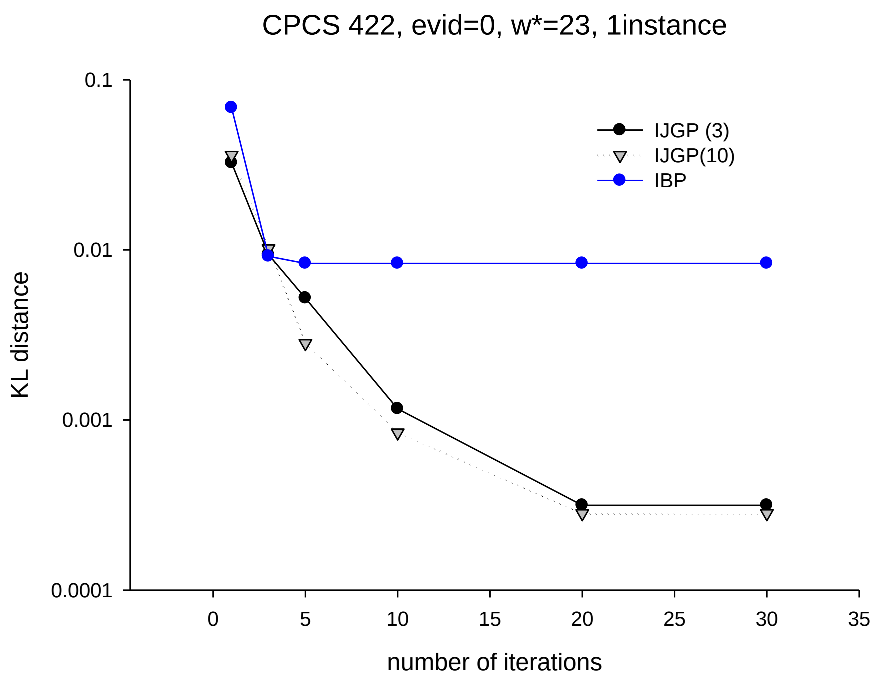


evidence=0

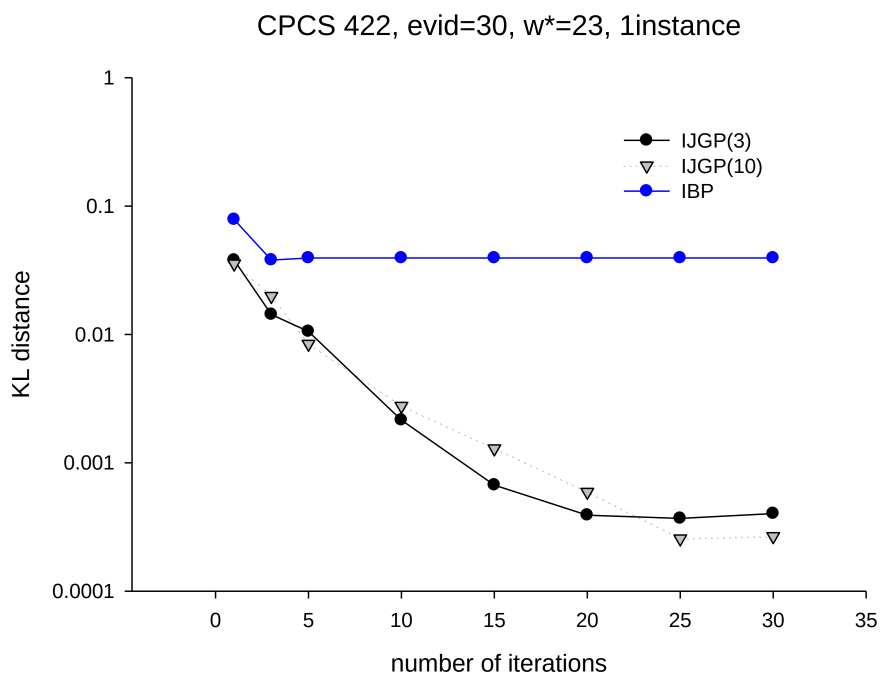


evidence=30

CPCS 422 – KL vs. Iterations

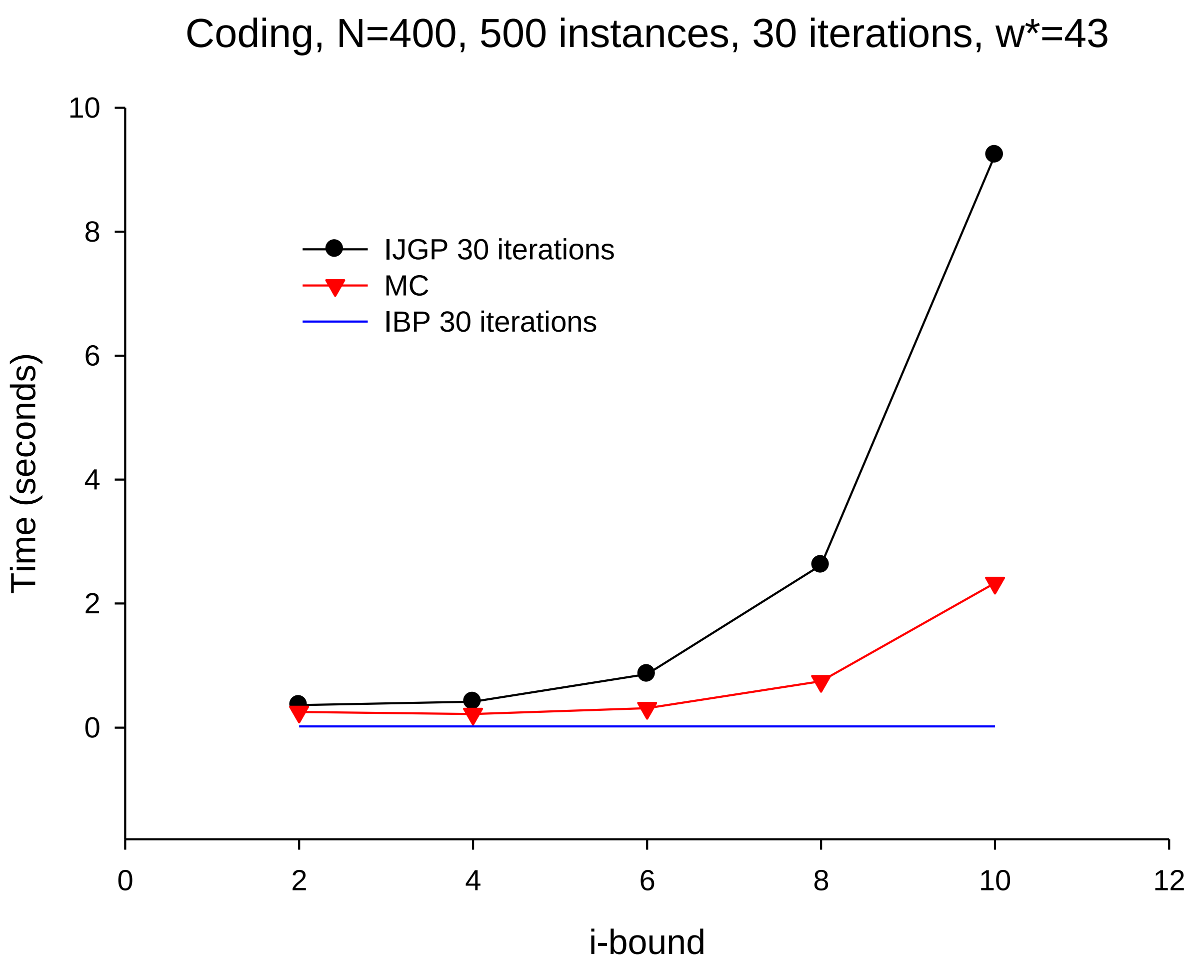


evidence=0



evidence=30

Coding networks - Time



More On the Power of Belief Propagation

- BP as local minima of KL distance (Read Darwiche)
- BP's power from constraint propagation perspective.

