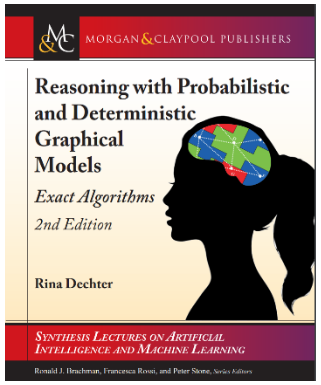
Causal and Probabilistic Reasoning

Slides Set 5:

Exact Inference Algorithms Bucket-elimination

Rina Dechter

(Dechter chapter 4, Darwiche chapter 6)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Reasoning with Probabilistic and Deterministic Graphical Models

Exact Algorithms

2nd Edition

Rina Dechter

Synthesis Lectures on Artificial Intelligence and Machine Learning

Ronald J. Brachman, Francesca Rossi, and Peter Stone, *Series Editors*

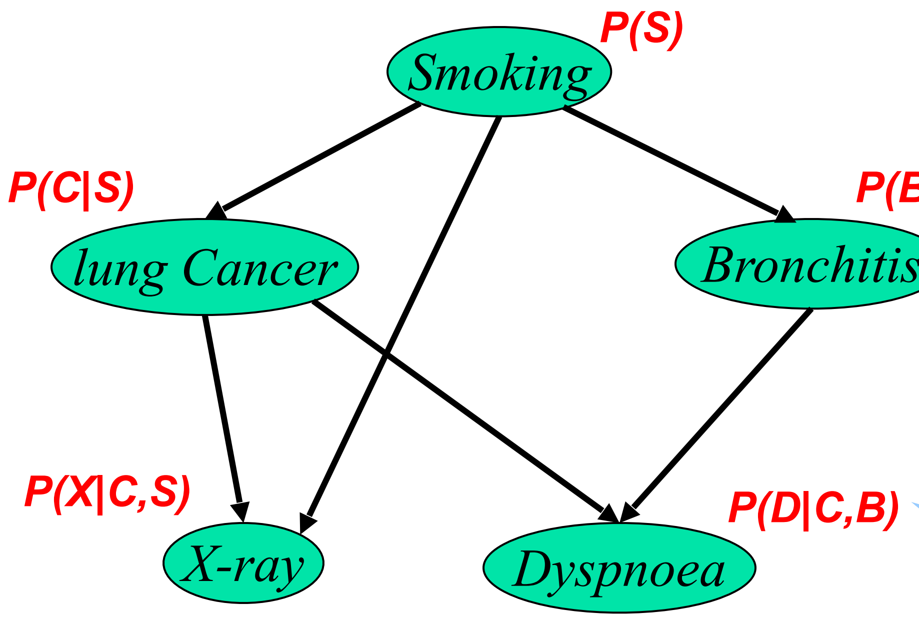
Inference for probabilistic networks

- Bucket elimination (Dechter chapter 4)
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→   Marginal Map)
  - Influence diagrams ?
- Induced-Width (Dechter, Chapter 3.4)

Inference for probabilistic networks

- Bucket elimination
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
- Induced-Width

Bayesian networks: example (Pearl, 1988)



CPD:

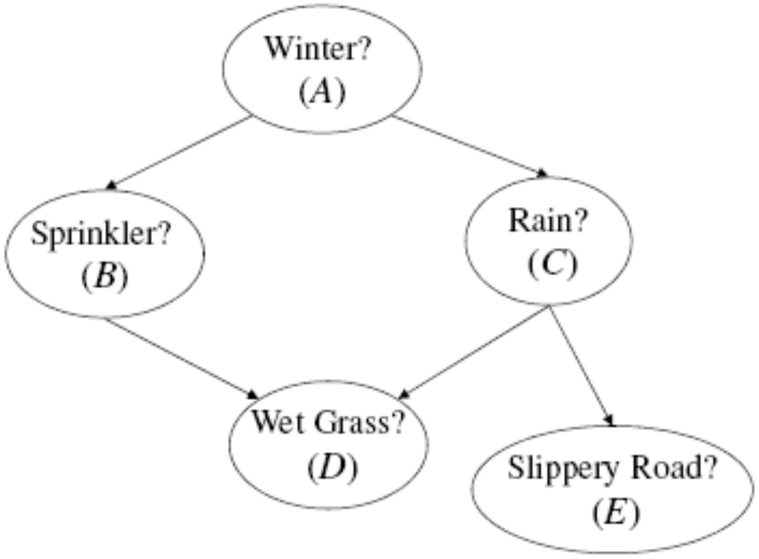
C	B	P(D|C,B)	
0	0	0.1	0.9
0	1	0.7	0.3
1	0	0.8	0.2
1	1	0.9	0.1

$$P(S, C, B, X, D) = P(S)\,P(C|S)\,P(B|S)\,P(X|C,S)\,P(D|C,B)$$

Belief Updating:

P (lung cancer=yes | smoking=no, dyspnoea=yes) = ?

A Bayesian Network



A	Θ_A
true	.6
false	.4

A	B	$\Theta_{B\|A}$
true	true	.2
true	false	.8
false	true	.75
false	false	.25

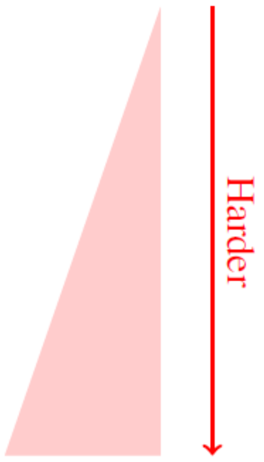
A	C	$\Theta_{C\|A}$
true	true	.8
true	false	.2
false	true	.1
false	false	.9

B	C	D	$\Theta_{D\|BC}$
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

C	E	$\Theta_{E\|C}$
true	true	.7
true	false	.3
false	true	0
false	false	1

Types of queries

Max-Inference:	$f(x^*) = \max_x \prod_\alpha f_\alpha(x_\alpha)$
Sum-Inference: (e.g., causal effects)	$Z = \sum_x \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MMAP): (optimal prediction)	$f_M(x_M^*) = \max_{x_M} \sum_{x_S} \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MEU): (e.g., decisions & planning)	$\text{MEU} = \max_{D_1,\ldots,D_m} \sum_{X_1,\ldots X_n} (\prod_{P_i \in P} P_i) \times (\sum_{r_i \in R} r_i)$



- **NP-hard**: exponentially many terms
- Difficulty (in part) due to restricted elimination orderings
- Focus is on **approximation** and Anytime algorithms
  - **Anytime**: very fast & very approximate ! Slower & more accurate

Belief updating is NP-hard

- Each SAT formula can be mapped into a belief updating query in a Bayesian network

- Example

A simple network

Given:

$$\begin{tikzcd} A \arrow[r, no head] & B \arrow[r] & C \arrow[r] & D \end{tikzcd}$$

- How can we compute P(D)?, P(D|A=0)? P(A|D=0)?
- Brute force O(k^4)
- Maybe O(4k^2)

A simple example

- Suppose we have two factors: $f(X) = f_{12}(X_1, X_2)\, f_{23}(X_2, X_3)$

- To compute the partition function (sum):

$$Z = \sum_{x_1, x_2, x_3} f(x_1, x_2, x_3) = f(0,0,0) + f(0,0,1) + f(0,0,2) + f(0,1,0) + \dots$$
$$+ f(1,0,0) + f(1,0,1) + f(1,0,2) + f(1,1,0) + \dots$$

  - Use the factorization of f(x):

$$Z = f_{12}(0,0)\, f_{23}(0,0) + f_{12}(0,0)\, f_{23}(0,1) + f_{12}(0,0)\, f_{23}(0,2) + f_{12}(0,1)\, f_{23}(1,0) + \dots$$
$$+ f_{12}(1,0)\, f_{23}(0,0) + f_{12}(1,0)\, f_{23}(0,1) + f_{12}(1,0)\, f_{23}(0,2) + f_{12}(1,1)\, f_{23}(1,0) + \dots$$

and apply the distributive rule:

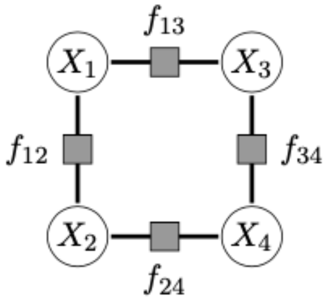
$$= f_{12}(0,0) \Big(f_{23}(0,0) + f_{23}(0,1) + f_{23}(0,2) \Big) + f_{12}(0,1) \Big(f_{23}(1,0) + \dots$$
$$+ f_{12}(1,0) \Big(f_{23}(0,0) + f_{23}(0,1) + f_{23}(0,2) \Big) + f_{12}(1,1) \Big(f_{23}(1,0) + \dots$$

We can pre-compute and re-use these terms in the sum!

$$\lambda(x_2) = \Big(\sum_{x_3} f_{23}(x_2, x_3) \Big)$$

$$Z = \sum_{x_1, x_2} f_{12}(x_1, x_2)\, \lambda(x_2)$$

Variable elimination



Product of factors:

$$p(X_1, X_2, X_3, X_4) = \frac{1}{Z} f_{12}(X_1, X_2) f_{13}(X_1, X_3) f_{24}(X_2, X_4) f_{34}(X_3, X_4).$$

Compute:

$$Z = \sum_{x_4} \sum_{x_3} \sum_{x_2} \sum_{x_1} f_{34}(x_3, x_4) f_{24}(x_2, x_4) f_{12}(x_1, x_2) f_{13}(x_1, x_3),$$

Collect terms involving x_1, then x_2, and so on:

$$Z = \sum_{x_4} \sum_{x_3} f_{34}(x_3, x_4) \sum_{x_2} f_{24}(x_2, x_4) \sum_{x_1} f_{12}(x_1, x_2) f_{13}(x_1, x_3),$$

"Bucket elimination":

$$\lambda_1(x_2, x_3) = \sum_{x_1} f_{12}(x_1, x_2) f_{13}(x_1, x_3),$$

Collect all factors with x_1 in a "bucket

$$\lambda_2(x_3, x_4) = \sum_{x_2} f_{24}(x_2, x_4) \lambda_1(x_2, x_3),$$

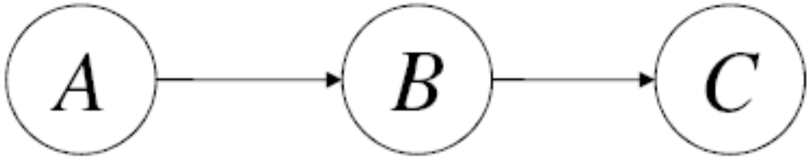
Collect all remaining factors with x_2

$$\lambda_3(x_4) = \sum_{x_3} f_{34}(x_3, x_4) \lambda_2(x_3, x_4),$$

Place intermediate calculations in bucket of their earliest argument

$$Z = \sum_{x_4} \lambda_3(x_4),$$

Elimination as a Basis for Inference



A	Θ_A
true	.6
false	.4

A	B	$\Theta_{B\|A}$
true	true	.9
true	false	.1
false	true	.2
false	false	.8

B	C	$\Theta_{C\|B}$
true	true	.3
true	false	.7
false	true	.5
false	false	.5

To compute the prior marginal on variable C, $\Pr(C)$

we first eliminate variable A and then variable B

Elimination as a Basis for Inference

- There are two factors that mention variable A, Θ_A and $\Theta_{B|A}$
- We multiply these factors first and then sum out variable A from the resulting factor.
- Multiplying Θ_A and $\Theta_{B|A}$:

A	B	$\Theta_A \Theta_{B\|A}$
true	true	.54
true	false	.06
false	true	.08
false	false	.32

- Summing out variable A:

B	$\sum_A \Theta_A \Theta_{B\|A}$
true	$.62 = .54 + .08$
false	$.38 = .06 + .32$

Elimination as a Basis for Inference

- We now have two factors, $\sum_A \Theta_A \Theta_{B|A}$ and $\Theta_{C|B}$, and we want to eliminate variable B
- Since B appears in both factors, we must multiply them first and then sum out B from the result.
- Multiplying:

B	C	$\Theta_{C\|B} \sum_A \Theta_A \Theta_{B\|A}$
true	true	.186
true	false	.434
false	true	.190
false	false	.190

- Summing out:

C	$\sum_B \Theta_{C\|B} \sum_A \Theta_A \Theta_{B\|A}$
true	.376
false	.624

Elimination as a Basis for Inference

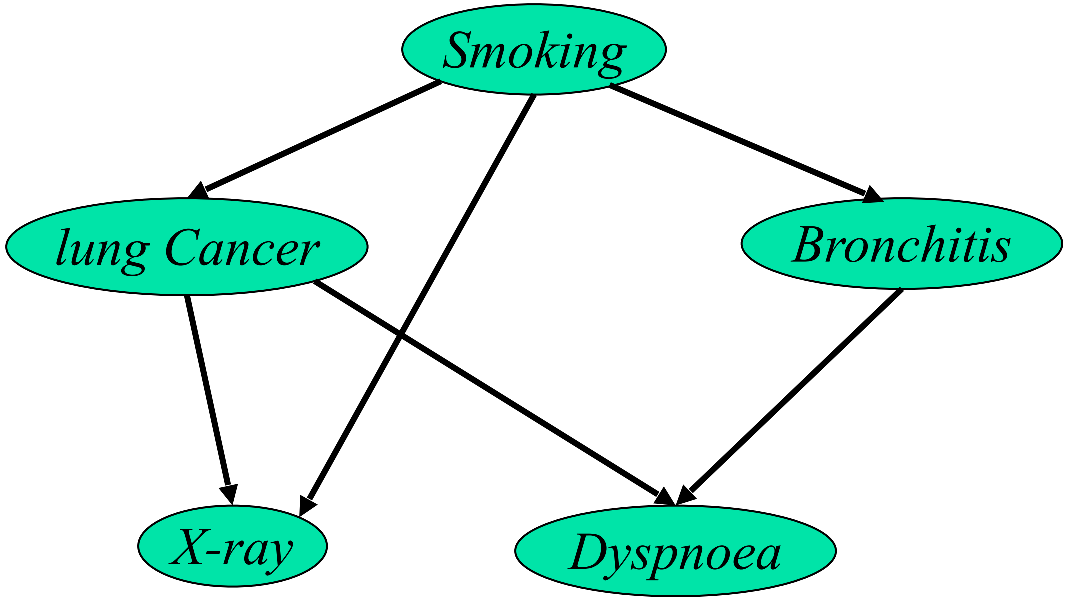
- We now have two factors, $\sum_A \Theta_A \Theta_{B|A}$ and $\Theta_{C|B}$, and we want to eliminate variable B
- Since B appears in both factors, we must multiply them first and then sum out B from the result.
- Multiplying:

B	C	$\Theta_{C\mid B} \sum_A \Theta_A \Theta_{B\mid A}$
true	true	.186
true	false	.434
false	true	.190
false	false	.190

- Summing out:

C	$\sum_B \Theta_{C\mid B} \sum_A \Theta_A \Theta_{B\mid A}$
true	.376
false	.624

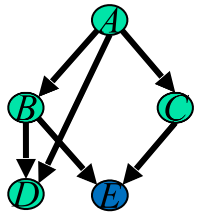
Belief updating



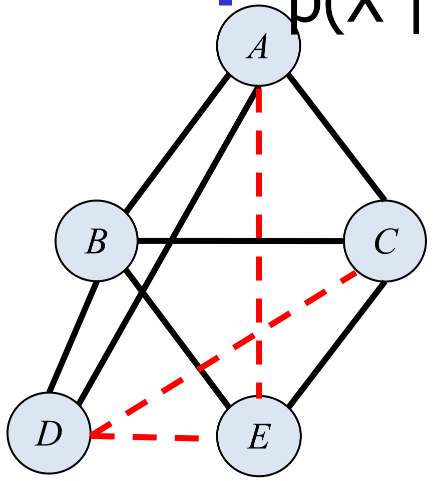
P (lung cancer=yes | smoking=no, dyspnoea=yes) = ?

