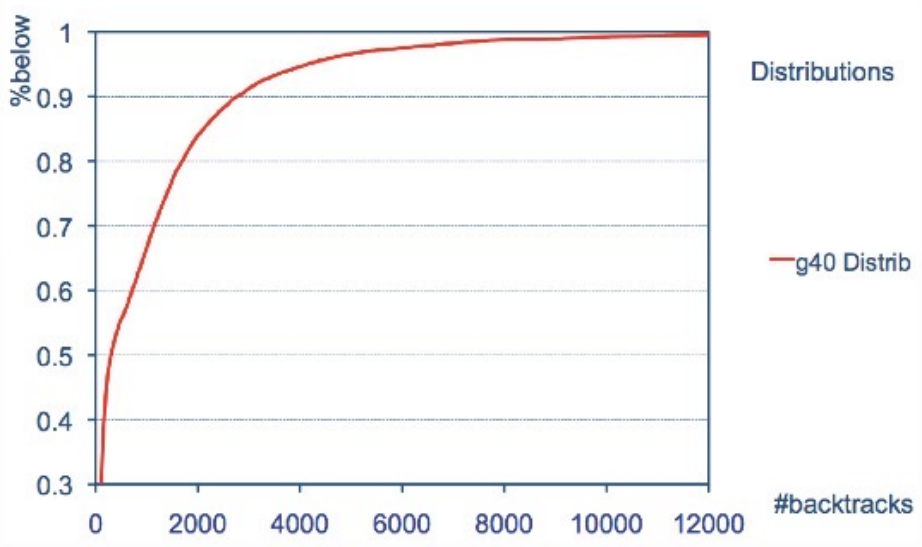
Outline

- Review: DPLL, Resolution
- Conflict-Directed Clause Learning (CDCL)
  - Implication graphs,
  - asserting clauses,
  - Unique Implication points (UIPs)
- Watch literals
- Restarts, Empirical evaluation