The Kullback-Leibler Divergence

The Kullback-Leibler divergence (KL–divergence)

$$\mathrm{KL}(\Pr'(\mathbf{X}|\mathbf{e}), \Pr(\mathbf{X}|\mathbf{e})) = \sum_{\mathbf{x}} \Pr'(\mathbf{x}|\mathbf{e}) \log \frac{\Pr'(\mathbf{x}|\mathbf{e})}{\Pr(\mathbf{x}|\mathbf{e})}$$

- $\mathrm{KL}(\Pr'(\mathbf{X}|\mathbf{e}), \Pr(\mathbf{X}|\mathbf{e}))$ is non-negative
- equal to zero if and only if $\Pr'(\mathbf{X}|\mathbf{e})$ and $\Pr(\mathbf{X}|\mathbf{e})$ are equivalent.

The Kullback-Leibler Divergence

KL–divergence is not a true distance measure in that it is not symmetric. In general:

$$\mathrm{KL}(\Pr'(\mathbf{X}|\mathbf{e}), \Pr(\mathbf{X}|\mathbf{e})) \neq \mathrm{KL}(\Pr(\mathbf{X}|\mathbf{e}), \Pr'(\mathbf{X}|\mathbf{e})).$$

- $\mathrm{KL}(\Pr'(\mathbf{X}|\mathbf{e}), \Pr(\mathbf{X}|\mathbf{e}))$ weighting the KL–divergence by the approximate distribution $\Pr'$
- We shall indeed focus on the KL–divergence weighted by the approximate distribution as it has some useful computational properties.

The Kullback-Leibler Divergence

Let $\Pr(\mathbf{X})$ be a distribution induced by a Bayesian network $\mathcal{N}$ having families $X\mathbf{U}$

The KL–divergence between $\Pr$ and another distribution $\Pr'$ can be written as a sum of three components:

$$
\begin{aligned}
& \mathrm{KL}(\Pr'(\mathbf{X}|\mathbf{e}), \Pr(\mathbf{X}|\mathbf{e})) \\
= \quad & -\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) - \sum_{X\mathbf{U}} \mathrm{AVG}'(\log \lambda_{\mathbf{e}}(X)\Theta_{X|\mathbf{U}}) + \log \Pr(\mathbf{e}),
\end{aligned}
$$

where

- $\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) = -\sum_{\mathbf{x}} \Pr'(\mathbf{x}|\mathbf{e}) \log \Pr'(\mathbf{x}|\mathbf{e})$ is the entropy of the conditioned approximate distribution $\Pr'(\mathbf{X}|\mathbf{e})$.
- $\mathrm{AVG}'(\log \lambda_{\mathbf{e}}(X)\Theta_{X|\mathbf{U}}) = \sum_{x\mathbf{u}} \Pr'(x\mathbf{u}|\mathbf{e}) \log \lambda_{\mathbf{e}}(x)\theta_{x|\mathbf{u}}$ is a set of expectations over the original network parameters weighted by the conditioned approximate distribution.

The Kullback-Leibler Divergence

Lambda is Grounding for evidence e)

A distribution $\Pr'(\mathbf{X}|\mathbf{e})$ minimizes the KL-divergence $\mathrm{KL}(\Pr'(\mathbf{X}|\mathbf{e}), \Pr(\mathbf{X}|\mathbf{e}))$ if it maximizes

$$\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) + \sum_{X\mathbf{U}} \mathrm{AVG}'(\log \lambda_{\mathbf{e}}(X)\Theta_{X|\mathbf{U}})$$

Competing properties of $\Pr'(\mathbf{X}|\mathbf{e})$ that minimize the KL–divergence:

- $\Pr'(\mathbf{X}|\mathbf{e})$ should match the original distribution by giving more weight to more likely parameters $\lambda_{\mathbf{e}}(x)\theta_{x|\mathbf{u}}$ (i.e, maximize the expectations).
- $\Pr'(\mathbf{X}|\mathbf{e})$ should not favor unnecessarily one network instantiation over another by being evenly distributed (i.e., maximize the entropy).

Optimizing the KL-Divergence

Theorem: Yedidia, Frieman and Weiss 2005

Let $\Pr(\mathbf{X})$ be a distribution induced by a Bayesian network $\mathcal{N}$ having families $X\mathbf{U}$. Then IBP messages are a fixed point if and only if IBP marginals $\mu_u = BEL(u)$ and $\mu_{x\mathbf{u}} = BEL(x\mathbf{u})$ are a stationary point of:

$$\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) + \sum_{X\mathbf{U}} \mathrm{AVG}'(\log \lambda_{\mathbf{e}}(X)\Theta_{X|\mathbf{U}})$$
$$= \quad -\sum_{X\mathbf{U}} \sum_{x\mathbf{u}} \mu_{x\mathbf{u}} \log \frac{\mu_{x\mathbf{u}}}{\prod_{u \sim \mathbf{u}} \mu_u} + \sum_{X\mathbf{U}} \sum_{x\mathbf{u}} \mu_{x\mathbf{u}} \log \lambda_{\mathbf{e}}(x)\theta_{x|\mathbf{u}},$$

under normalization constraints:

$$\sum_u \mu_u = \sum_{x\mathbf{u}} \mu_{x\mathbf{u}} = 1$$

for each family $X\mathbf{U}$ and parent U, and under consistency constraints:

$$\sum_{x\mathbf{u} \sim y} \mu_{x\mathbf{u}} = \mu_y$$

for each family instantiation $x\mathbf{u}$ and value y of family member $Y \in X\mathbf{U}$.