From essai

Belief updating



- p(X | Evidence) = ?



"primal" graph

$$p(A|E=0)$$

$$\propto p(A, E=0)$$

$$= \sum_{e,d,c,b} p(A)\, \underbrace{p(b|A)}\, p(c|A)\, \underbrace{p(d|b,A)\, p(e|b,c)}\, \mathbb{1}[e=0]$$

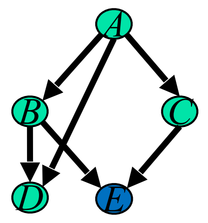
$$p(A) \sum_{e} \sum_{d} \sum_{c} p(c|A)\, \mathbb{1}[e=0]\, \underbrace{\sum_{b} p(b|A)\, p(d|b,A)\, p(e|b,c)}_{\lambda_{B\to C}(a,d,c,e)}$$

$\boxed{\textit{Variable Elimination}}$

From essai

Bucket elimination

Algorithm BE-bel [Dechter 1996]



$$p(A|E=0) = \alpha \sum_{e,d,c,b} p(A)\, p(b|A)\, p(c|A)\, p(d|A,b)\, p(e|b,c)\, \mathbb{1}[e=0]$$

$$\sum_b \prod \longleftarrow$$ *Elimination & combination operators*

bucket B: $\overbrace{p(b|A)\;\; p(d|b,A)\;\; p(e|b,c)}$

bucket C: $p(c|A) \quad \lambda_{B\to C}(\underbrace{A,d,c,e})$

bucket D: $\lambda_{C\to D}(A,d,e)$

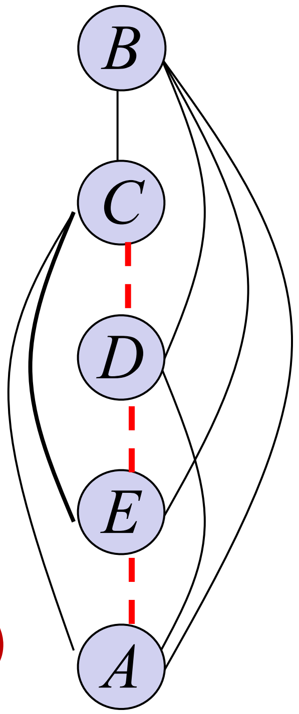
bucket E: $\mathbb{1}[E=0]\ \lambda_{D\to E}(A,e)$

bucket A: $p(A) \quad \lambda_{E\to A}(A)$

$p(E=0)$

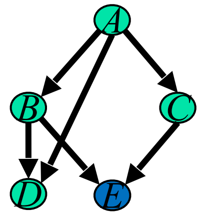
$$p(A|E=0) = p(A,E=0)\,/\,p(E=0)$$

W=4*
"induced width"
(max clique size)



Bucket elimination

Algorithm BE-bel [Dechter 1996]



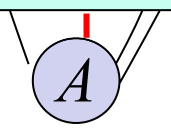
$$p(A|E=0) = \alpha \sum_{e,d,c,b} p(A)\, p(b|A)\, p(c|A)\, p(d|A,b)\, p(e|b,c)\, \mathbb{1}[e=0]$$

$$\sum_b \prod \longleftarrow$$ *Elimination & combination operators*

Time and space exponential in the induced-width / treewidth

bucket A: $p(A)$ $\lambda_{E\to A}(A)$

induced width
(max clique size)

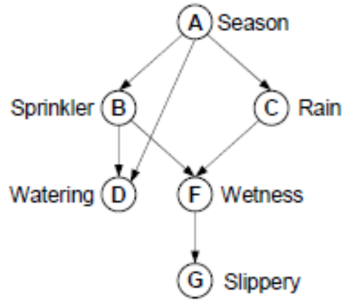


$p(E=0)$

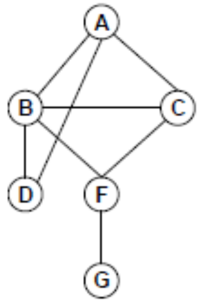
$p(A|E=0) = p(A,E=0)\ /\ p(E=0)$

A Bayesian network

ordering: A,C,B,F,D,G



(a) Directed acyclic graph



(b) Moral graph

$$P(a, g=1) = \sum_{c,b,e,d,g=1} P(a,b,c,d,e,g) = \sum_{c,b,f,d,g=1} P(g|f)P(f|b,c)P(d|a,b)P(c|a)P(b|a)P(a).$$

$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\sum_{f} P(f|b,c)\sum_{d} P(d|b,a)\sum_{g=1} P(g|f). \quad (4.1)$$

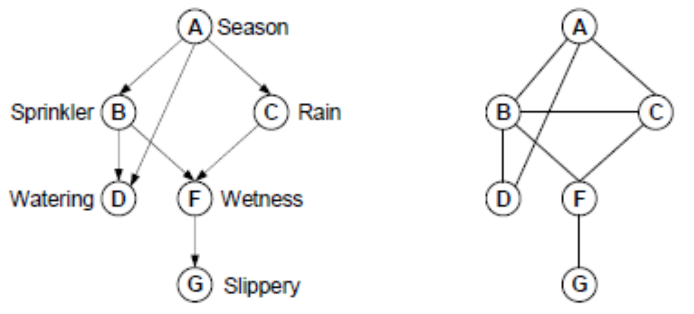
$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\sum_{f} P(f|b,c)\lambda_G(f)\sum_{d} P(d|b,a). \quad (4.2)$$

$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\lambda_D(a,b)\sum_{f} P(f|b,c)\lambda_G(f) \quad (4.3)$$

$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\lambda_D(a,b)\lambda_F(b,c) \quad (4.4)$$

$$P(a, g=1) = P(a)\sum_{c} P(c|a)\lambda_B(a,c) \quad (4.5)$$

A Bayesian network
ordering: A,C,B,F,D,G



(a) Directed acyclic graph (b) Moral graph

$$P(a, g=1) = \sum_{c,b,e,d,g=1} P(a,b,c,d,e,g) = \sum_{c,b,f,d,g=1} P(g|f)P(f|b,c)P(d|a,b)P(c|a)P(b|a)P(a).$$

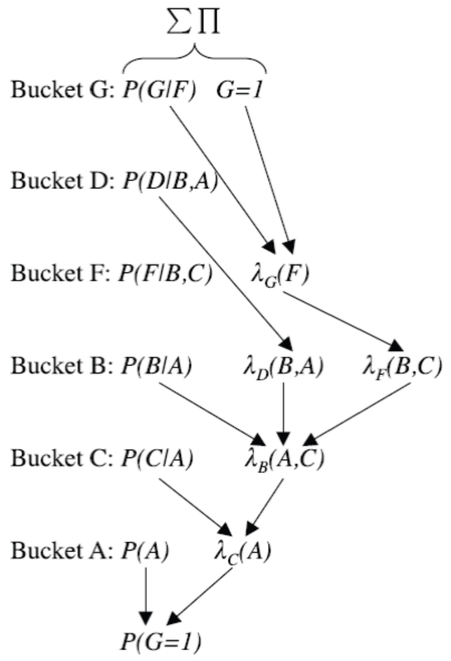
$$P(a, g=1) = P(a)\sum_c P(c|a)\sum_b P(b|a)\sum_f P(f|b,c)\sum_d P(d|b,a)\boxed{\sum_{g=1} P(g|f)} \quad (4.1)$$

$$P(a, g=1) = P(a)\sum_c P(c|a)\sum_b P(b|a)\sum_f P(f|b,c)\lambda_G(f)\boxed{\sum_d P(d|b,a).} \quad (4.2)$$

$$P(a, g=1) = P(a)\sum_c P(c|a)\sum_b P(b|a)\lambda_D(a,b)\boxed{\sum_f P(f|b,c)\lambda_G(f)} \quad (4.3)$$

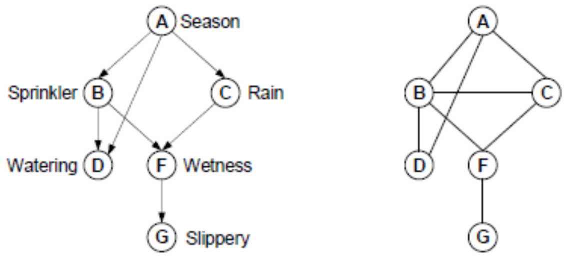
$$P(a, g=1) = P(a)\sum_c P(c|a)\boxed{\sum_b P(b|a)\lambda_D(a,b)\lambda_F(b,c)} \quad (4.4)$$

$$P(a, g=1) = P(a)\sum_c P(c|a)\lambda_B(a,c) \quad (4.5)$$



ΣΠ
Bucket G: P(G|F) G=1
Bucket D: P(D|B,A)
Bucket F: P(F|B,C) λ_G(F)
Bucket B: P(B|A) λ_D(B,A) λ_F(B,C)
Bucket C: P(C|A) λ_B(A,C)
Bucket A: P(A) λ_C(A)
P(G=1)

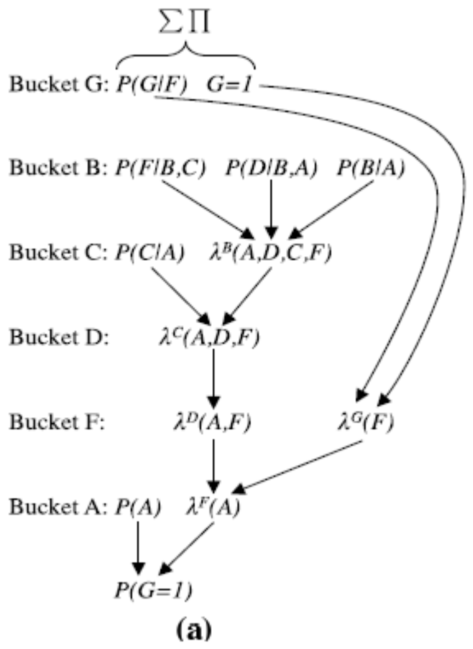
A different ordering



(a) Directed acyclic graph (b) Moral graph

Ordering: A,F,D,C,B,G

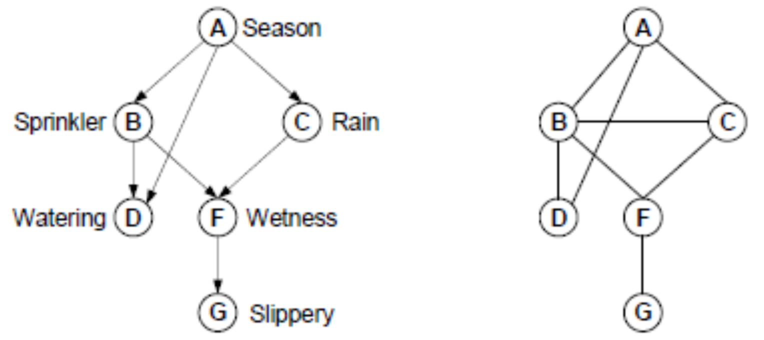
$$P(a, g=1) = P(a)\sum_f \sum_d \sum_c P(c|a) \sum_b P(b|a)\ P(d|a,b)P(f|b,c) \sum_{g=1} P(g|f)$$
$$= P(a)\sum_f \lambda_G(f) \sum_d \sum_c P(c|a) \sum_b P(b|a)\ P(d|a,b)P(f|b,c)$$
$$= P(a)\sum_f \lambda_G(f) \sum_d \sum_c P(c|a)\lambda_B(a,d,c,f)$$
$$= P(a)\sum_f \lambda_g(f) \sum_d \lambda_C(a,d,f)$$
$$= P(a)\sum_f \lambda_G(f)\lambda_D(a,f)$$
$$= P(a)\lambda_F(a)$$



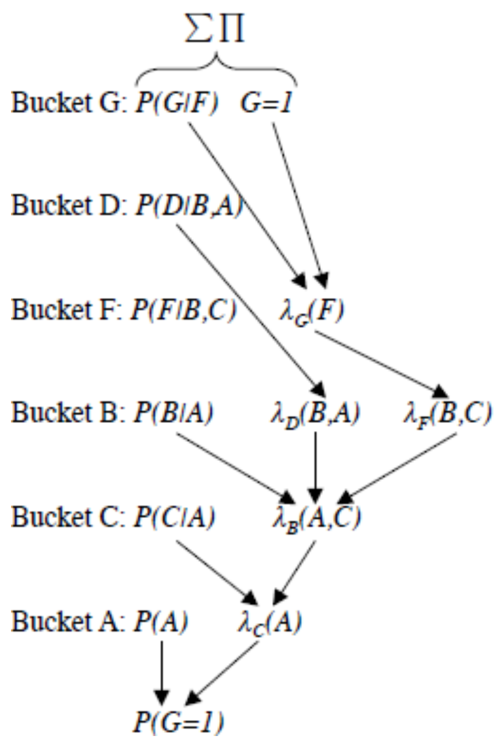
ΣΠ
Bucket G: P(G|F) G=1
Bucket B: P(F|B,C) P(D|B,A) P(B|A)
Bucket C: P(C|A) λ^B(A,D,C,F)
Bucket D: λ^C(A,D,F)
Bucket F: λ^D(A,F) λ^G(F)
Bucket A: P(A) λ^F(A)
P(G=1)
(a)

Figure 4.3: The bucket's output when processing along $d_2 = A, F, D, C, B, G$

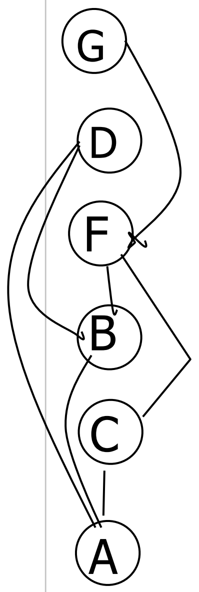
A Bayesian network processed along two orderings