Restarts I

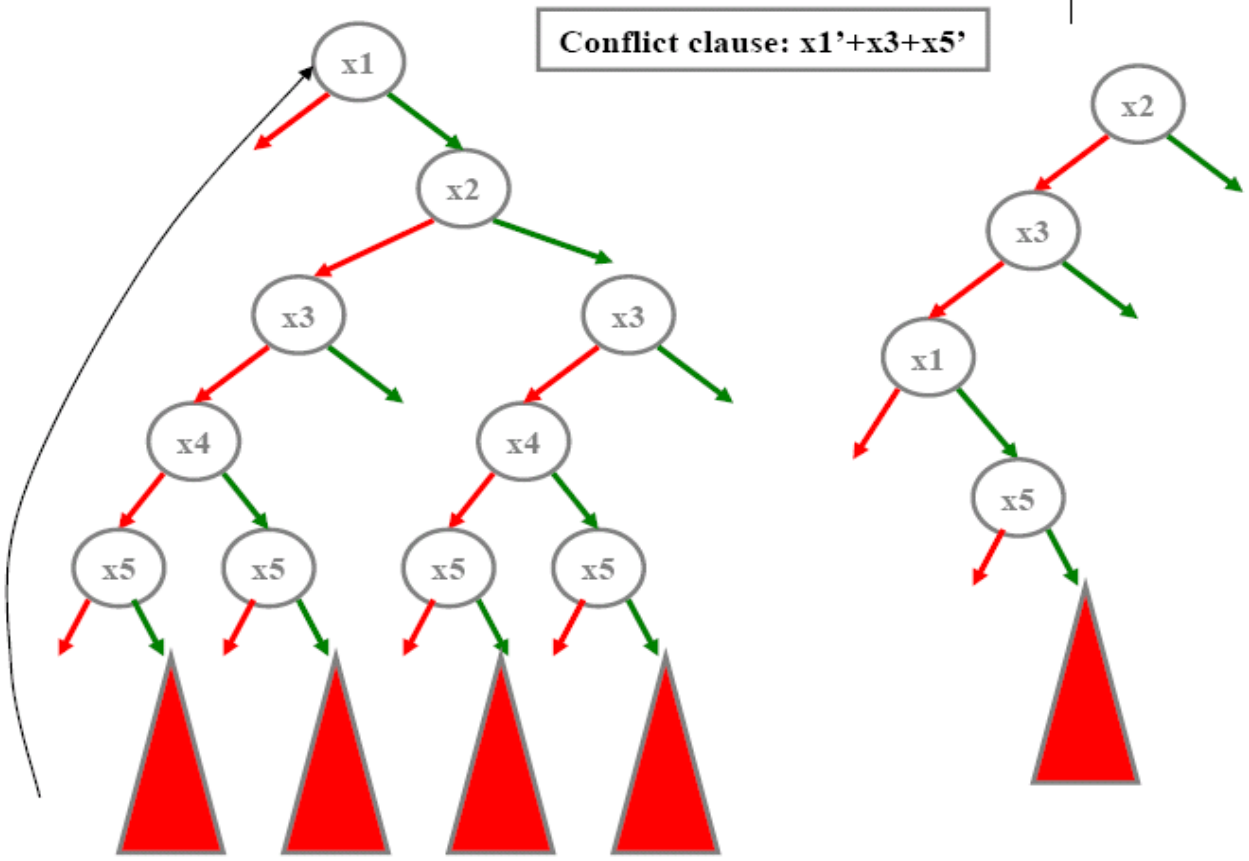


- Plot for processor verification instance with branching randomization and 10000 runs
  - More than 50% of the runs require less than 1000 backtracks
  - A small percentage requires more than 10000 backtracks
- Run times of backtrack search SAT solvers characterized by heavy-tail distributions

[Gomes et al.'98]

Restart

- Abandon the current search tree and reconstruct a new one
- Helps reduce variance - adds to robustness in the solver
- The clauses learned prior to the restart are *still there* after the restart and can help pruning the search space



Evolution of SAT Solvers

Instance	Posit'94	Grasp'96	Chaff'03	Minisat'03	Picosat'08
ssa2670-136	13.57	0.22	0.02	0.00	0.01
bf1355-638	310.93	0.02	0.02	0.00	0.03
design_3	> 1800	3.93	0.18	0.17	0.93
design_1	> 1800	34.55	0.35	0.11	0.68
4pipe_4_ooo	> 1800	> 1800	17.47	110.97	44.95
fifo8_300	> 1800	> 1800	348.50	53.66	39.31
w08_15	> 1800	> 1800	> 1800	99.10	71.89
9pipe_9_ooo	> 1800	> 1800	> 1800	> 1800	> 1800
c6288	> 1800	> 1800	> 1800	> 1800	> 1800

- Modern SAT algorithms can solve instances with hundreds of thousands of variables and tens of millions of clauses

Outline

- Review: DPLL, Resolution
- Conflict-Directed Clause Learning (CDCL)
  - Implication graphs,
  - asserting clauses,
  - Unique Implication points (UIPs)
- Watch literals
- Restarts, Empirical evaluation