Optimizing the KL-Divergence

- IBP fixed points are stationary points of the KL–divergence: they may only be local minima, or they may not be minima.
- When IBP performs well, it will often have fixed points that are indeed minima of the KL–divergence.
- For problems where IBP does not behave as well, we will next seek approximations $\Pr'$ whose factorizations are more expressive than that of the polytree-based factorization.

terative Joingraph Propagation

Let $\Pr(\mathbf{X})$ be a distribution induced by a Bayesian network $\mathcal{N}$ having families $X\mathbf{U}$, and let $\mathbf{C}_i$ and $\mathbf{S}_{ij}$ be the clusters and separators of a joingraph for $\mathcal{N}$.
Then messages M_{ij} are a fixed point of `IJGP` if and only if `IJGP` marginals $\mu_{\mathbf{c}_i} = BEL(\mathbf{c}_i)$ and $\mu_{\mathbf{s}_{ij}} = BEL(\mathbf{s}_{ij})$ are a stationary point of:

$$\begin{aligned} &\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) + \sum_{\mathbf{C}_i} \mathrm{AVG}'(\log \Phi_i) \\ = \quad & -\sum_{\mathbf{C}_i}\sum_{\mathbf{c}_i} \mu_{\mathbf{c}_i} \log \mu_{\mathbf{c}_i} + \sum_{\mathbf{S}_{ij}}\sum_{\mathbf{s}_{ij}} \mu_{\mathbf{s}_{ij}} \log \mu_{\mathbf{s}_{ij}} + \sum_{\mathbf{C}_i}\sum_{\mathbf{c}_i} \mu_{\mathbf{c}_i} \log \Phi_i(\mathbf{c}_i), \end{aligned}$$

under normalization constraints:

$$\sum_{\mathbf{c}_i} \mu_{\mathbf{c}_i} = \sum_{\mathbf{s}_{ij}} \mu_{\mathbf{s}_{ij}} = 1$$

for each cluster $\mathbf{C}_i$ and separator $\mathbf{S}_{ij}$, and under consistency constraints:

$$\sum_{\mathbf{c}_i \sim \mathbf{s}_{ij}} \mu_{\mathbf{c}_i} = \mu_{\mathbf{s}_{ij}} = \sum_{\mathbf{c}_j \sim \mathbf{s}_{ij}} \mu_{\mathbf{c}_j}$$

for each separator $\mathbf{S}_{ij}$ and neighboring clusters $\mathbf{C}_i$ and $\mathbf{C}_j$.

Summary of IJGP so far

A spectrum of approximations.

IBP: results from applying IJGP to the dual joingraph.

Jointree algorithm: results from applying IJGP to a jointree (as a joingraph).

In between these two ends, we have a spectrum of joingraphs and corresponding factorizations, where IJGP seeks stationary points of the KL–divergence between these factorizations and the original distribution.

Outline

- Mini-bucket elimination
- Weighted Mini-bucket
- Mini-clustering
- Iterative Belief propagation
- Iterative-join-graph propagation
- Re-parameterization, cost-shifting

Cost-Shifting

(Reparameterization)

$+\lambda(B)$

A	B	f(A,B)
b	b	6 + 3
b	g	0 – 1
g	b	0 + 3
g	g	6 – 1

A	B	C	f(A,B,C)
b	b	b	12
b	b	g	6
b	g	b	0
b	g	g	6
g	b	b	6
g	b	g	0
g	g	b	6
g	g	g	12

= 0 + 6

$-\lambda(B)$

B	C	f(B,C)
b	b	6 – 3
b	g	0 – 3
g	b	0 + 1
g	g	6 + 1

B	λ(B)
b	3
g	-1

Modify the individual functions

- but –

keep the sum of functions the same

Tightening the bound

- Reparameterization (or, “cost shifting”)
  - Decrease bound without changing overall function

A	B	f_1(A,B)
0	0	2.0
1	0	3.5
0	1	1.0
1	1	3.0

+

B	C	f_2(B,C)
0	0	1.0
0	1	0.0
1	0	1.0
1	1	3.0

=

$$f_{AB}(a,b) + f_{BC}(b,c)$$

A	B	C	F(A,B,C)
0	0	0	3.0
0	0	1	2.0
0	1	0	2.0
0	1	1	4.0
1	0	0	4.5
1	0	1	3.5
1	1	0	4.0
1	1	1	6.0

$$\max_{a,b} f_1(a,b) + \lambda_{B\to AB}(b) \quad + \quad \max_{b,c} f_2(b,c) + \lambda_{B\to BC}(b)$$

$$\lambda_{B\to AB}(b) + \lambda_{B\to BC}(b) = 0$$

(Adjusting functions cancel each other)

=

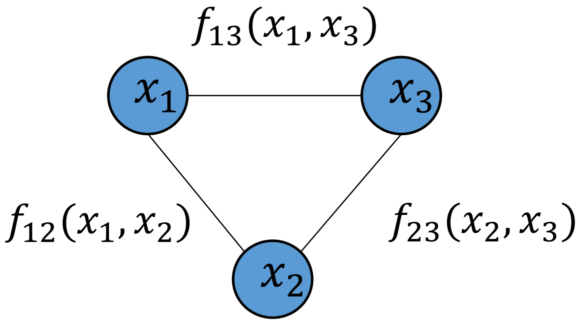
A	B	f_1(A,B)	¸(B)
0	0	2.0	0
1	0	3.5	
0	1	1.0	+1
1	1	3.0	

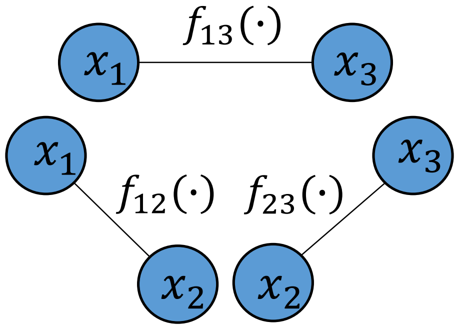
+

B	C	f_2(B,C)	-¸(B)
0	0	1.0	0
0	1	0.0	
1	0	1.0	-1
1	1	3.0	

(Decomposition bound is exact)

Dual Decomposition

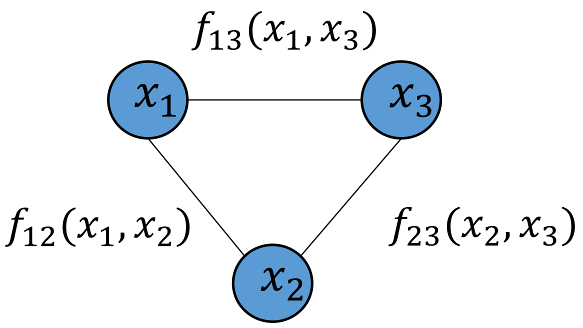


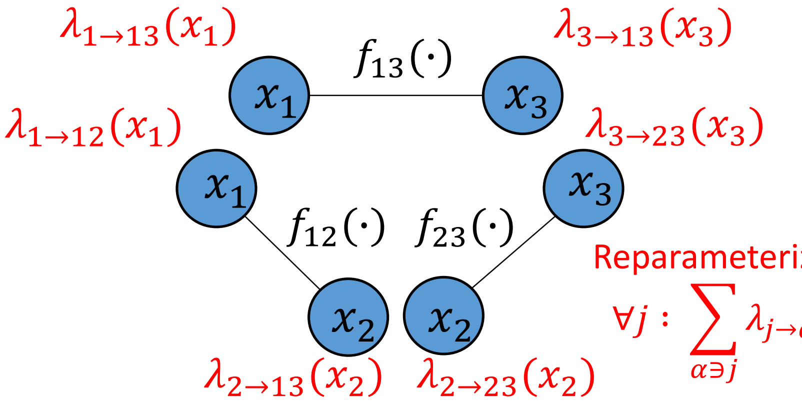


$$F^* = \min_x \sum_\alpha f_\alpha(x) \qquad \geq \qquad \sum_\alpha \min_x \; f_\alpha(x)$$