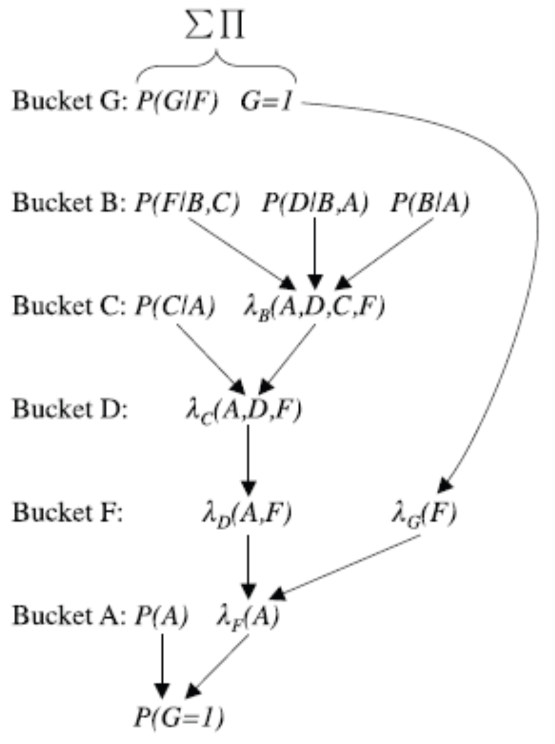


(a) Directed acyclic graph (b) Moral graph



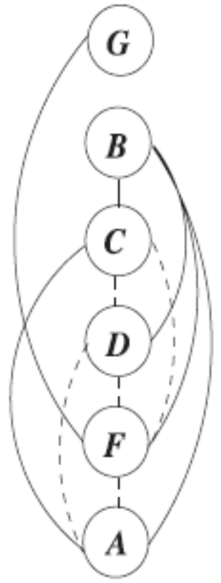
$\Sigma\Pi$
Bucket G: $P(G|F)$ $G=1$
Bucket D: $P(D|B,A)$
Bucket F: $P(F|B,C)$ $\lambda_G(F)$
Bucket B: $P(B|A)$ $\lambda_D(B,A)$ $\lambda_F(B,C)$
Bucket C: $P(C|A)$ $\lambda_B(A,C)$
Bucket A: $P(A)$ $\lambda_C(A)$
$P(G=1)$





$\Sigma\Pi$
Bucket G: $P(G|F)$ $G=1$
Bucket B: $P(F|B,C)$ $P(D|B,A)$ $P(B|A)$
Bucket C: $P(C|A)$ $\lambda_B(A,D,C,F)$
Bucket D: $\lambda_C(A,D,F)$
Bucket F: $\lambda_D(A,F)$ $\lambda_G(F)$
Bucket A: $P(A)$ $\lambda_F(A)$
$P(G=1)$

(a)



(b)

d1=A,C,B,F,D,G

d2: A,F,D,C,B,G

The operation in a bucket

- Multiplying functions
- Marginalizing (summing-out) functions

Combination of Cost Functions

A	B	f(A,B)
b	b	0.4
b	g	0.1
g	b	0
g	g	0.5

●

B	C	f(B,C)
b	b	0.2
b	g	0
g	b	0
g	g	0.8

A	B	C	f(A,B,C)
b	b	b	0.1
b	b	g	0
b	g	b	0
b	g	g	0.08
g	b	b	0
g	b	g	0
g	g	b	0
g	g	g	0.4

= *0.1 x 0.8*

Elimination in a factor

A	B	f(A,B)
b	b	4
b	g	6
b	r	1
g	b	2
g	g	6
g	r	3
r	b	1
r	g	1
r	r	6

Elim = sum

$\xrightarrow{\text{Elim}(\mathbf{f},\mathbf{B})}$

A	g(A)
b	11
g	11
r	8

$\xrightarrow{\text{Elim}(\mathbf{g},\mathbf{A})}$

$h_\emptyset$
30

Factors: Sum-Out Operation

The result of summing out variable X from factor $f(\mathbf{X})$

is another factor over variables $\mathbf{Y} = \mathbf{X} \setminus \{X\}$:

$$\left(\sum_X f\right)(\mathbf{y}) \overset{def}{=} \sum_x f(x, \mathbf{y})$$

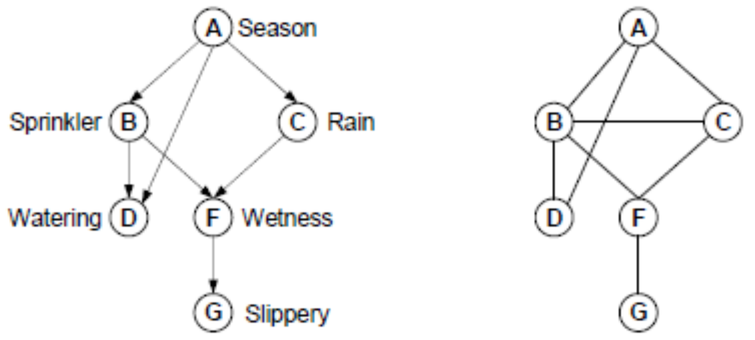
B	C	D	f_1
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

B	C	$\sum_D f_1$
true	true	1
true	false	1
false	true	1
false	false	1

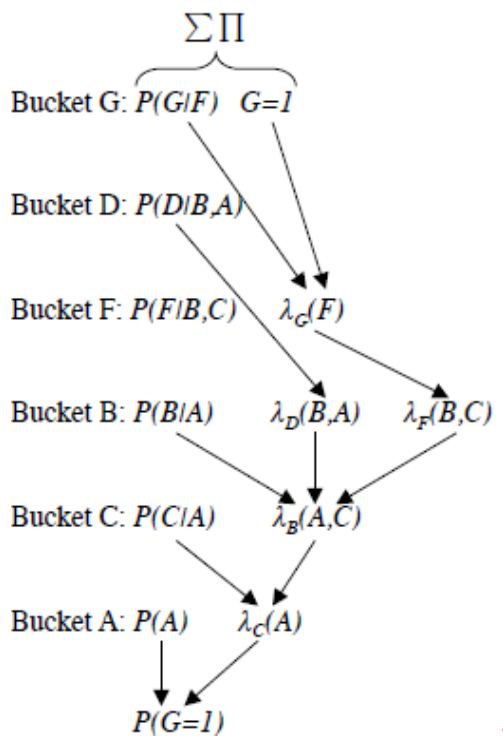
	$\sum_B \sum_C \sum_D f_1$
$\top$	4

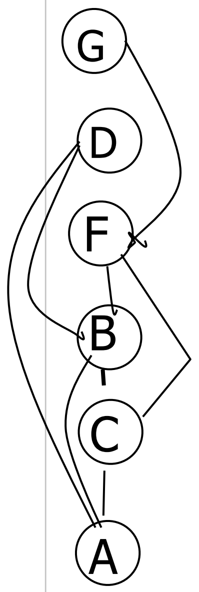
Thanks to Darwiche

Bucket elimination and induced-width



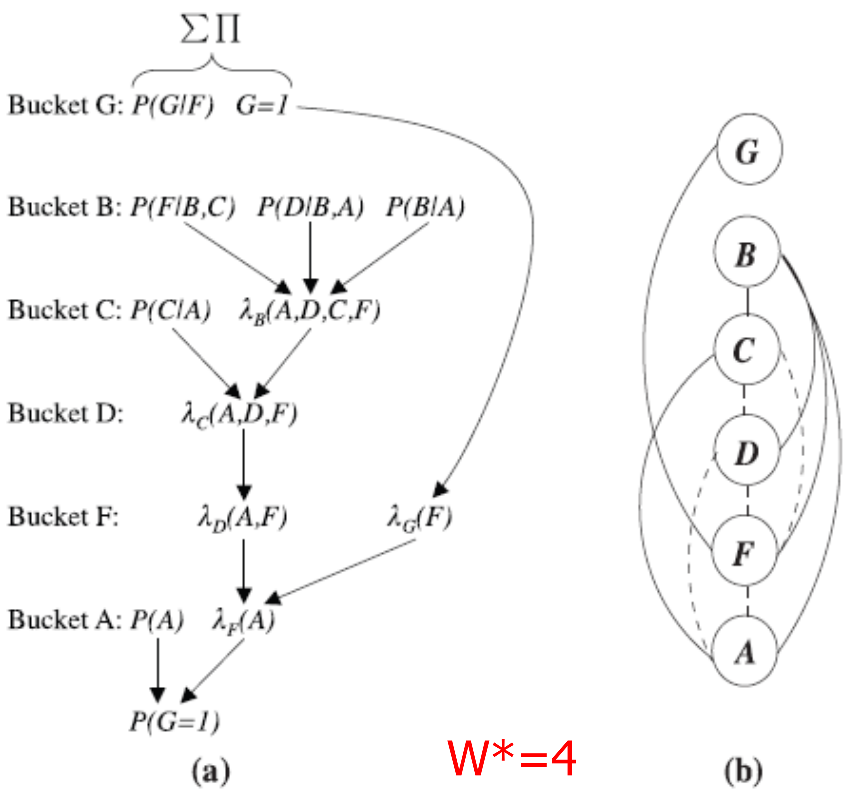
(a) Directed acyclic graph (b) Moral graph


$\Sigma\Pi$
Bucket G: $P(G|F)$ $G=1$
Bucket D: $P(D|B,A)$
Bucket F: $P(F|B,C)$ $\lambda_G(F)$
Bucket B: $P(B|A)$ $\lambda_D(B,A)$ $\lambda_F(B,C)$
Bucket C: $P(C|A)$ $\lambda_B(A,C)$
Bucket A: $P(A)$ $\lambda_C(A)$
$P(G=1)$



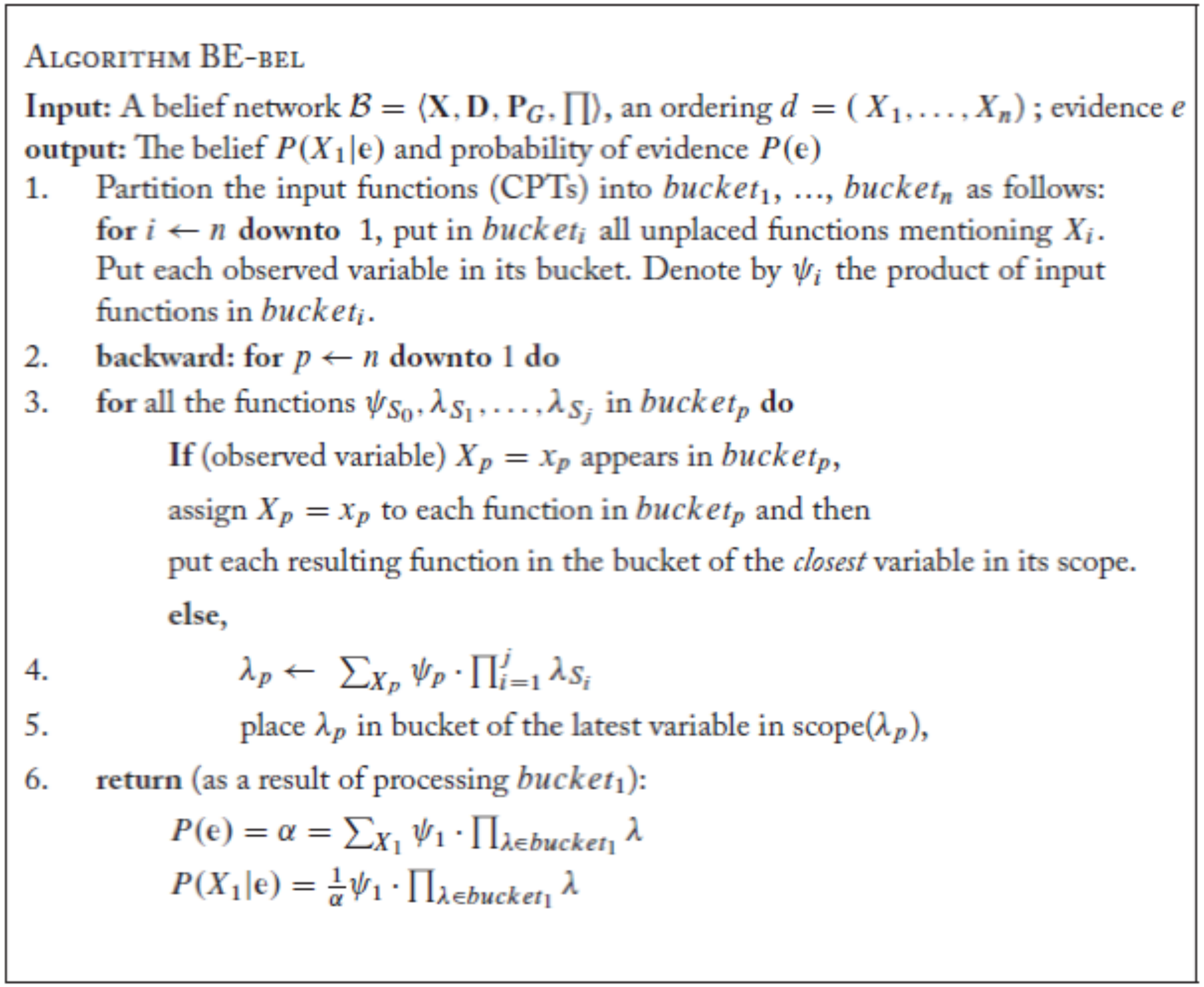
W*=2

d1=A,C,B,F,D,G


$\Sigma\Pi$
Bucket G: $P(G|F)$ $G=1$
Bucket B: $P(F|B,C)$ $P(D|B,A)$ $P(B|A)$
Bucket C: $P(C|A)$ $\lambda_B(A,D,C,F)$
Bucket D: $\lambda_C(A,D,F)$
Bucket F: $\lambda_D(A,F)$ $\lambda_G(F)$
Bucket A: $P(A)$ $\lambda_F(A)$
$P(G=1)$
(a)
G
B
C
D
F
A
(b)