CompSci 275, Constraint Networks

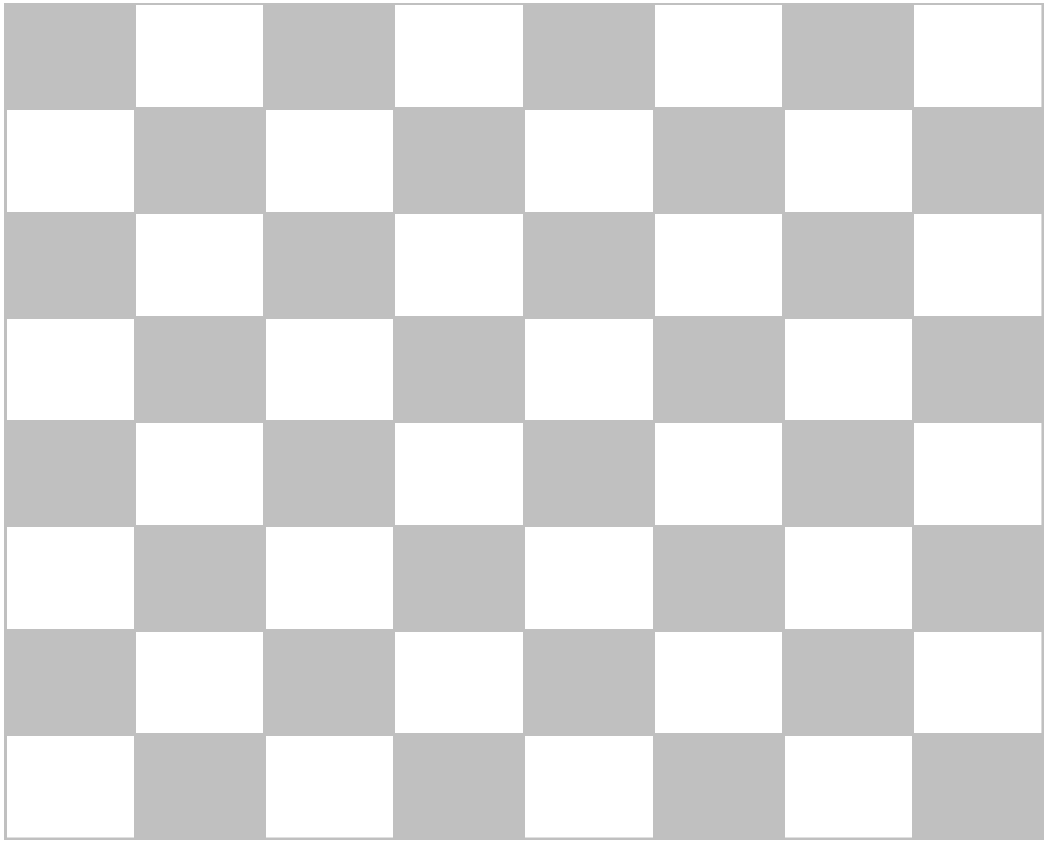
Rina Dechter, Winter 2026

Chapter 7
Stochastic Greedy Local Search

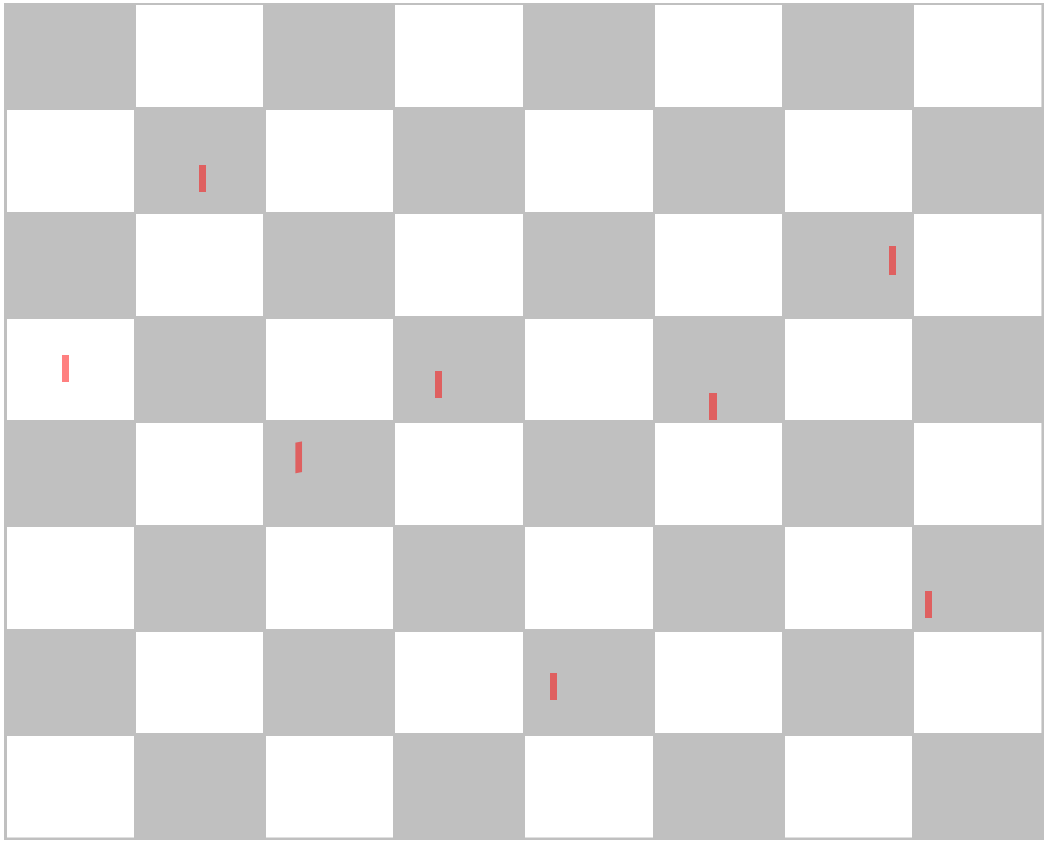
Outline

- Greedy local search
- Random Walk Strategies; e.g. walksat
- Hybrid of local search and Inference
  - The impact of constraint propagation
  - Local search on the cycle-cutset