- Bound solution using decomposed optimization
- Solve independently: optimistic bound

Dual Decomposition





Reparameterization:

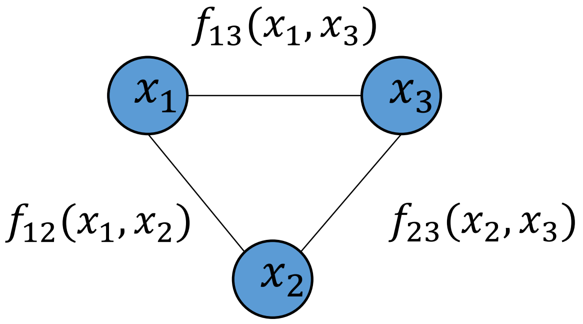
$$\forall j : \sum_{\alpha \ni j} \lambda_{j\to\alpha}(x_j) = 0$$

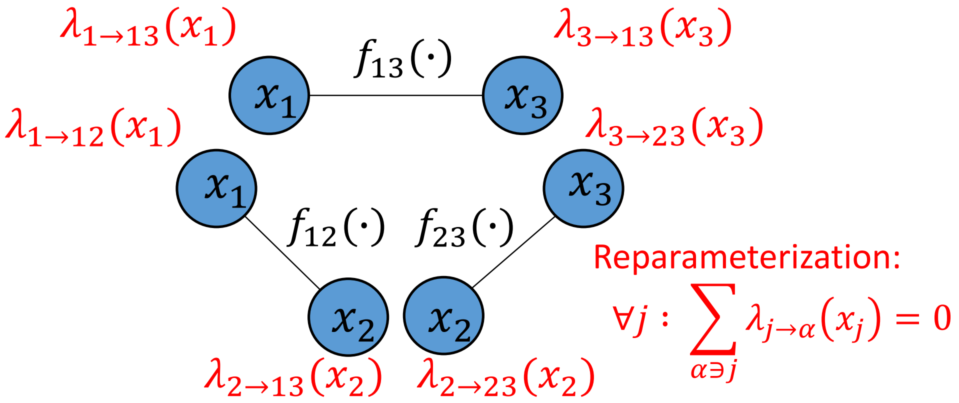
$$F^* = \min_x \sum_\alpha f_\alpha(x) \qquad \geq \qquad \max_{\lambda_{i\to\alpha}} \sum_\alpha \min_x \left[f_\alpha(x) + \sum_{i\in\alpha} \lambda_{i\to\alpha}(x_i) \right]$$

- Bound solution using decomposed optimization
- Solve independently: optimistic bound

- Tighten the bound by reparameterization
  - Enforce lost equality constraints via Lagrange multipliers

Dual Decomposition



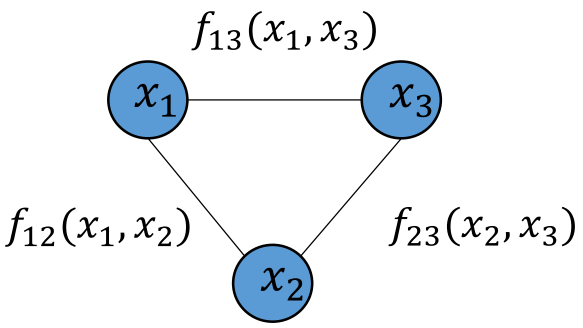


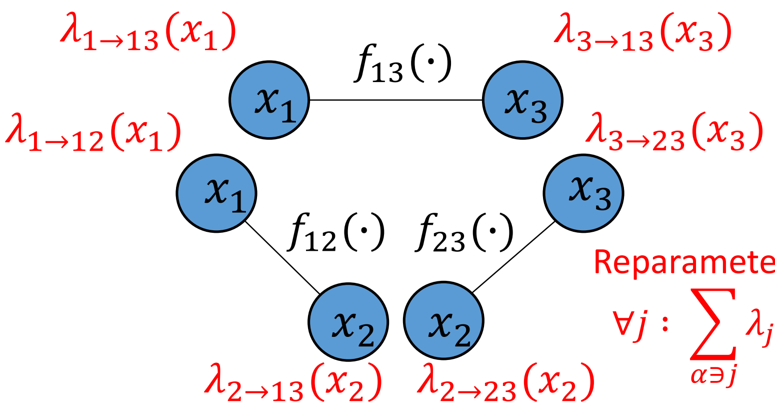
$$F^* = \min_x \sum_\alpha f_\alpha(x) \qquad \geq \qquad \max_{\lambda_{i\to\alpha}} \sum_\alpha \min_x \left[f_\alpha(x) + \sum_{i\in\alpha} \lambda_{i\to\alpha}(x_i) \right]$$

Many names for the same class of bounds:

- Dual decomposition [Komodakis et al. 2007]
- TRW, MPLP [Wainwright et al. 2005; Globerson & Jaakkola, 2007]
- Soft arc consistency [Cooper & Schiex, 2004]
- Max-sum diffusion [Warner 2007]

Dual Decomposition





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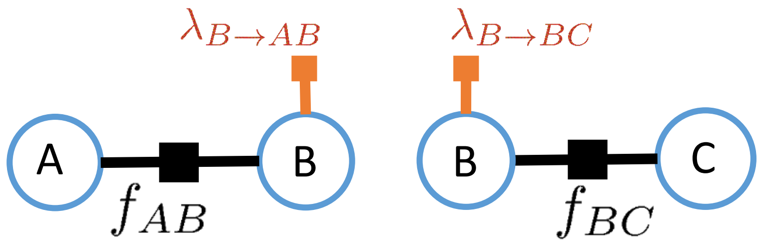
Many ways to optimize the bound:

- Sub-gradient descent [Komodakis et al. 2007; Jojic et al. 2010]
- Coordinate descent [Warner 2007; Globerson & Jaakkola 2007; Sontag et al. 2009; Ihler et al. 2012]
- Proximal optimization [Ravikumar et al, 2010]
- ADMM [Meshi & Globerson 2011; Martins et al. 2011; Forouzan & Ihler 2013]

Optimizing the bound

- Can optimize the bound in various ways:
  - (Sub-)gradient descent



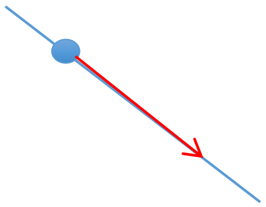
=

A	B	f_1(A,B)	.(B)
0	0	1.0	0
1	0	0.0	
0	1	0.0	0
1	1	2.5	
0	2	1.0	0
1	2	3.0	

+

B	C	f_2(B,C)	-.(B)
0	0	5.0	0
0	1	2.0	
1	0	1.0	0
1	1	1.5	
2	0	0.2	0
2	1	0.0	

$$\max_x f_1(a,b) + \lambda_{B\to AB}(b) \quad + \quad \max_x f_2(b,c) + \lambda_{B\to BC}(b)$$