W*=4

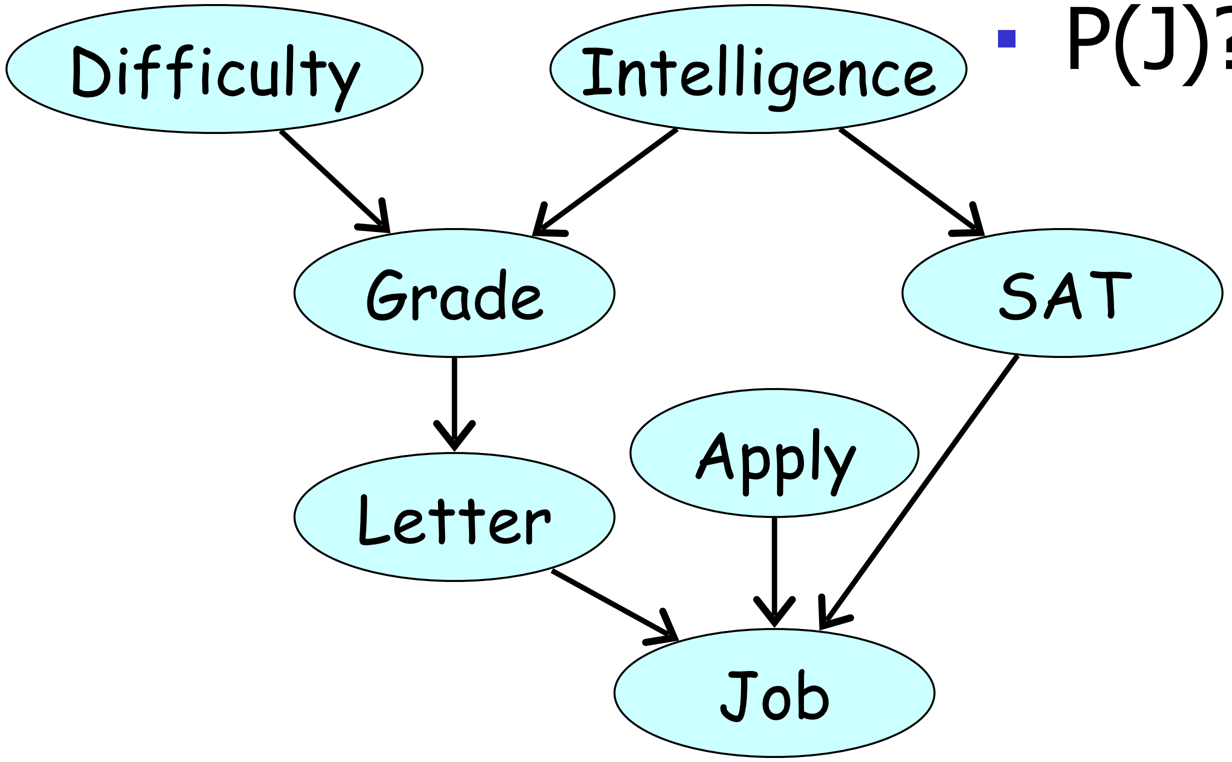
d2: A,F,D,C,B,G



```
ALGORITHM BE-BEL
Input: A belief network $\mathcal{B} = \langle \mathbf{X}, \mathbf{D}, \mathbf{P}_G, \prod \rangle$, an ordering $d = (X_1, \ldots, X_n)$ ; evidence $e$
output: The belief $P(X_1|\mathrm{e})$ and probability of evidence $P(\mathrm{e})$
1.  Partition the input functions (CPTs) into $bucket_1$, …, $bucket_n$ as follows:
    for $i \leftarrow n$ downto 1, put in $bucket_i$ all unplaced functions mentioning $X_i$.
    Put each observed variable in its bucket. Denote by $\psi_i$ the product of input
    functions in $bucket_i$.
2.  backward: for $p \leftarrow n$ downto 1 do
3.  for all the functions $\psi_{S_0}, \lambda_{S_1}, \ldots, \lambda_{S_j}$ in $bucket_p$ do
        If (observed variable) $X_p = x_p$ appears in $bucket_p$,
        assign $X_p = x_p$ to each function in $bucket_p$ and then
        put each resulting function in the bucket of the *closest* variable in its scope.
        else,
4.          $\lambda_p \leftarrow \sum_{X_p} \psi_p \cdot \prod_{i=1}^{j} \lambda_{S_i}$
5.          place $\lambda_p$ in bucket of the latest variable in scope($\lambda_p$),
6.  return (as a result of processing $bucket_1$):
        $P(\mathrm{e}) = \alpha = \sum_{X_1} \psi_1 \cdot \prod_{\lambda \in bucket_1} \lambda$
        $P(X_1|\mathrm{e}) = \frac{1}{\alpha} \psi_1 \cdot \prod_{\lambda \in bucket_1} \lambda$
```

Figure 4.5: BE-bel: a sum-product bucket-elimination algorithm.

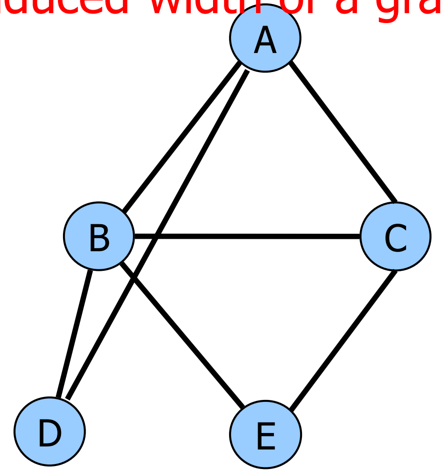
Student network example



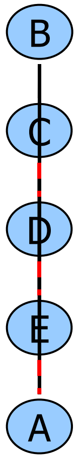
- P(J)?

Induced width

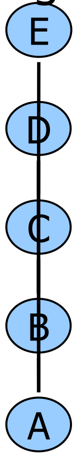
- Width is the max number of parents in the ordered graph
- Induced-width is the width of the induced ordered graph: recursively connecting parents going from last node to first.
- Induced-width w*(d) is the max induced-width over all nodes in ordering d
- Induced-width of a graph, w* is the min w*(d) over all orderings d



primal graph



$w^*(d_1) = 4$



$w^*(d_2) = 2$

Complexity of bucket elimination

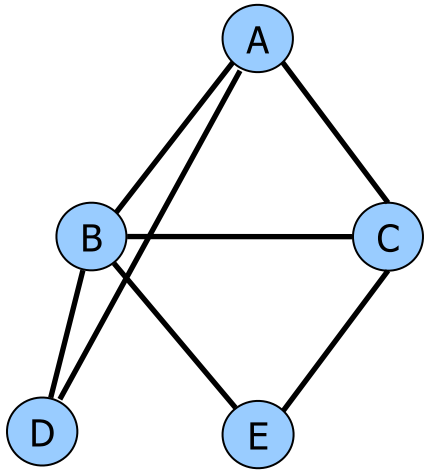
Bucket-Elimination is **time** and **space**

$$O\left(r \exp(w_d^*)\right)$$

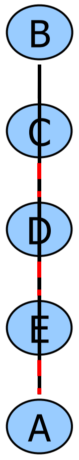
w_d^*: the induced width of the primal graph along ordering d

r = number of functions

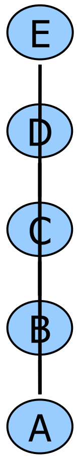
The effect of the ordering:



primal graph



$w^*(d_1) = 4$



$w^*(d_2) = 2$

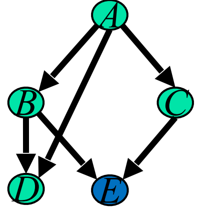
Finding smallest induced-width is hard!

Inference for probabilistic networks

- Bucket elimination
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→   Marginal Map)
- Induced-Width

The impact of evidence?

Algorithm BE-bel



Elimination operator

bucket B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$ B=1

bucket C: $P(c|a)$

bucket D:

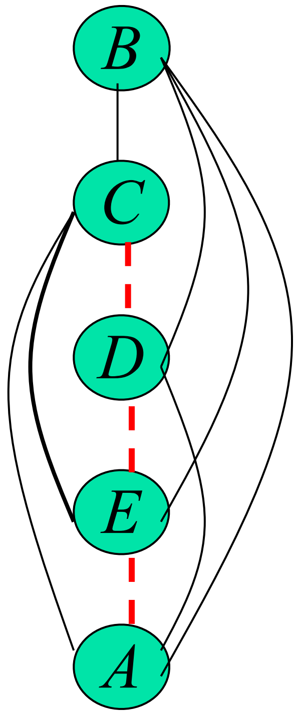
bucket E: $e=0$

bucket A: $P(a)$

$P(e=0)$

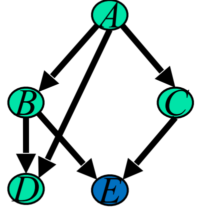
$P(a|e=0)$

W*=4
"induced width"
(max clique size)



The impact of evidence?

Algorithm BE-bel



P(A|E=0,B=1)?

Elimination operator

bucket B: $\underbrace{P(b|a) \quad P(d|b,a) \quad P(e|b,c)}$ B=1

bucket C: $P(c|a)$ P(e|b=1,c)

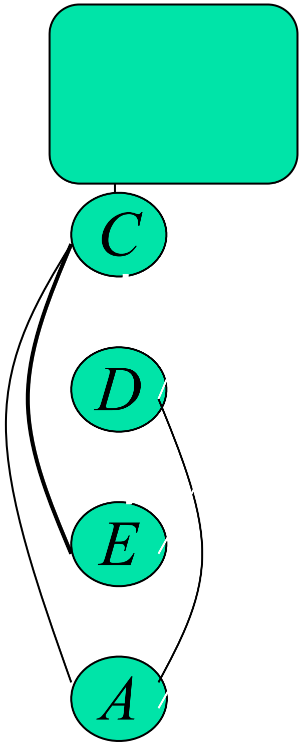
bucket D: P(d|b=1,a)

bucket E: $e=0$

bucket A: $P(a)$ P(b=1|a)

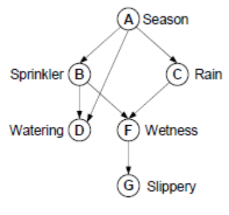
$\mathbf{P(e=0)}$

$\mathbf{P(a|e=0)}$

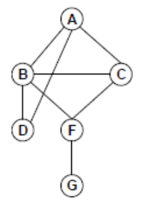


$$P(a|e=0) = \frac{P(a,e=0)}{P(e=0)}$$

The impact of observations



(a) Directed acyclic graph



(b) Moral graph

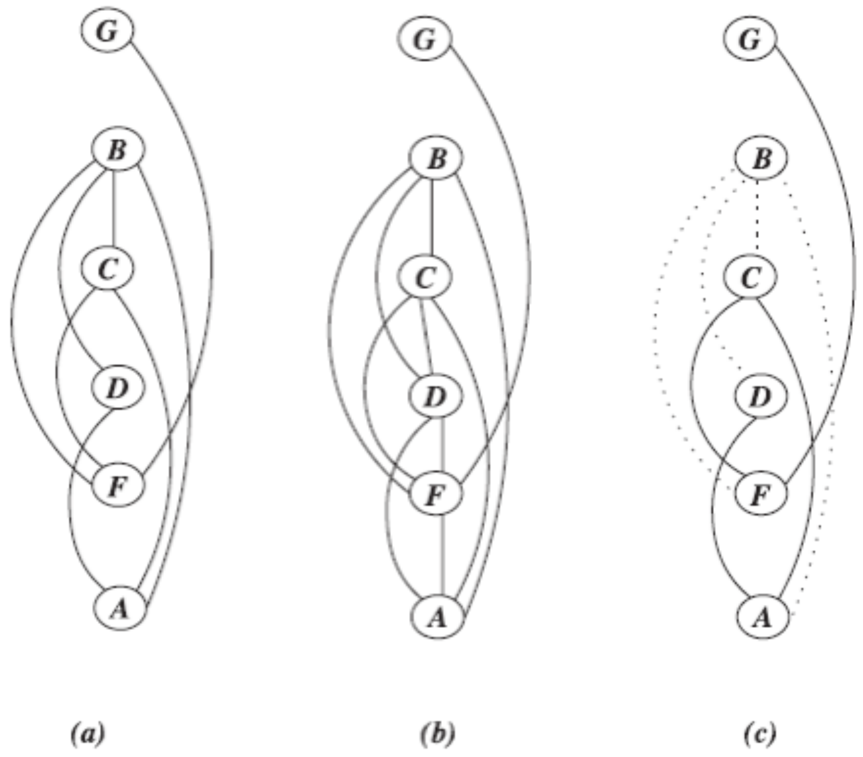
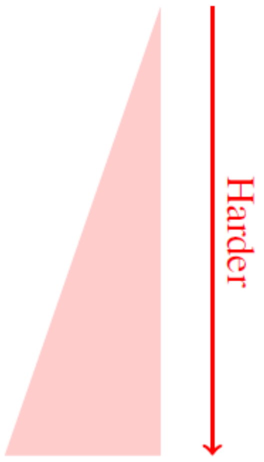


Figure 4.9: Adjusted induced graph relative to observing B.

Ordered graph   Induced graph   Ordered conditioned graph

Types of queries

Max-Inference:	$f(x^*) = \max_x \prod_\alpha f_\alpha(x_\alpha)$
Sum-Inference: (e.g., causal effects)	$Z = \sum_x \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MMAP): (optimal prediction)	$f_M(x_M^*) = \max_{x_M} \sum_{x_S} \prod_\alpha f_\alpha(x_\alpha)$
Mixed-Inference (MEU): (e.g., decisions & planning)	$\text{MEU} = \max_{D_1,\ldots,D_m} \sum_{X_1,\ldots X_n} (\prod_{P_i \in P} P_i) \times (\sum_{r_i \in R} r_i)$



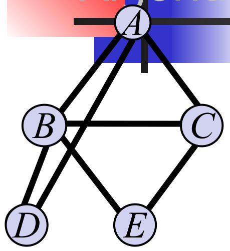
- **NP-hard**: exponentially many terms
- Difficulty (in part) due to restricted elimination orderings
- We will focus on **approximation** algorithms
  - **Anytime**: very fast & very approximate ! Slower & more accurate

Inference for Probabilistic Networks

- Bucket elimination
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
- Induced-Width

Finding MPE/MAP

Algorithm BE-mpe (Dechter 1996, Bertele



$$\text{MPE} = \max_{a,e,d,c,b} p(a)\,p(c|a)\,p(b|a)\,p(\ldots$$

$$= \max_b p(b|a) \cdot p(d|b,a) \cdot p(e|b,c)$$

$$\overbrace{\qquad\qquad\qquad}^{\max_x \prod}$$

bucket B: $p(b|a)\ p(d|b,a)\ p(e|b,c)$

bucket C: $p(c|a)\quad \lambda_{B\to C}\underbrace{(a,d,c,e)}$

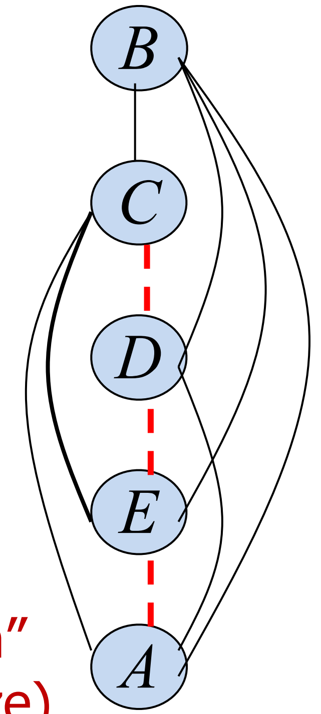
bucket D: $\lambda_{C\to D}(a,d,e)$

bucket E: $\mathbb{1}[e=0]\quad \lambda_{D\to E}(a,e)$

bucket A: $p(a)\quad \lambda_{E\to A}(a)$

OPT

W*=4
"induced width"
(max clique size)