Example: 8-queen problem



Example: 8-queen problem



Example: 8-queen problem

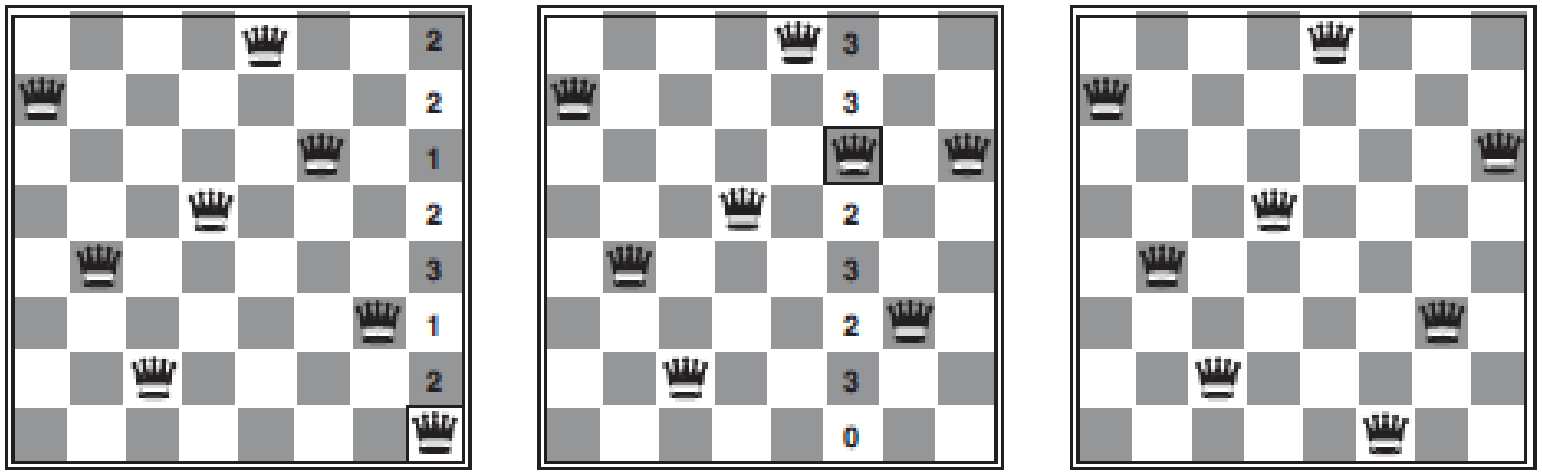


Figure 6.9 A two-step solution using min-conflicts for an 8-queens problem. At each stage, a queen is chosen for reassignment in its column. The number of conflicts (in this case, the number of attacking queens) is shown in each square. The algorithm moves the queen to the min-conflicts square, breaking ties randomly.

Hopfield networks

Find an assignment of 0, 1 to the unit that minimize the energy.

$$E = \sum_{i<j} w_{ij}\, s_i s_j + \sum_i b_i s_i$$

$$\mathrm{E}(s_i = 1) - E(S_i = 0) = b_i + \sum_j w_{ij} s_j$$

Activation rule: s_i ← 1 iff $\sum_j w_{ij} s_j > 0$
0 otherwise

Greedy local search

- Choose a full assignment and iteratively improve it towards a solution
- Requires a cost function: number of unsatisfied constraints or clauses.
- Neural (Hopfield) networks use energy minimization
- Drawback: local minima's
- Remedy: introduce a random element
- Cannot decide inconsistency

Algorithm Stochastic Local search (SLS)

Procedure SLS
Input: A constraint network $\mathcal{R} = (X, D, C)$, number of tries MAX_TRIES. A cost function.
Output: A solution iff the problem is consistent, "false" otherwise.