Optimizing the bound

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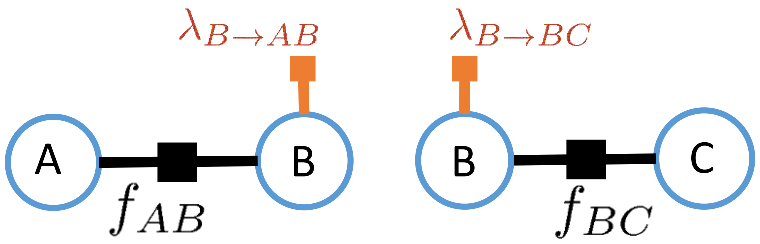
  - (Sub-)gradient descent



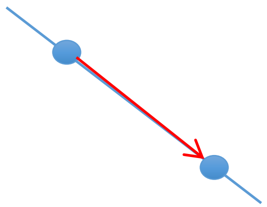
=

A	B	f$_1$(A,B)	¸(B)
0	0	1.0	+1
1	0	0.0	
0	1	0.0	0
1	1	2.5	
0	2	1.0	-1
1	2	3.0	

+

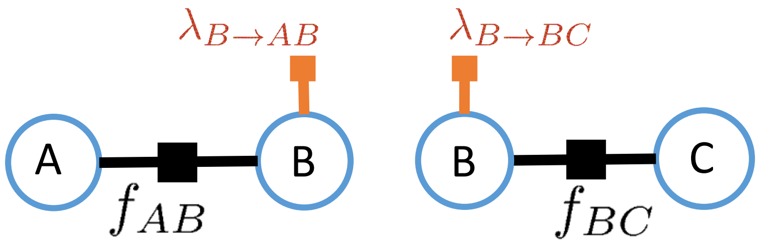
B	C	f$_2$(B,C)	-¸(B)
0	0	5.0	-1
0	1	2.0	
1	0	1.0	0
1	1	1.5	
2	0	0.2	+1
2	1	0.0	

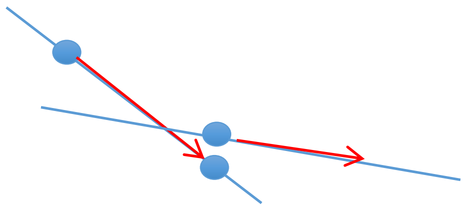
$$\max_x f_1(a,b) + \lambda_{B\to AB}(b) \quad + \quad \max_x f_2(b,c) + \lambda_{B\to BC}(b)$$



Optimizing the bound

- Can optimize the bound in various ways:
  - (Sub-)gradient descent





=

A	B	f₁(A,B)	λ(B)
0	0	1.0	+1
1	0	0.0	
0	1	0.0	0
1	1	2.5	
0	2	1.0	-1
1	2	3.0	

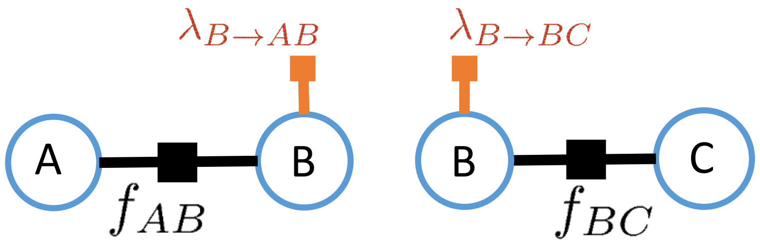
+

B	C	f₂(B,C)	-λ(B)
0	0	5.0	-1
0	1	2.0	
1	0	1.0	0
1	1	1.5	
2	0	0.2	+1
2	1	0.0	

$$\max_x f_1(a,b) + \lambda_{B\to AB}(b) \quad + \quad \max_x f_2(b,c) + \lambda_{B\to BC}(b)$$

Optimizing the bound

- Can optimize the bound in various ways:
  - (Sub-)gradient descent



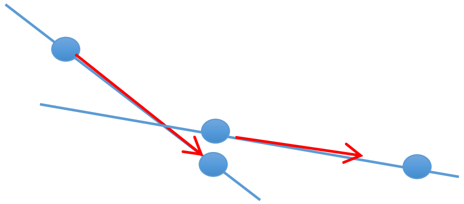
=

A	B	f$_1$(A,B)	¸(B)
0	0	1.0	+2
1	0	0.0	
0	1	0.0	-1
1	1	2.5	
0	2	1.0	-1
1	2	3.0	

+

B	C	f$_2$(B,C)	-¸(B)
0	0	5.0	-2
0	1	2.0	
1	0	1.0	+1
1	1	1.5	
2	0	0.2	+1
2	1	0.0	

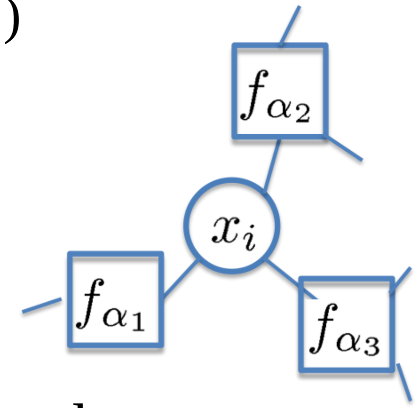
$$\max_x f_1(a,b) + \lambda_{B\to AB}(b) \quad + \quad \max_x f_2(b,c) + \lambda_{B\to BC}(b)$$



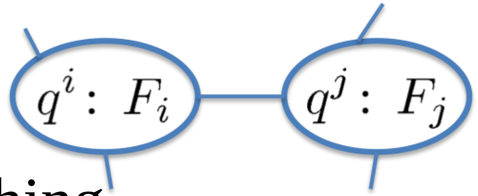
Various Update Schemes

- Can use any decomposition updates
  - (message passing, subgradient, augmented, etc.)

- FGLP: Update the original factors



- JGLP: Update clique function of the join graph



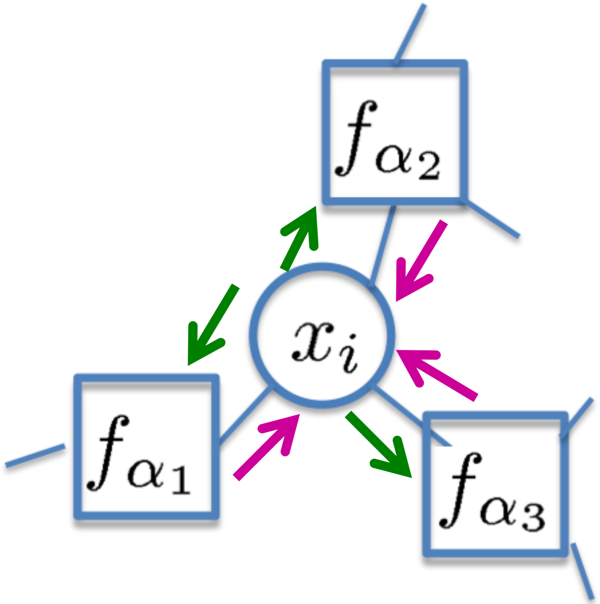
- **MBE-MM:** Mini-bucket with moment matching
  - Apply cost-shifting within each bucket only