Generating the optimal assignment

- Given BE messages, select optimum config in reverse order

$$\mathbf{b}^* = \arg\max_b \, p(b|a^*)\, p(d^*|b,a^*)\, p(e^*|b,c^*)$$

$$\mathbf{c}^* = \arg\max_c \, p(c|a^*)\, \lambda_{B\to C}(a^*, c, d^*, e^*)$$

$$\mathbf{d}^* = \arg\max_d \, \lambda_{C\to D}(a^*, d, e^*)$$

$$\mathbf{e}^* = \arg\max_e \, \mathbb{1}[e=0]\, \lambda_{D\to E}(a^*, e)$$

$$\mathbf{a}^* = \arg\max_a \, p(a) \cdot \lambda_{E\to A}(a)$$

$$\begin{tikzcd}
\textbf{B:} & \underbrace{p(b|a)\;\; p(d|b,a)\;\; p(e|b,c)} \arrow[d] \\
\textbf{C:} & \underbrace{p(c|a) \qquad \lambda_{B\to C}(a,c,d,e)} \arrow[d] \\
\textbf{D:} & \underbrace{\lambda_{C\to D}(a,d,e)} \arrow[d] \\
\textbf{E:} & \underbrace{\mathbb{1}[e=0] \quad \lambda_{D\to E}(a,e)} \arrow[d, dash] \\
\textbf{A:} & \underbrace{p(a)\;\; \lambda_{E\to A}(a)} \arrow[d] \\
 & \textbf{OPT = optimal value}
\end{tikzcd}$$

Return optimal configuration (a*,b*,c*,d*,e*)

Complexity of bucket elimination

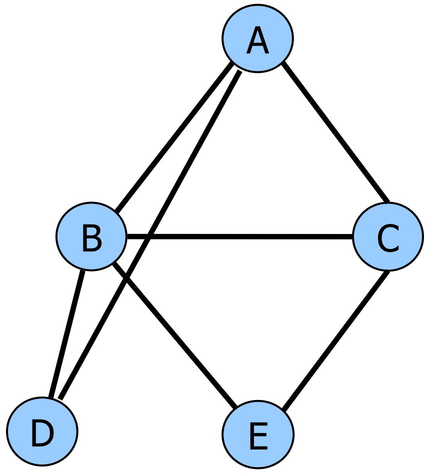
Bucket-Elimination is **time** and **space**

$$O\left(r \exp(w_d^*)\right)$$

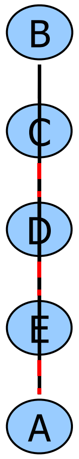
w_d^* : the induced width of the primal graph along ordering d

r = number of functions

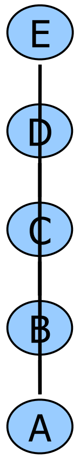
The effect of the ordering:



primal graph



$w^*(d_1) = 4$

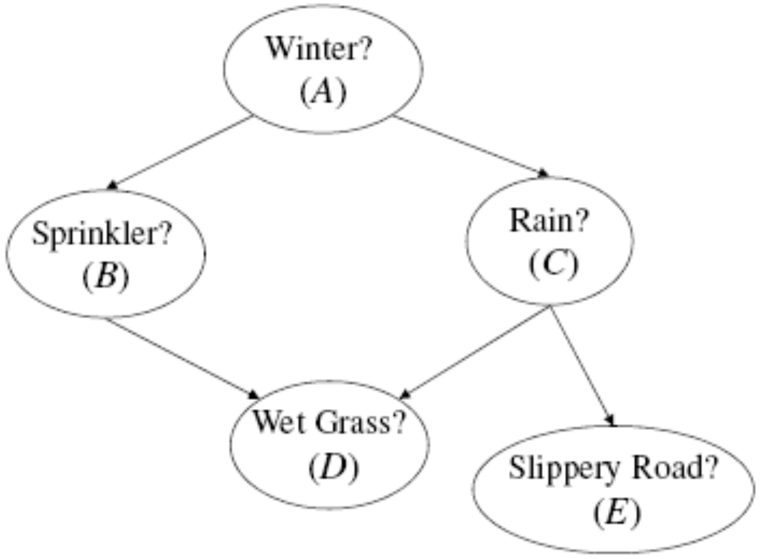


$w^*(d_2) = 2$

Finding smallest induced-width is hard!

A Bayesian Network

Example with mpe?



A	Θ_A
true	.6
false	.4

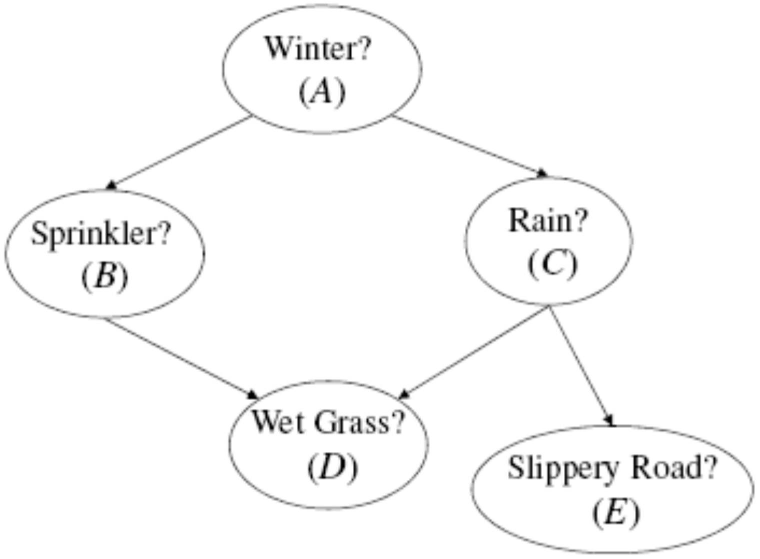
A	B	$\Theta_{B\|A}$
true	true	.2
true	false	.8
false	true	.75
false	false	.25

A	C	$\Theta_{C\|A}$
true	true	.8
true	false	.2
false	true	.1
false	false	.9

B	C	D	$\Theta_{D\|BC}$
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

C	E	$\Theta_{E\|C}$
true	true	.7
true	false	.3
false	true	0
false	false	1

Try to compute MPE when E=0



A	Θ_A
true	.6
false	.4

A	B	$\Theta_{B\|A}$
true	true	.2
true	false	.8
false	true	.75
false	false	.25

A	C	$\Theta_{C\|A}$
true	true	.8
true	false	.2
false	true	.1
false	false	.9

B	C	D	$\Theta_{D\|BC}$
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

C	E	$\Theta_{E\|C}$
true	true	.7
true	false	.3
false	true	0
false	false	1

Complexity of bucket-elimination

- **Theorem:**

BE is O(n exp(w*+1)) time and O(n exp(w*)) space, when w* is the induced-width of the moral graph along d when evidence nodes are processed (edges from evidence nodes to earlier variables are removed.)

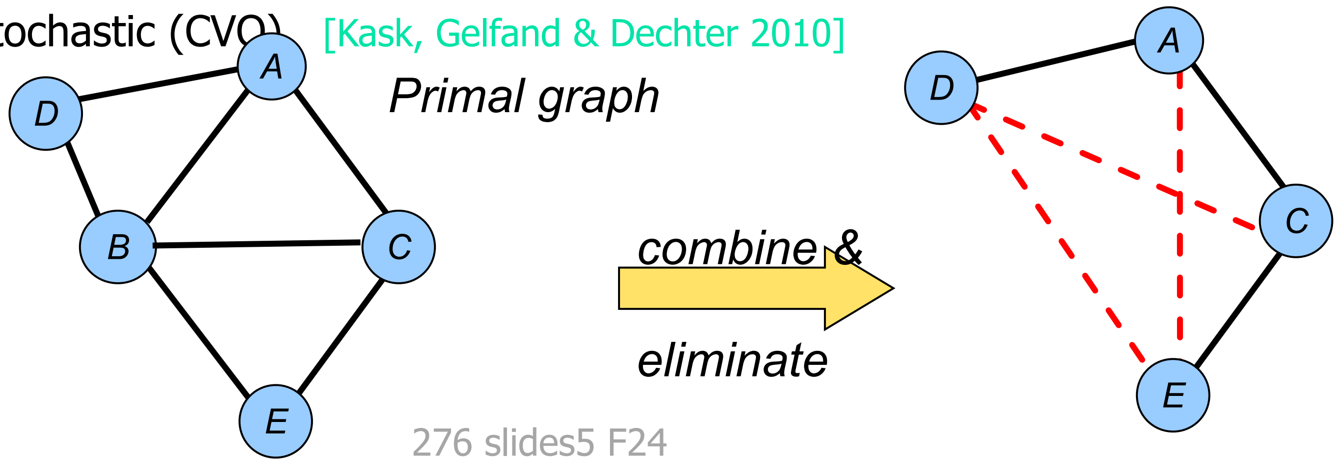
More accurately: O(r exp(w*(d)) where r is the number of CPTs.
For Bayesian networks r=n. For Markov networks?

Inference for probabilistic networks

- Bucket elimination
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→   Marginal Map)
- Induced-Width (Dechter 3.4,3.5)

Variable ordering heuristics

- What makes a good order?
  - Low induced width
  - Elimination creates a function over neighbors
- Finding the best order is hard (NP-complete!)
  - But we can do well with simple heuristics
    - Min-induced-width, Min-Fill, …
  - Anytime algorithms
    - Search-based [Gogate & Dechter 2003]
    - Stochastic (CVO) [Kask, Gelfand & Dechter 2010]



Min-width ordering

```
MIN-WIDTH (MW)
input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: A min-width ordering of the nodes $d = (v_1, ..., v_n)$.
1.  for $j = n$ to 1 by -1 do
2.          $r \leftarrow$ a node in $G$ with smallest degree.
3.          put $r$ in position $j$ and $G \leftarrow G - r$.
            (Delete from $V$ node $r$ and from $E$ all its adjacent edges)
4.  endfor
```

Proposition: algorithm min-width finds a min-width ordering of a graph

What is the Complexity of MW?

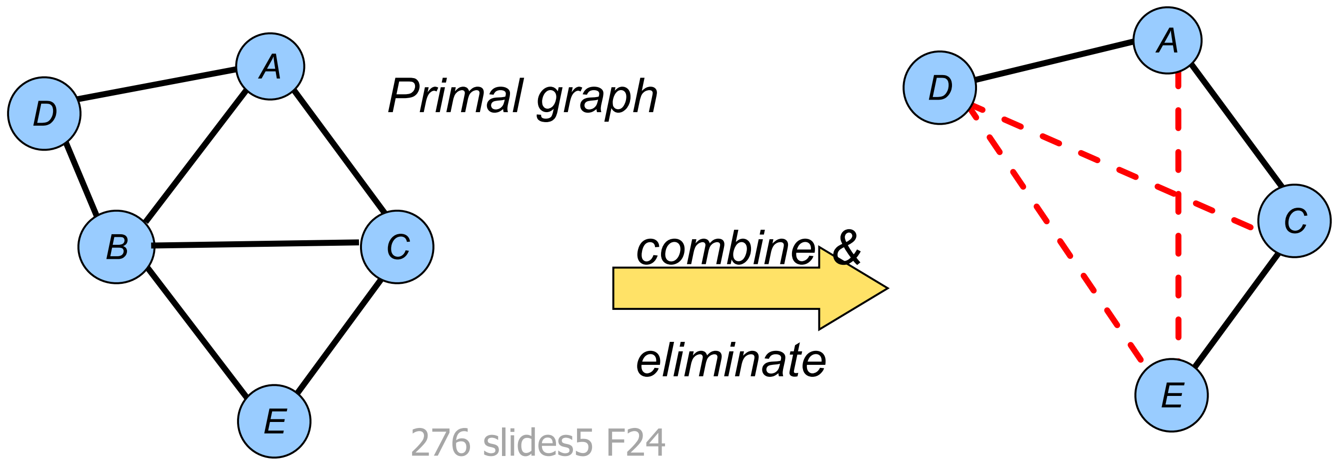
O(e)

Variable ordering heuristics

- Min (induced) width heuristic

```
1. for i=1 to n   ( # of variables)
2.     Select a node $X_i$ with smallest degree as next eliminated
3.     Connect Xi's neighbors:
4.             E = E + { $(X_j, X_k)$ : $(X_i, X_j)$ and $(X_i, X_k)$ in E }
5.     Remove $X_i$ from the graph:  V = V – $\{X_i\}$
6. end
```

("Weighted" version: weight edges by domain size

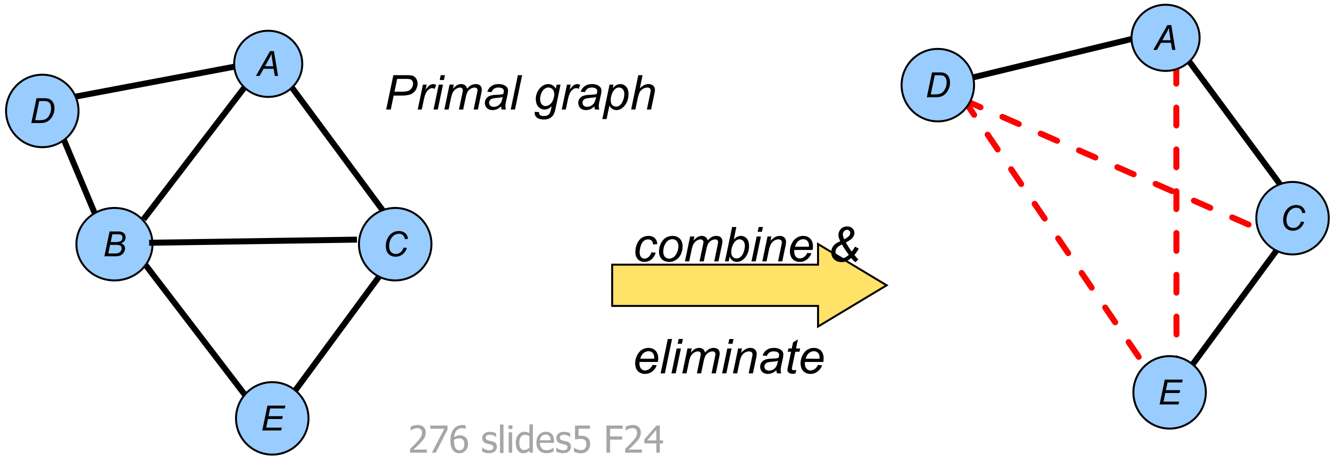


Variable ordering heuristics

- Min fill heuristic

```
1.  for i=1 to n   ( # of variables)
2.      Select a node $X_i$ with smallest "fill edges" as next eliminated
3.      Connect Xi's neighbors:
4.          E = E + { $(X_j, X_k)$ : $(X_i, X_j)$ and $(X_i, X_k)$ in E }
5.      Remove $X_i$ from the graph:  V = V – $\{X_i\}$
6.  end
```