1. **for** i=1 to MAX_TRIES
   - **initialization: let** $\bar{a} = (a_1, ..., a_n)$ be a random initial assignment to all variables.
   - **repeat**
     (a) **if** $\bar{a}$ is consistent, return $\bar{a}$ as a solution.
     (b) **else** let $Y = \{< x_i, a'_i >\}$ be the set of variable-value pairs that when x_i is assigned a'_i, give a maximum improvement in the cost of the assignment; pick a pair $< x_i, a'_i > \in Y$,
     $\bar{a} \leftarrow (a_1, ..., a_{i-1}, a'_i, a_{i+1}, ..., a_n)$ (just flip a_i to a'_i).
   - **until** the current assignment cannot be improved.
2. **endfor**
3. return **false**

GSAT: SLS when the cost is the number of unsatisfied clauses.

Example of GSAT: CNF

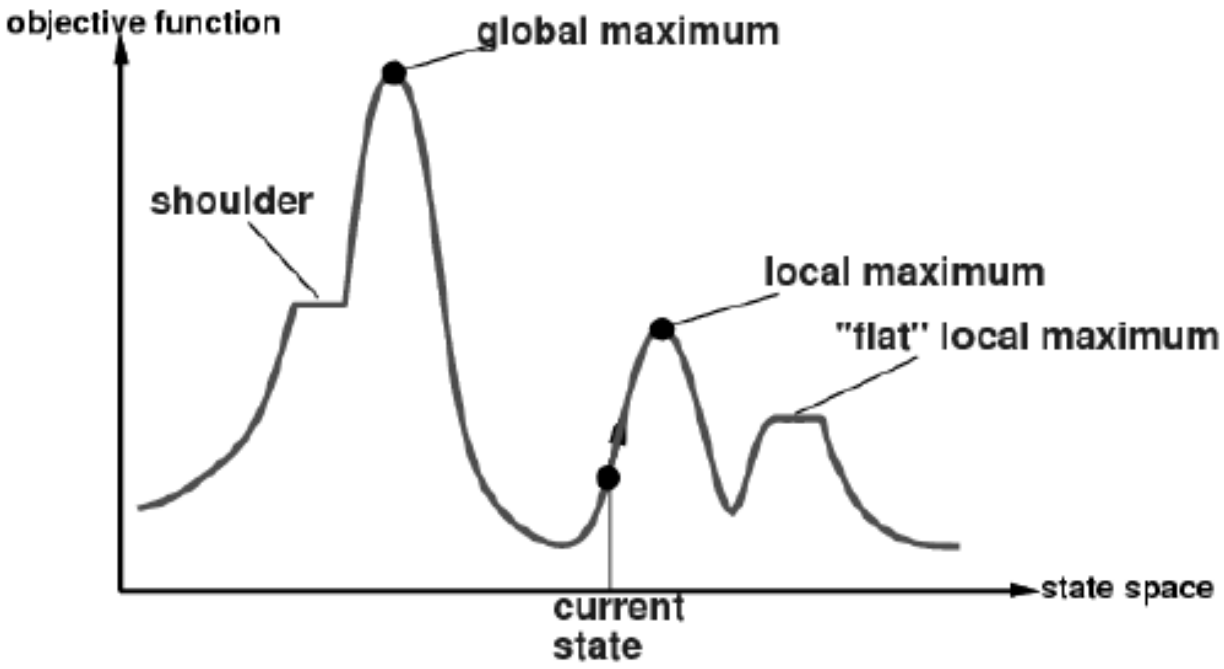
Example 7.1 Consider the formula $\varphi = \{(\neg C)(\neg A \vee \neg B \vee C)(\neg A \vee D \vee E)(\neg B \vee \neg C)\}$. Assume that in the initial assignment all variables are assigned the value "1". This

Example of GSAT: CNF

Example 7.1 Consider the formula $\varphi = \{(\neg C)(\neg A \vee \neg B \vee C)(\neg A \vee D \vee E)(\neg B \vee \neg C)\}$. Assume that in the initial assignment all variables are assigned the value "1". This assignment violates two clauses, the first and the last, so the cost is 2. Next we see that flipping A, E or D will not remove any inconsistency. Flipping C to "0" will satisfy the two violated clauses but will violate the clause $(\neg A \vee \neg B \vee C)$, yielding a cost of 1. Flipping B to $\neg B$ will remove one inconsistency and has a cost of 1 as well. If we flip C to $\neg C$, and subsequently flipping $B$to $\neg B$ yields a cost of 0 – and a solution. □