Factor graph Linear Programming

- Update the original factors (FGLP)
  - Tighten all factors over over x_i simultaneously
  - Compute max-marginals
  - & update:

$$\forall \alpha, \; \gamma_\alpha(x_i) = \max_{x_\alpha \setminus x_i} f_\alpha$$

$$\forall \alpha, \; f_\alpha(x_\alpha) \leftarrow f_\alpha(x_\alpha) - \gamma_\alpha(x_i) + \frac{1}{|F_i|} \sum_\beta \gamma_\beta(x_i)$$



Mini-Bucket as Decomposition [Ihler et al. 2012]

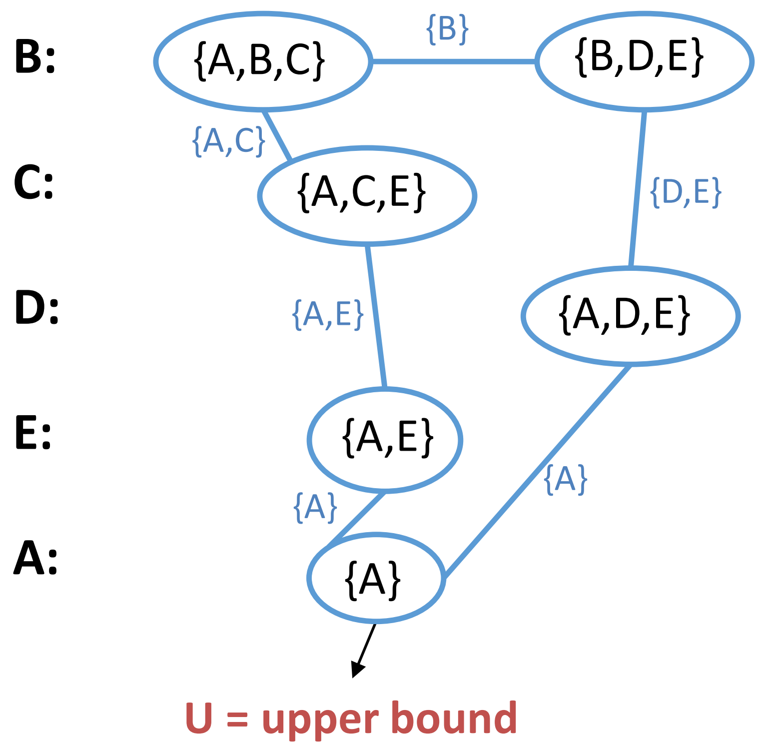
- Downward pass as cost shifting

- Can also do cost shifting within mini-buckets:
  "Join graph" message passing

- "Moment-matching" version:
  One message exchange within each bucket, during downward sweep

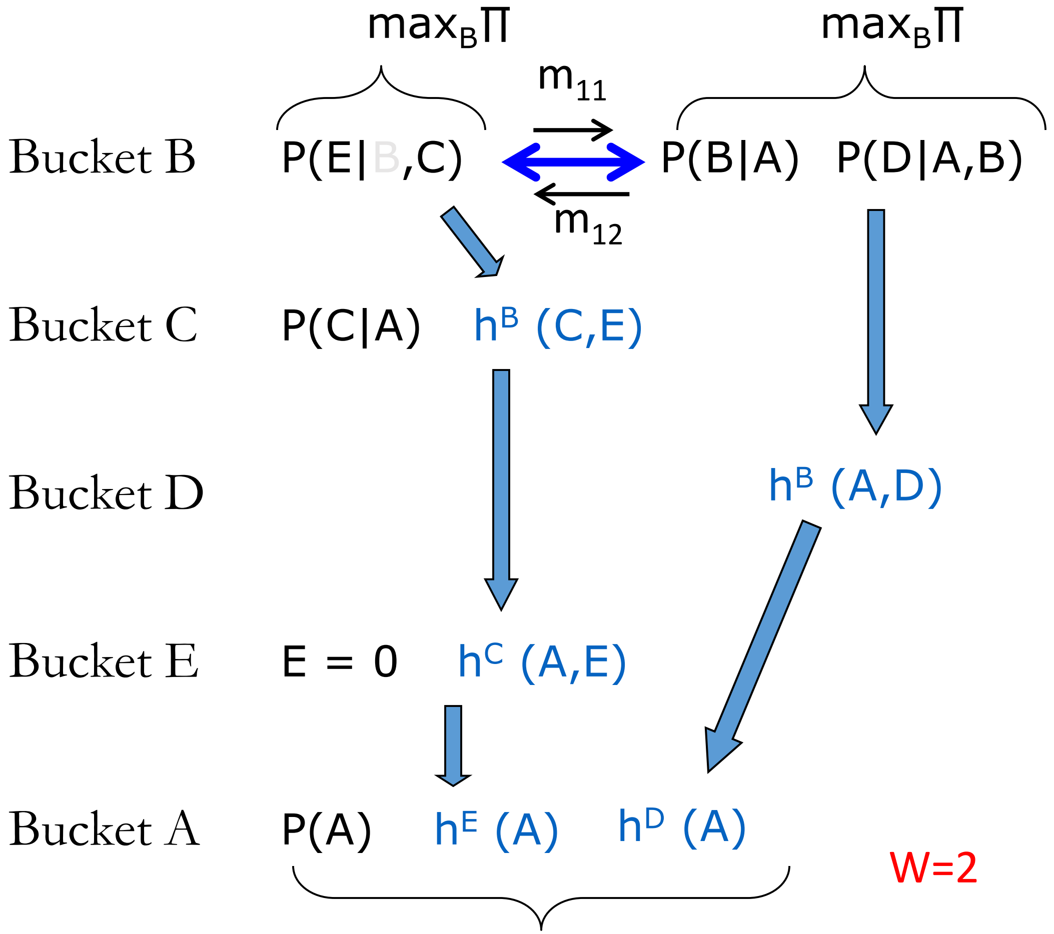
- Optimal bound defined by cliques ("regions") and cost-shifting f'n scopes ("coordinates")

Join graph:

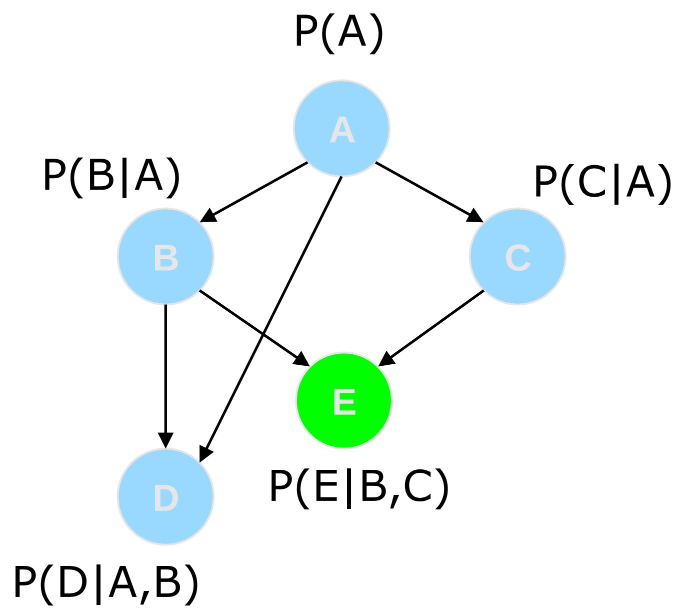


MBE-MM: MBE with moment matching

m_{11}, m_{12}- moment-matching messages



W=2



MPE* is an upper bound on MPE --U
Generating a solution yields a lower bound--L

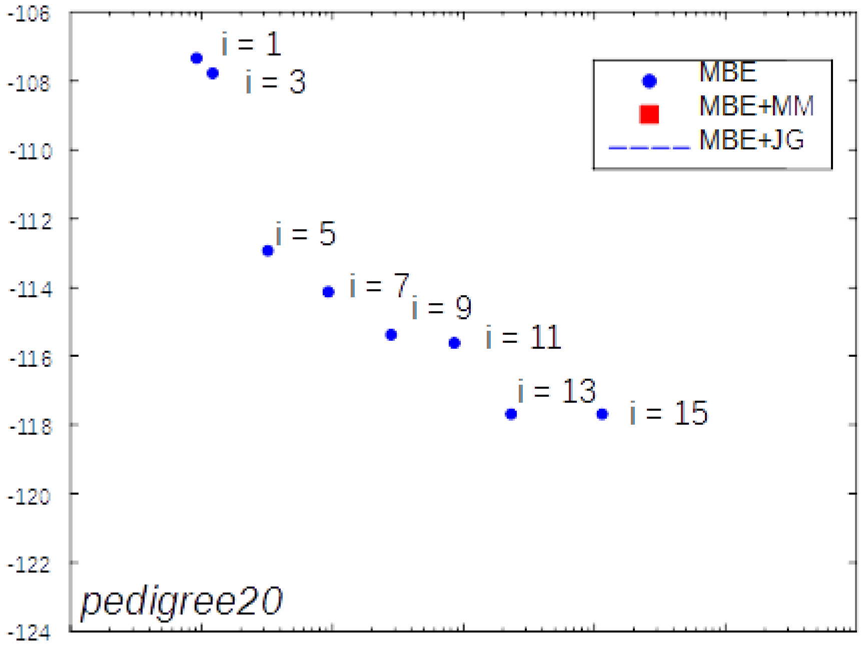
MBE-MM (MBE with Moment-Matching)

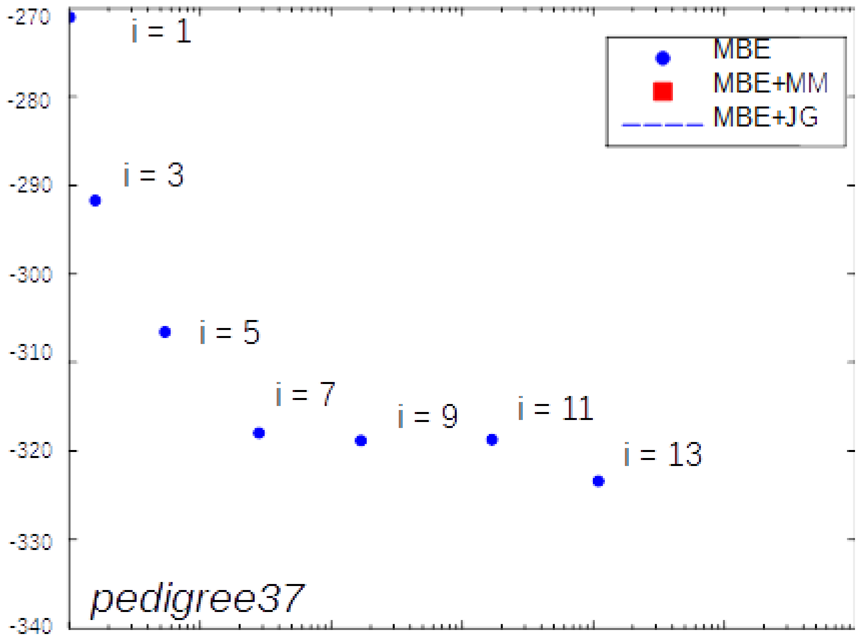
Algorithm 26: Algorithm MBE-MM

Input: A graphical model $\mathcal{M} = \langle \mathbf{X}, \mathbf{D}, \mathbf{F}, \sum \rangle$, variable order $o = \{X_1, \ldots, X_n\}$, i-bound parameter i
Output: Upper bound on the optimum value of MPE cost
//**Initialize:**
Partition the functions in $\mathbf{F}$ into $\mathbf{B}_{X_1}, \ldots, \mathbf{B}_{X_n}$, where $\mathbf{B}_{X_k}$ contains all functions f_j whose highest variable is X_k;
//processing bucket $\mathbf{B}_{X_k}$
for $k \leftarrow n$ *down to 1* **do**
    Partition functions g (both original and messages generated in previous buckets) in $\mathbf{B}_{X_k}$ into the mini-buckets defined $Q_{X_k} = \{q_k^1, \ldots, q_k^t\}$, where each q_k^p has no more than $i+1$ variables;
    Find the set of variables common to all the mini-buckets of variable X_k:
    $\mathbf{S}_k = Scope(q_k^1) \cap \cdots \cap Scope(q_k^t)$;
    Find the function of each mini-bucket
    q_k^p: $F_k^p \leftarrow \prod_{g \in q_k^p} g$;
    Find the max-marginals of each mini-bucket
    q_k^p: $\gamma^p_{X_k} = \max_{Scope(q_k^p) \setminus \mathbf{S}_k} (F_k^p)$;
    Update functions of each mini-bucket
    $F_k^p \leftarrow F_k^p - \gamma^p_{X_k} + \frac{1}{t} \sum_{j=1}^{t} \gamma^j_{X_k}$;
    Generate messages $h^p_{X_k \to X_m} = \max_{X_k} F_k^p$ and place each in the bucket of highest in the ordering o variable X_m in $Scope(q_k^p)$;
return *All the buckets and the cost bound from* $\mathbf{B}_1$;

Theorem 5.3 (**Complexity of MBE-MM**). *Given a problem with n variables having domain of size k and an i-bound i, the worst-case time complexity of MBE-MM is $O(n \cdot Q \cdot k^{i+1})$ and its space complexity is $O(n \cdot k^i)$, where Q bounds the number of functions having the same variable X_i in their scopes.*