("Weighted" version: weight edges by domain size

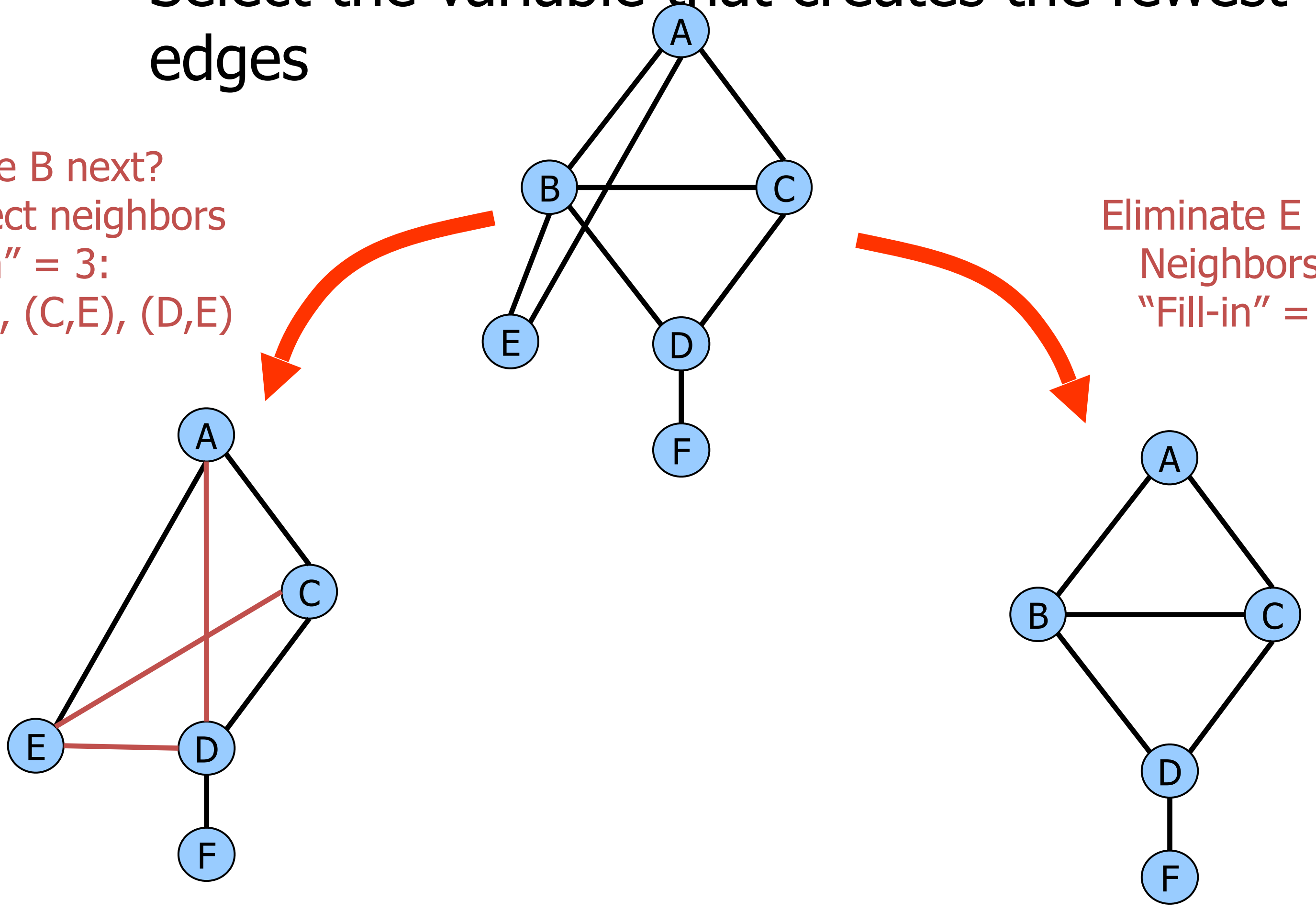


Min-Fill heuristic

- Select the variable that creates the fewest "fill-in" edges

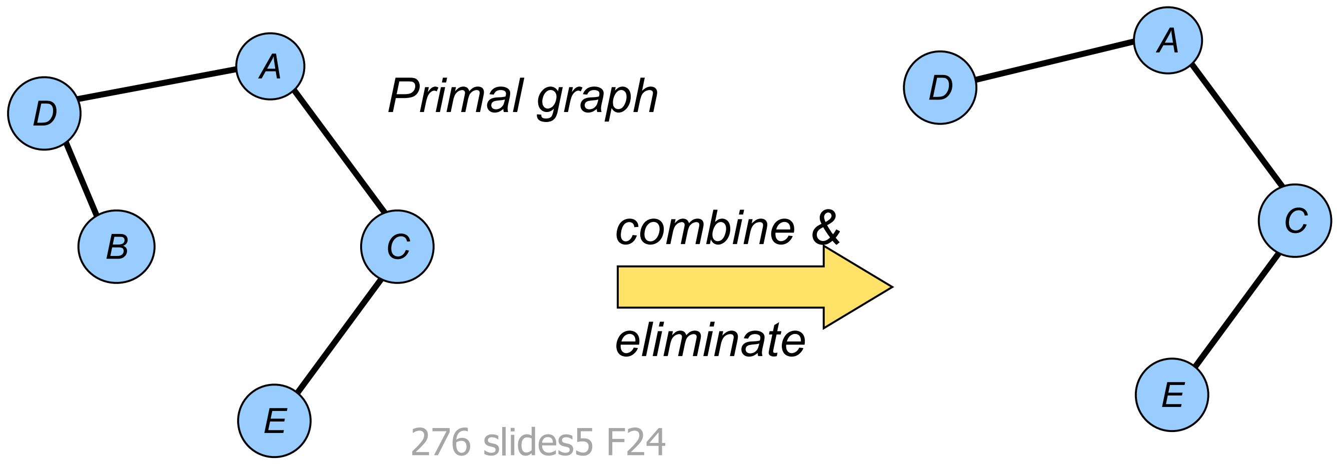
Eliminate B next?
Connect neighbors
"Fill-in" = 3:
(A,D), (C,E), (D,E)

Eliminate E next?
Neighbors already connected
"Fill-in" = 0



Tree-structured graphs

- If the graph is a tree, the best ordering is easy:
  - B, E have only one neighbor; no "fill"
  - Select one to eliminate; remove it
  - Now D or E have only one neighbor; no "fill"…
- Order
  - leaves to root
  - never increases the size of the factors

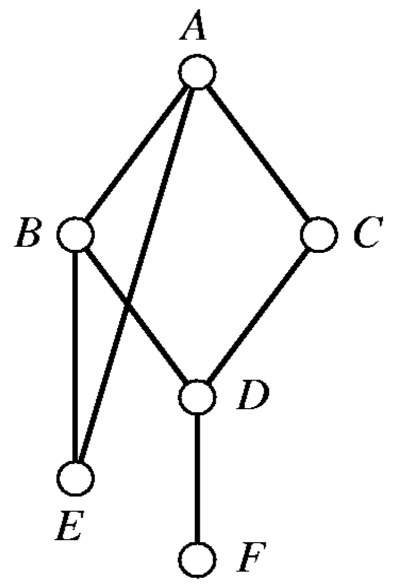


Greedy orderings heuristics

- Min-induced-width
  - From last to first, pick a node with smallest width, then connect parent and remove
- Min-Fill
  - From last to first, pick a node with smallest fill-edges

Complexity? O(n^3)

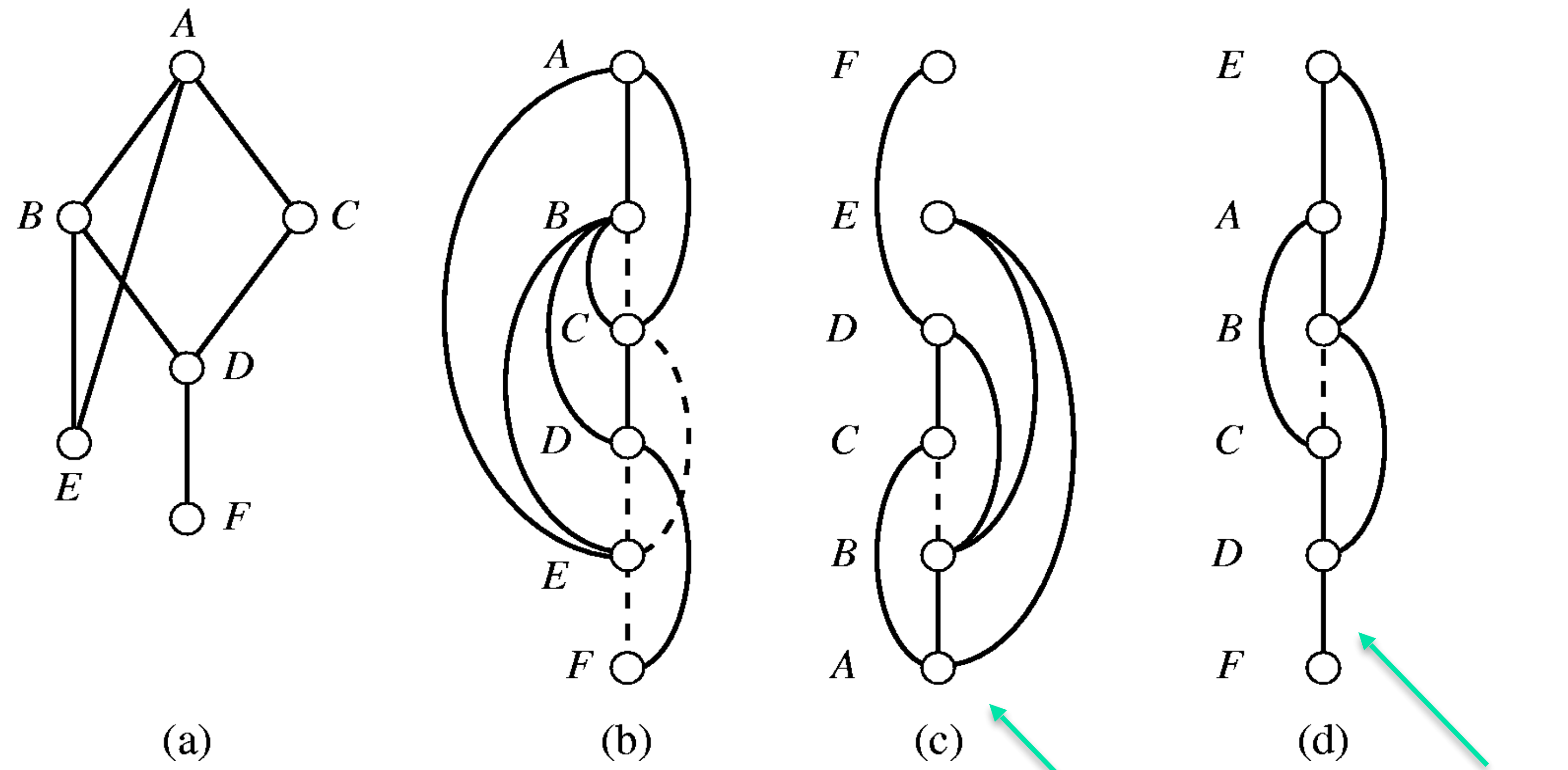
Different induced-graphs



(a)

Let’s find a miw ordering and a min-fill ordering

Different induced-graphs



A Miw ordering

A Min-fill ordering

Which greedy algorithm is best?

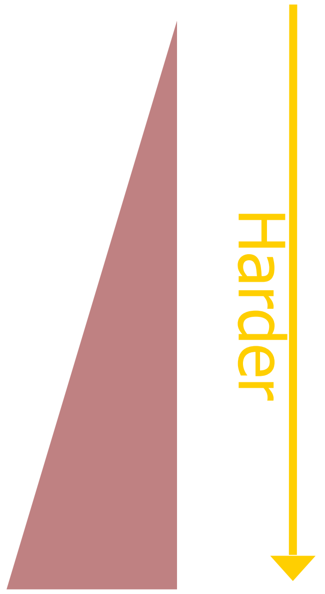
- Min-Fill, prefers a node who adds the least number of fill-in arcs.
- Empirically, fill-in is the best among the greedy algorithms (MW,MIW,MF,MC)
- Complexity of greedy orderings?
- MW is O(e), MIW: O(n^3) MF O(n^3) MC is O(e+n)

Inference for probabilistic networks

- Bucket elimination (Dechter chapter 4)
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
  - Influence diagrams ?
- Induced-Width (Dechter, Chapter 3.4)

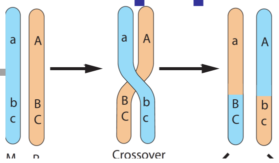
Marginal Map

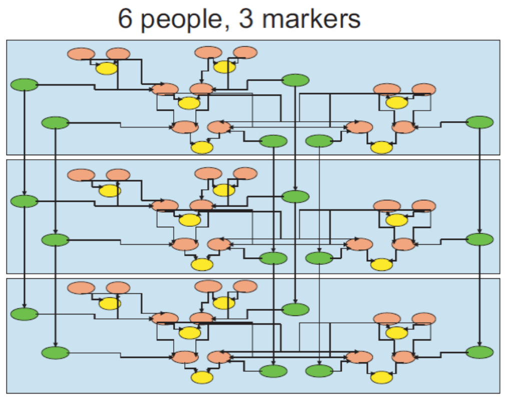
Max-Inference	$f(\mathbf{x}^*) = \max_{\mathbf{x}} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$
Sum-Inference	$Z = \sum_{\mathbf{x}} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$
Mixed-Inference	$f(\mathbf{x}_M^*) = \max_{\mathbf{x}_M} \sum_{\mathbf{x}_S} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$



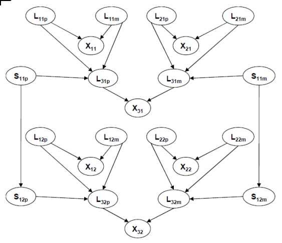
- **NP-hard**: exponentially many terms

Example for MMAP Applications





- Haplotype in Family pedigrees



- Coding networks

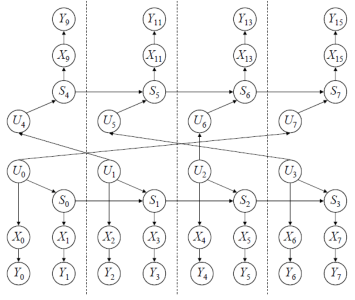
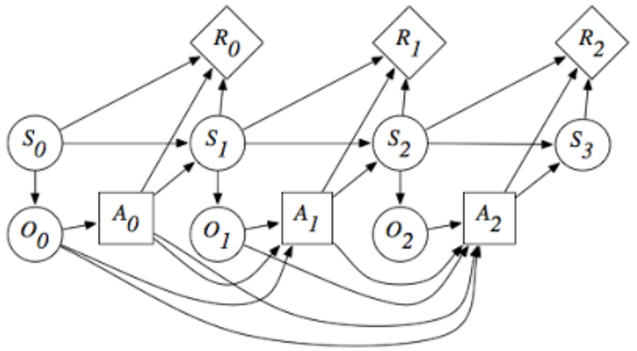
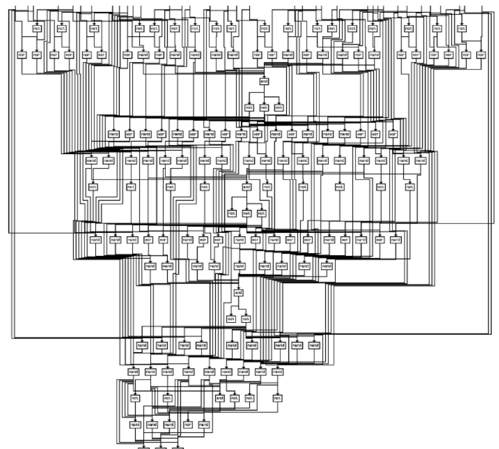


Figure 5.24: A Bayesian network for a turbo code.