- Example: z divides y,x,t
- z = {2,3,5}, x,y = {2,3,4}, t={2,5,6}

Local Minima



Heuristics for improving local search

- Plateau search: at local minima continue search sideways.
- Constraint weighting: use weighted cost function
  - The cost C_i is 1 if C_i is violated. At local minima increase the weights of violating constraints. Then Choose (V,v) pair that leads to largest reduction of F

$$F(\bar{a}) = \sum w_i C_i(\bar{a})$$

- Tabu search:
  - prevent backwards moves by keeping a list of assigned variable-values. Tie-breaking rule may be conditioned on historic information: select the value that was flipped least recently
- Automating Max-flips:
  - Based on experimenting with a class of problems
  - Given a progress in the cost function, allow the same number of flips used up to current progress.

Random walk strategies

- Combine random walk with greediness
  - At each step:
    - choose randomly an unsatisfied clause.
    - with probability p flip a random variable in the clause, with (1-p) do a greedy step minimizing the breakout value: the number of new constraints that are unsatisfied

Figure 7.2: Algorithm WalkSAT

Procedure WalkSAT
Input: A network $\mathcal{R} = (X, D, C)$, number of flips MAX_FLIPS, MAX_TRIES, probability p.
Output: True iff the problem is consistent, false otherwise.

1. For i= 1 to MAX_TRIES do
2. Compare best assignment with $\bar{a}$ and retain the best.
   (a) **start** with a random initial assignment $\bar{a}$.
   (b) **for** i=1 to MAX_FLIPS
      - **if** $\bar{a}$ is a solution, return **true** and $\bar{a}$.
      - **else,**
        i. **pick** a violated constraint C, randomly
        ii. **choose** with probability p a variable-value pair $< x, a' >$ for $x \in scope(C)$, or, with probability $1-p$, choose a variable-value pair $< x, a' >$ that minimizes the number of new constraints that break when the value of x is changed to a', (minus 1 if the current constraint is satisfied).
        iii. Change x's value to a'.
3. **endfor**
4. return **false** and the best current assignment.

Example of walkSAT:
start with assignment of true to all vars

Example 7.2 Following our earlier example 7.1.1, we will first select an unsatisfied clause, such as $(\neg B \vee \neg C)$, and then select a variable.

$$(\neg C), (\neg A \vee \neg B \vee C)(\neg A \vee D \vee E)(\neg B \vee \neg C)$$

Example of walkSAT:
start with assignment of true to all vars

Example 7.2 Following our earlier example 7.1.1, we will first select an unsatisfied clause, such as $(\neg B \vee \neg C)$, and then select a variable. If we try to minimize the number of additional constraints that would be broken, we will select B and flip its value. Subsequently, the only unsatisfied clause is $\neg C$ which is selected and flipped. □

$$(\neg C), (\neg A \vee \neg B \vee C)(\neg A \vee D \vee E)(\neg B \vee \neg C)$$

Simulated Annealing
(Kirkpatric, Gellat and Vecchi (1983))

- Pick randomly a variable and a value and compute delta: the change in the cost function when the variable is flipped to the value.
- If change improves execute it,
- Otherwise it is executed with probability $e^{-\frac{\delta}{T}}$ where T is a temperature parameter.
- The algorithm will converge if T is reduced gradually.

Properties of local search

- Guarantee to terminate at local minima
- Random walk on 2-sat is guaranteed to converge with probability 1 after N^2 steps, when N is the number of variables.
- Proof:
  - A random assignment is on the average N^2 flips away from a satisfying assignment.
  - There is at least ½ chance that a flip of a 2-clause will reduce the distance to a given satisfying assignment by 1 (because the satisfying assignment assigns true to at least one literal in a randomly picked unsatisfied 2-clause, so at least 50% chance a flip will satisfy it).
  - Random walk will cover this distance in N^2 steps on the average.
- Analysis breaks for 3-SAT
- Empirical evaluation shows good performance compared with complete algorithms (see chapter and numerous papers)

Comparing various styles of SLS

The *random walk strategy* (i.e., GSAT +walk) augments GSAT as follows:

- with a probability *p*, pick a variable occurring in an unsatisfied clause and flip its truth value. With probability 1-*p* do a regular greedy step.