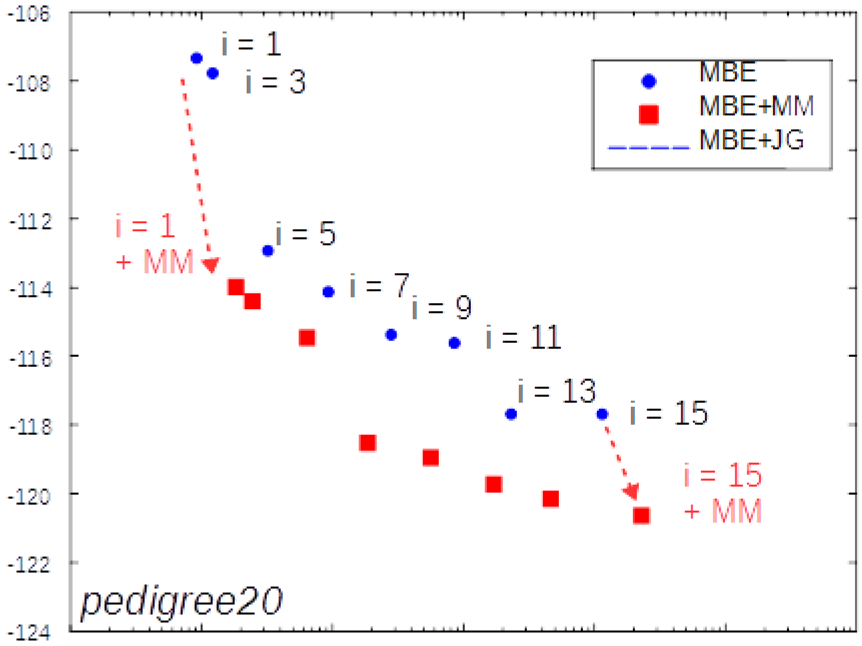
Anytime Approximation

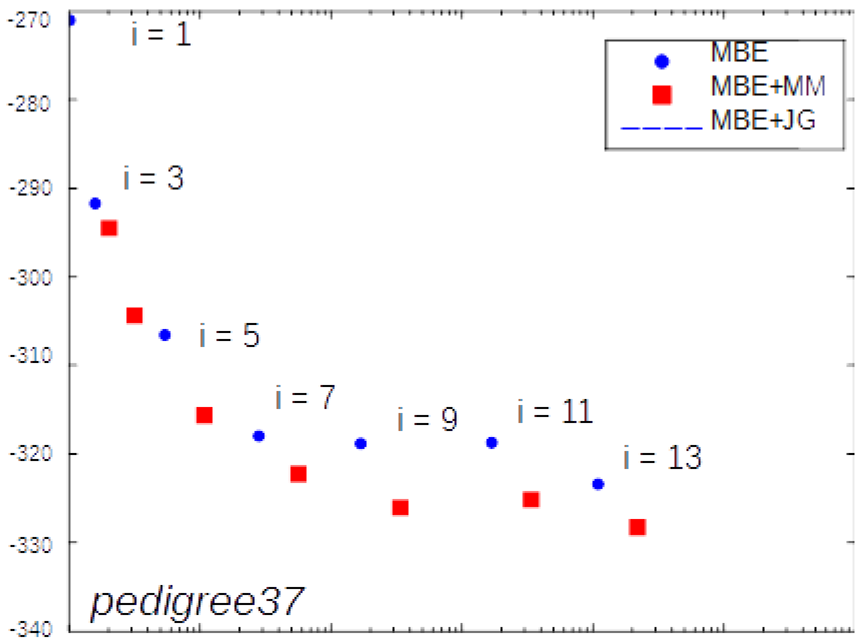




- Can tighten the bound in various ways
  - Cost-shifting (improve consistency between cliques)
  - Increase i-bound (higher order consistency)
- Simple moment-matching step improves bound significantly

Anytime Approximation





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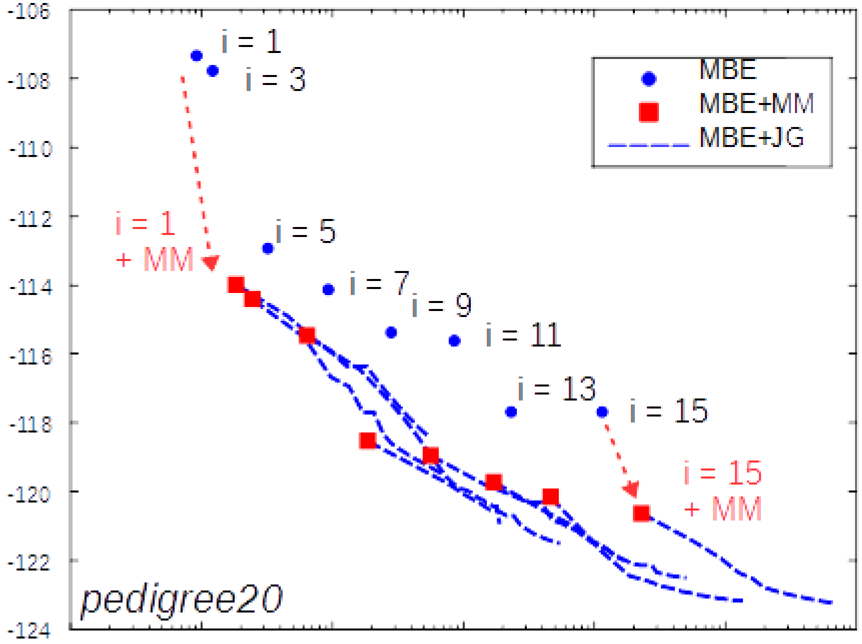

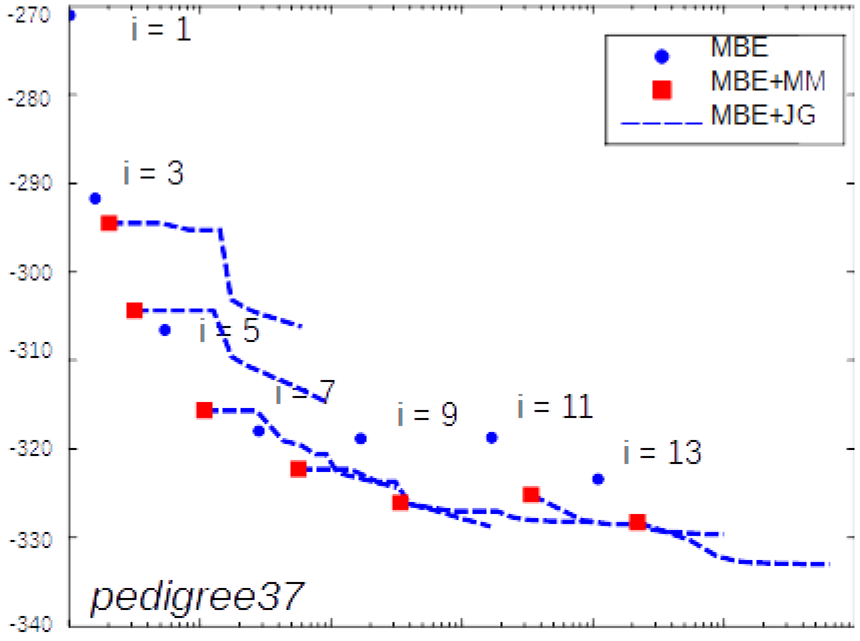

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