- Probabilistic planning

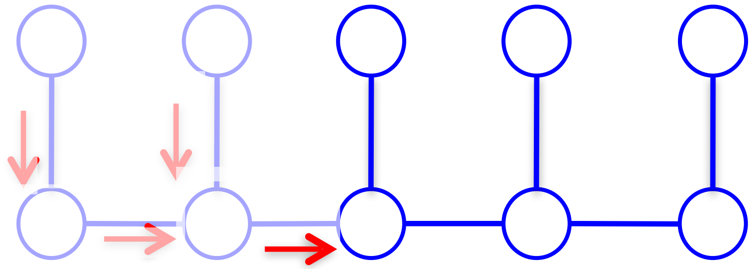


- Diagnosis



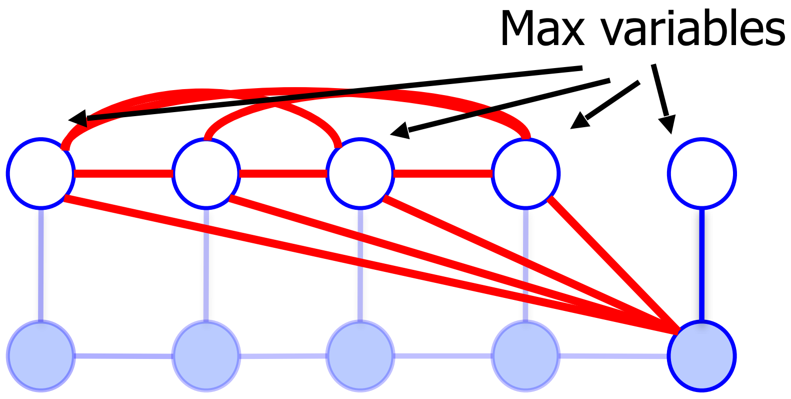
Marginal MAP is not easy on trees

- Pure MAP or summation tasks
  - Dynamic programming
  - Ex: efficient on trees



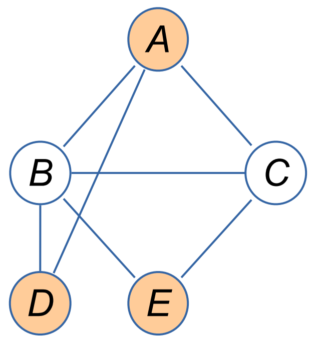
- Marginal MAP
  - Operations do not commute:
  - Sum must be done first!

$$\sum \max \neq \max \sum$$



Bucket elimination for MMAP

Bucket Elimination



$\mathbf{X}_M = \{A, D, E\}$

$\mathbf{X}_S = \{B, C\}$

$$\max_{\mathbf{X}_M} \sum_{\mathbf{X}_S} P(\mathbf{X})$$

constrained elimination order

SUM

B: $\underbrace{f(A,B)\, f(B,C)\, f(B,D) f(B,E)}$

Σ_B

C: $\underbrace{\lambda^B(A,C,D,E) f(A,C)\; f(C,E)}$

Σ_C

MAX

D: $\underbrace{\lambda^C(A,D,E)\; f(A,D)}$

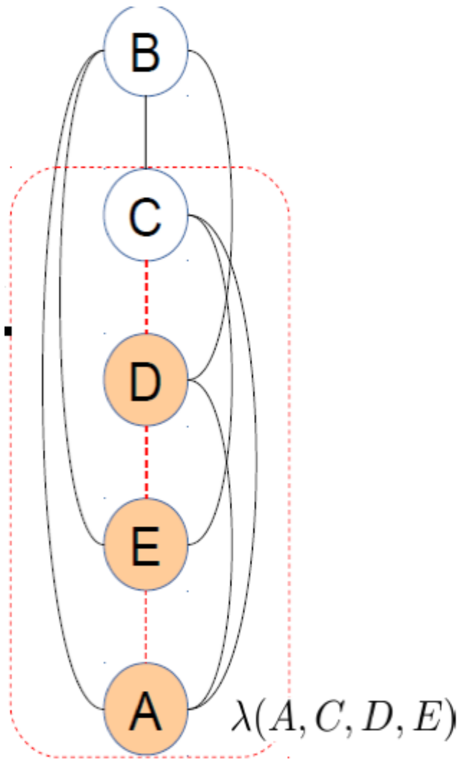
max_D

E: $\underbrace{\lambda^D(A,E)}$

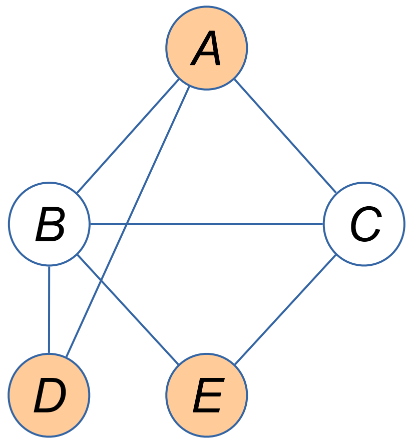
max_E

A: $\underbrace{\lambda^E(A)}$

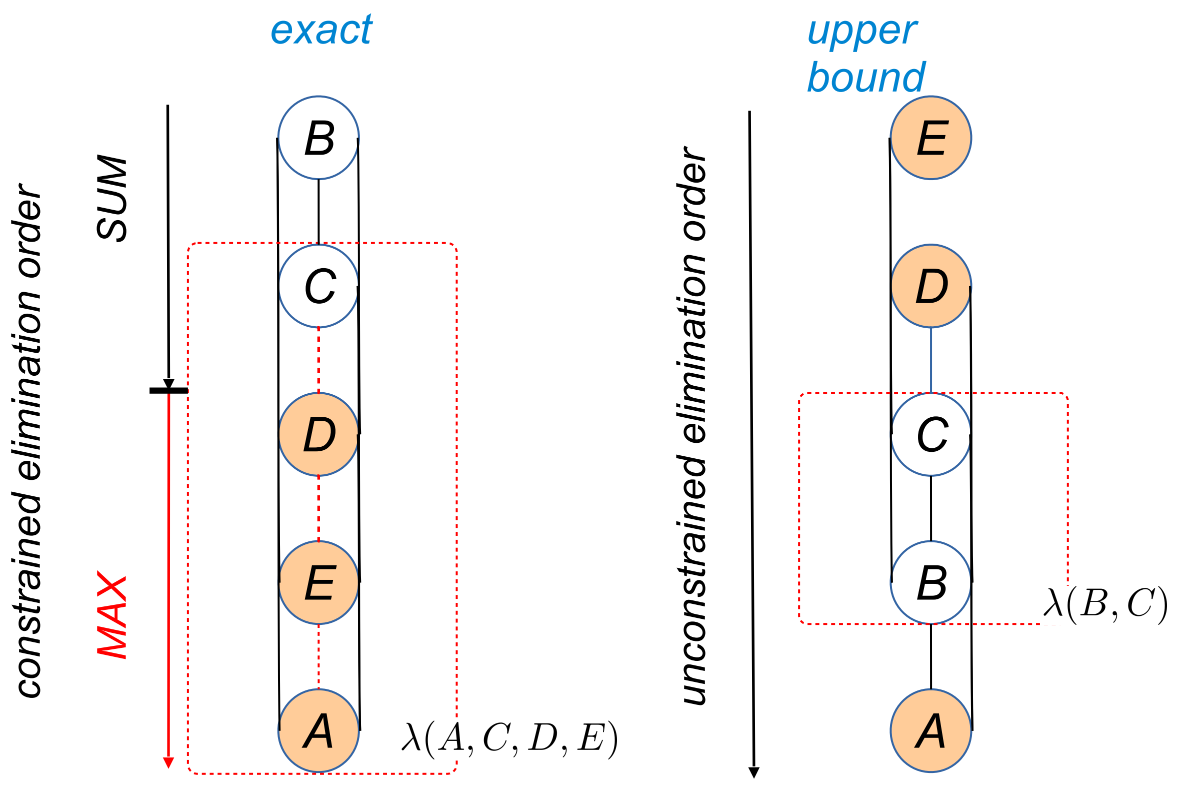
MAP is the marginal MAP value*



Why is MMAP harder?



$\mathbf{X}_M = \{A, D, E\}$
$\mathbf{X}_S = \{B, C\}$



$w^* = 4$ $w^* = 2$

In practice, constrained induced is much larger!

$$\max_X \sum_Y \phi \le \sum_Y \max_X \phi$$

(Park & Darwiche, 2003)
(Yuan & Hansen, 2009)

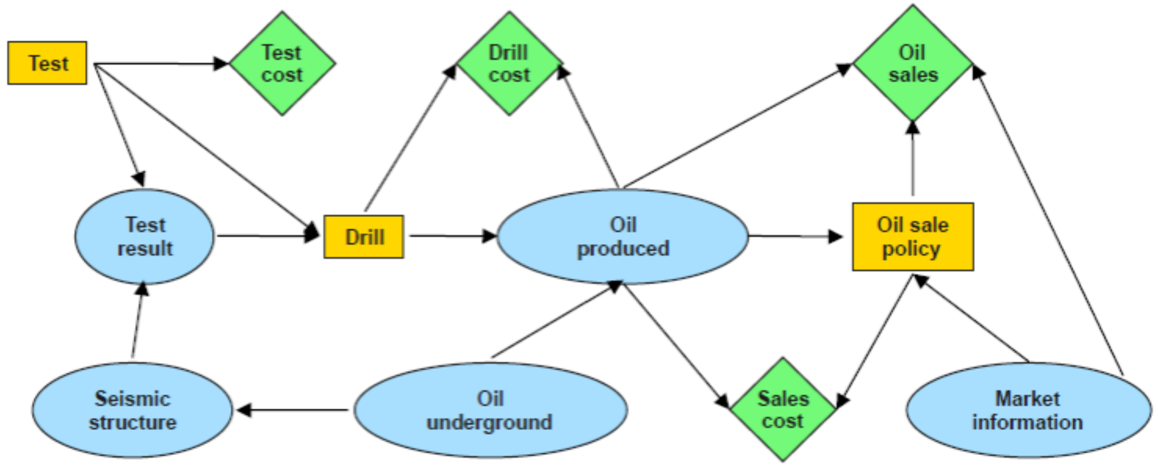
Inference for probabilistic networks

- Bucket elimination (Dechter chapter 4)
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
- Induced-Width (Dechter, Chapter 3.4)
- Mixed networks
- Influence diagrams ?

Ex: “oil wildcatter”

e.g., [Raiffa 1968; Shachter 1986]

- Influence diagram:



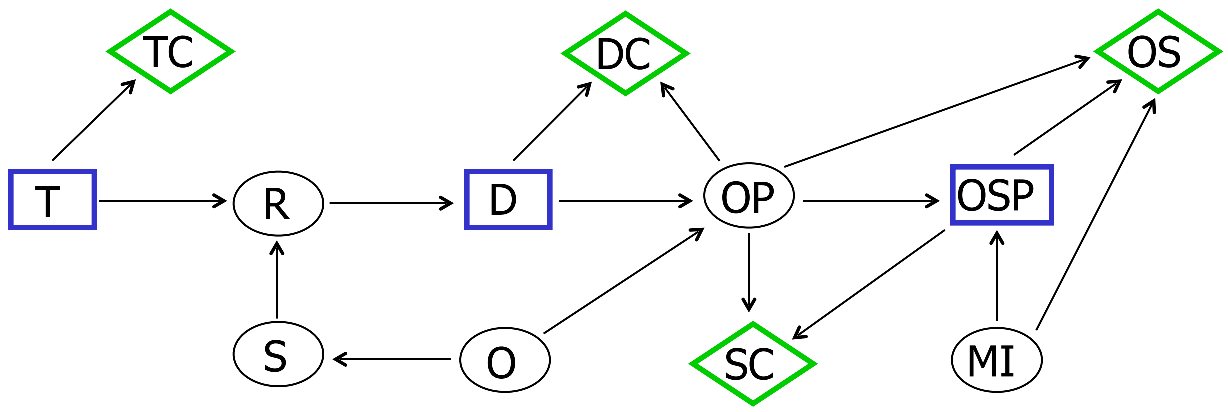
- Three actions: test, drill, sales policy
- Chance variables:

  P(oil) P(seismic|oil) P(result | seismic, test) P(produced | oil, drill) P(market)
- Utilities capture costs of actions, rewards of sale

  Oil sales - Test cost - Drill cost - Sales cost

Influence Diagrams

***Influence diagram** ID = (X,D,P,R).*



Chance variables ***over domains.***

Decision variables

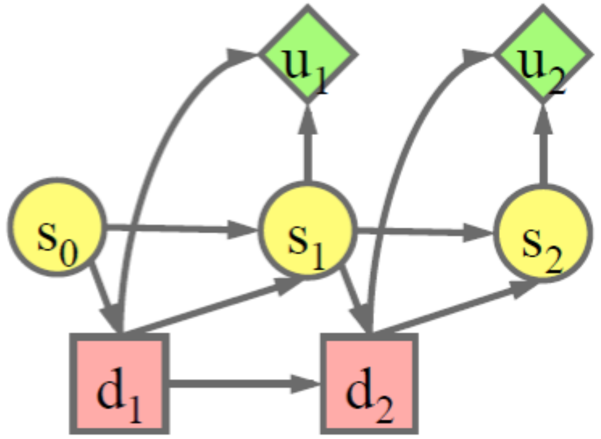
CPT's for chance variables

Reward components

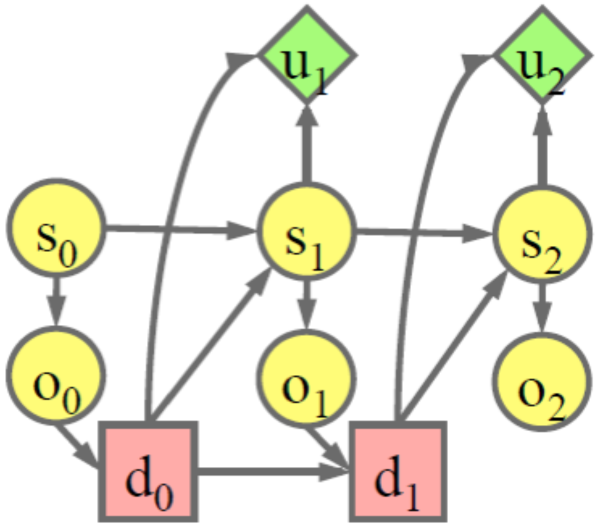
Utility function

Common examples

- Markov decision process
  - Markov chain state sequence
  - Actions “di” influence state transition
  - Rewards based on action, new state
  - Temporally homogeneous



- Partially observable MDP
  - Hidden Markov chain state sequence
  - Generate observations
  - Actions based on observations



Influence Diagrams (continue)

A decision rule *for is a mapping:*

where is the cross product of domains in S.