The *random noise* strategy is the same except the variable can be picked from *any* clause.

Both random walk and random noise differ from WalkSAT in a subtle way.

formula		GSAT									Simul. Ann.		
		basic			walk			noise					
vars	clauses	time	flips	R	time	flips	R	time	flips	R	time	flips	R
100	430	.4	7554	8.3	.2	2385	1.0	.6	9975	4.0	.6	4748	1.1
200	860	22	284693	143	4	27654	1.0	47	396534	6.7	21	106643	1.2
400	1700	122	$2.6\text{x}10^6$	67	7	59744	1.1	95	892048	6.3	75	552433	1.1
600	2550	1471	$30\text{x}10^6$	500	35	241651	1.0	929	$7.8\text{x}10^6$	20	427	$2.7\text{x}10^6$	3.3
800	3400	*	*	*	286	$1.8\text{x}10^6$	1.1	*	*	*	*	*	*
1000	4250	*	*	*	1095	$5.8\text{x}10^6$	1.2	*	*	*	*	*	*
2000	8480	*	*	*	3255	$23\text{x}10^6$	1.1	*	*	*	*	*	*

Table 7.1: Comparing noise strategies on hard random 3CNF instances.

Comparing DPLL and local search on circuit synthesis problems

formula			DP	GSAT+w	WSAT
id	vars	clauses	time	time	time
2bitadd_12	708	1702	*	0.081	0.013
2bitadd_11	649	1562	*	0.058	0.014
3bitadd_32	8704	32316	*	94.1	1.0
3bitadd_31	8432	31310	*	456.6	0.7
2bitcomp_12	300	730	23096	0.009	0.002
2bitcomp_5	125	310	1.4	0.009	0.001

Table 7.2: Comparing complete DPLL method (DP) with local search strategies on circuit synthesis problems. (Timings in seconds.)

Outline

- Greedy local search
- Random Walk Strategies; e.g. walksat
- Hybrid of local search and Inference
  - The impact of constraint propagation
  - Local search on the cycle-cutset

Hybrids of local search and inference

- We can use exact hybrids of search+inference and replace search by SLS (Kask and Dechter 1996)
  - Good when cutset is small
- The effect of preprocessing by constraint propagation on SLS (Kask and Dechter 1995)
  - Great improvement on structured problems
  - Not so much on uniform problems

SLS and Local Consistency

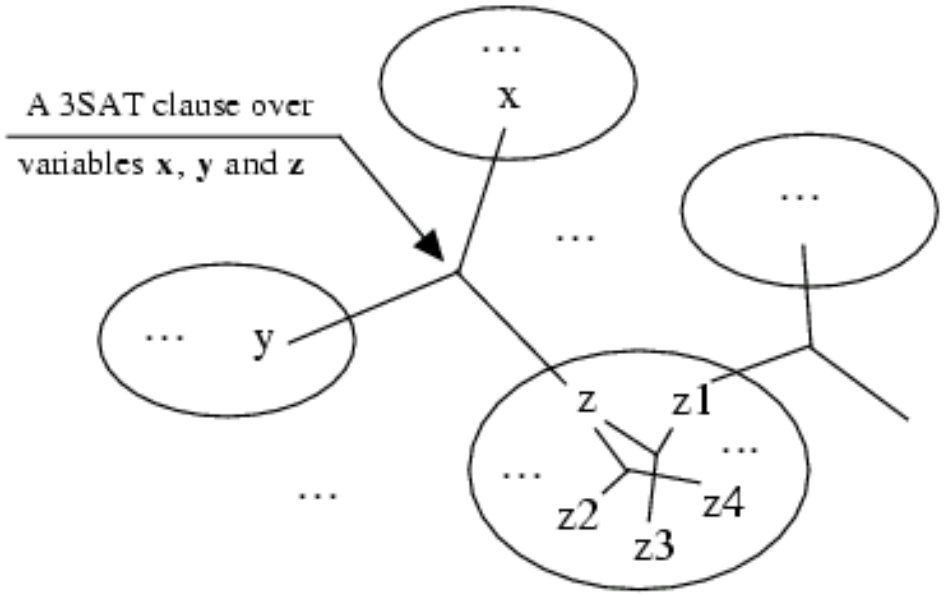
- **Structured** (hierarchical 3SAT cluster structures) vs. **(uniform) random**.