A policy *is a list of decision rules*

Task: *Find an optimal policy that maximizes the expected utility.*

The Car Example (Howard 1976)

A car buyer needs to buy one of two used cars. The buyer can carry out tests with various costs, and then, decide which car to buy.

T: Test variable (t_0, t_1, t_2) (t_1 test car 1, t_2 test car 2)

D: the decision of which car to buy, $D \in \{buy1, buy2\}$

C_i: the quality of car i, $C_i \in \{q_1, q_2\}$

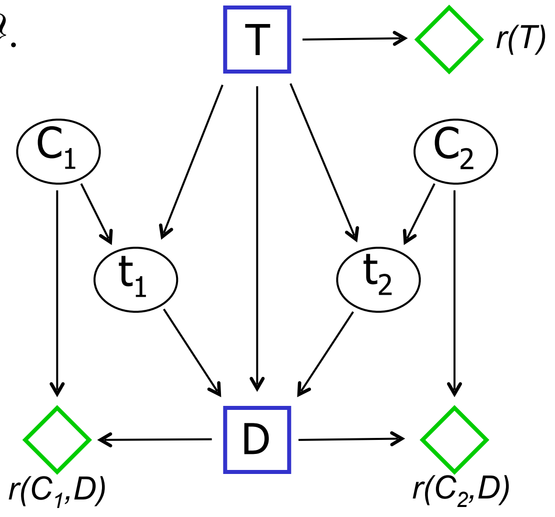
t_i: the outcome of the test on car i, $t_i \in \{pass, fail, null\}$.

$r(T)$: The cost of testing,

$r(C_1,D)$, $r(C_2,D)$: the reward in buying cars 1 and 2.

The utility is: $r(T) + r(C_1,D) + r(C_2,D)$.

Task: determine decision rules T and D such that:



Bucket Elimination for meu (Algorithm Elim-meu-id)

Input: *An Influence diagram* $ID = \{P_1, \ldots, P_n, r_1, \ldots, r_j\}$

Output: *Meu and optimizing policies.*

1. ***Order the variables and partition into buckets.***
2. ***Process buckets from last to first:***

$o = T, t_2, t_2, D, C_2, C_1$

$bucket(C_1): \underbrace{P(C_1), P(t_1|C_1,T), r(C_1,D)}$

$bucket(C_2): \underbrace{P(C_2), P(t_2|C_2,T), r(C_2,D)}$

$bucket(D): \underbrace{\theta_{C_1}(t_1,T,D), \theta_{C_2}(t_2,T,D)}$

$bucket(t_1): \underbrace{\lambda_{C_1}(t_1,T) \quad \theta_D(t_1,t_2,T),} \delta(t_1,t_2,T)$

$bucket(t_2): \underbrace{\lambda_{C_2}(t_2,T) \quad \theta_{t_1}(t_2,T)}$

$bucket(T): r(T) \quad \underbrace{\lambda_{t1}(T) \quad \lambda_{t2}(T) \quad \theta_{t1}(T)}$

$\theta_{T'}, \delta_T$

3. ***Forward:*** *Assign values in ordering* d

The Bucket Description

Final buckets: *(λs or Ps) utility components (θ's or r's).*

$bucket(C_1)$: $P(C_1), P(t_1|C_1,T), r(C_1,D)$

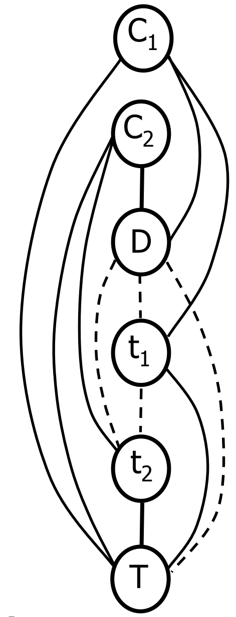
$bucket(C_2)$: $P(C_2), P(t_2|C_2,T), r(C_2,D)$

$bucket(D)$:

$bucket(t_1)$:

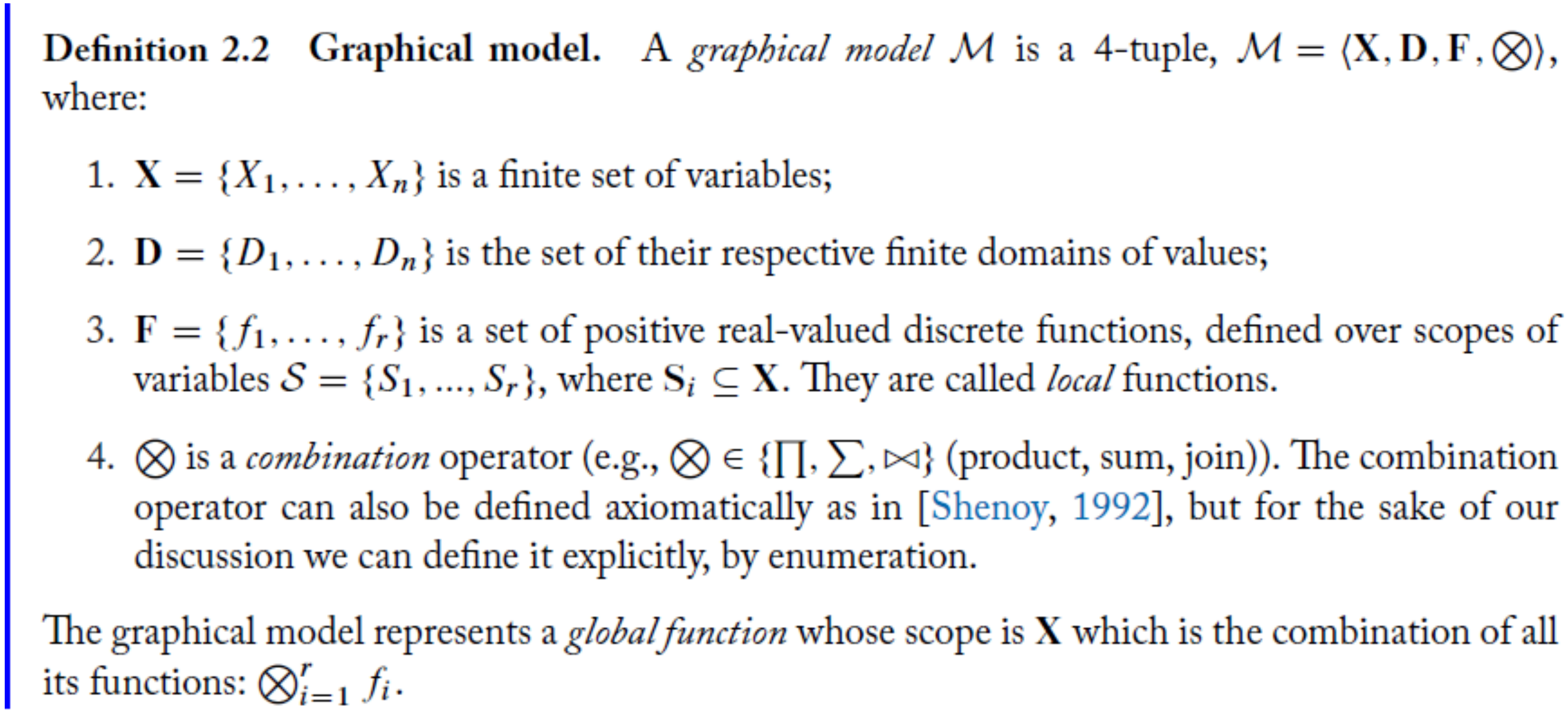
$bucket(t_2)$:

$bucket(T)$: $r(T)$



Optimizing policies: *is argmax of* *computed in* $bucket(T)$, *and* *in* $bucket(t_1)$.

General Graphical Models

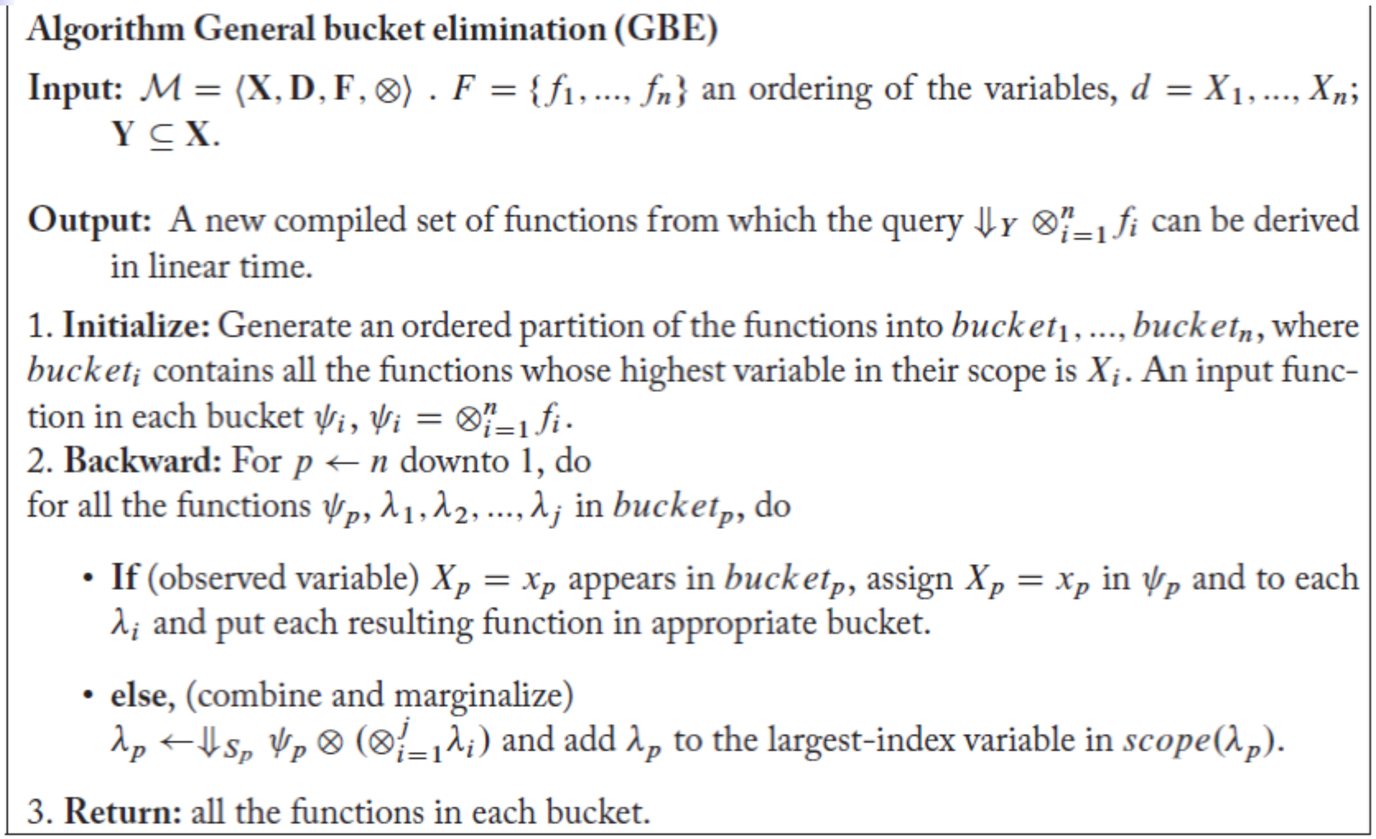


Definition 2.2 Graphical model. A *graphical model* $\mathcal{M}$ is a 4-tuple, $\mathcal{M} = \langle \mathbf{X}, \mathbf{D}, \mathbf{F}, \bigotimes \rangle$, where:

1. $\mathbf{X} = \{X_1, \ldots, X_n\}$ is a finite set of variables;
2. $\mathbf{D} = \{D_1, \ldots, D_n\}$ is the set of their respective finite domains of values;
3. $\mathbf{F} = \{f_1, \ldots, f_r\}$ is a set of positive real-valued discrete functions, defined over scopes of variables $\mathcal{S} = \{S_1, ..., S_r\}$, where $\mathbf{S}_i \subseteq \mathbf{X}$. They are called *local* functions.
4. $\bigotimes$ is a *combination* operator (e.g., $\bigotimes \in \{\prod, \sum, \bowtie\}$ (product, sum, join)). The combination operator can also be defined axiomatically as in [Shenoy, 1992], but for the sake of our discussion we can define it explicitly, by enumeration.

The graphical model represents a *global function* whose scope is $\mathbf{X}$ which is the combination of all its functions: $\bigotimes_{i=1}^{r} f_i$.

General bucket elimination

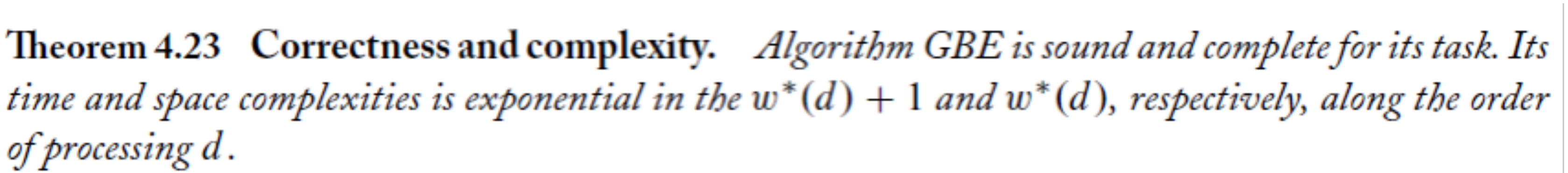


Algorithm General bucket elimination (GBE)

Input: $\mathcal{M} = \langle \mathbf{X}, \mathbf{D}, \mathbf{F}, \otimes \rangle$. $F = \{f_1, ..., f_n\}$ an ordering of the variables, $d = X_1, ..., X_n$; $\mathbf{Y} \subseteq \mathbf{X}$.

Output: A new compiled set of functions from which the query $\Downarrow_Y \otimes_{i=1}^n f_i$ can be derived in linear time.

```
1. Initialize: Generate an ordered partition of the functions into $bucket_1, ..., bucket_n$, where
   $bucket_i$ contains all the functions whose highest variable in their scope is $X_i$. An input func-
   tion in each bucket $\psi_i$, $\psi_i = \otimes_{i=1}^n f_i$.
2. Backward: For $p \leftarrow n$ downto 1, do
   for all the functions $\psi_p, \lambda_1, \lambda_2, ..., \lambda_j$ in $bucket_p$, do
      • If (observed variable) $X_p = x_p$ appears in $bucket_p$, assign $X_p = x_p$ in $\psi_p$ and to each
        $\lambda_i$ and put each resulting function in appropriate bucket.
      • else, (combine and marginalize)
        $\lambda_p \leftarrow \Downarrow_{S_p} \psi_p \otimes (\otimes_{i=1}^j \lambda_i)$ and add $\lambda_p$ to the largest-index variable in $scope(\lambda_p)$.
3. Return: all the functions in each bucket.
```



Theorem 4.23 Correctness and complexity. *Algorithm GBE is sound and complete for its task. Its time and space complexities is exponential in the* $w^*(d) + 1$ *and* $w^*(d)$, *respectively, along the order of processing* d.