Basic scheme :

- Apply preprocessing (resolution, path consistency)
- Run SLS
- Compare against SLS alone

What can we say about local search when we have the minimal network?

SLS and Local Consistency



[Kask and Dechter, Ijcai 1995]

It was shown that certain classes of structured problems, which are very easy for systematic backtracking algorithms, are quite hard for local search, In particular, 3-SAT problems, having tightly connected clusters of variables which, in turn, are loosely connected by another set of constraints, can be extremely hard for even the best implementations of SLS.

However, these problems often become trivial for SLS once a restricted form of resolution is enforced. Moreover, the overhead associated with enforcing this kind of local consistency is still much less than the computation needed to solve the problem without it.

A possible explanation: enforcing local consistency changes the search space of local-search methods by eliminating many near solutions. Namely, assignments that satisfy *almost* all clauses, for which the value of the cost function is therefore almost zero, become assignments whose cost is high due to consistency enforcing.

SLS and SLS on structured problems

Solvable 3SAT cluster structures, 100 instances, MaxFlips = 512K
5 variables per cluster, 50 clusters, 200 clauses between clusters
Restricted Bound-3 Resolution : only original clauses resolved
Running times, number of flips and clauses added are given as an average per problem solved

	Before Resolution			After Resolution					
C/cluster	Solved	Time	Flips	Solved	RBR-3 Time	Total Time	Flips	New Clauses	DP
30	100	0.52 sec	4.5K	100	3.6 sec	3.7 sec	189	1736	1.03 sec
31	100	0.71	5.1K	100	3.88	3.91	176	1731	1.04
32	100	1.03	8.4K	100	4.16	4.20	162	1722	1.09
33	100	1.54	12K	100	4.36	4.39	155	1708	1.11
34	100	3.44	26K	100	4.66	4.70	151	1690	1.15
35	100	6.38	49K	100	4.92	4.95	140	1668	1.18
36	90	21.7	161K	100	5.23	5.26	135	1640	1.19
37	41	35.5	252K	100	5.42	5.45	131	1609	1.23
38	3	28.4	202K	100	5.94	5.97	125	1574	1.27
39	0	-	-	100	5.95	5.98	121	1540	1.29
40	0	-	-	100	6.13	6.17	115	1503	1.29

Table 1: Bound-3 Resolution and GSAT

http://www.ics.uci.edu/%7Ecsp/r34-gsat-local-consistency.pdf

SLS and Local Consistency

N=100, K=8, T=32/64, 200 instances, MaxFlips = 512K								
C	Solvable	Algorithm	Solved	Tries	Flips	PPC Time	Total Time	BJ-DVO
265	88.5 %	GSAT	139	336	147K	0 sec	36 sec	19 min
		PPC + GSAT	152	292	140K	8 sec	66 sec	
270	66 %	GSAT	78	406	191K	0 sec	45 sec	33 min
		PPC + GSAT	83	381	195K	14 sec	92 sec	
N=30, K=64, T=2048/4096, 100 instances, MaxFlips = 128K, C_{crit}=180?								
163		PPC + GSAT	56	276	59K	56 sec	153 sec	*
		GSAT	58	247	53K	0 sec	89 sec	*

Table 2: Partial Path Consistency and GSAT

Uniform random 3SAT, N=600, C=2550, 100 instances, MaxFlips = 512K					
Algorithm	Solved	Tries	Flips	BR-3 Time	Total Time
GSAT	36	63	176K	0 sec	15.3 sec
BR-3 + GSAT	31	45	125K	0.3 sec	15.0 sec

Table 3: Bound-3 Resolution and GSAT

SLS and Local Consistency

Summary:

- For structured problems, enforcing local consistency will improve SLS
- For uniform CSPs, enforcing local consistency is not cost effective: performance of SLS is improved, but not enough to compensate for the preprocessing cost.

Outline

- Greedy local search
- Random Walk Strategies; e.g. walksat
- Hybrid of local search and Inference
  - The impact of constraint propagation
  - Local search on the cycle-cutset

SLS and Cutset Conditioning

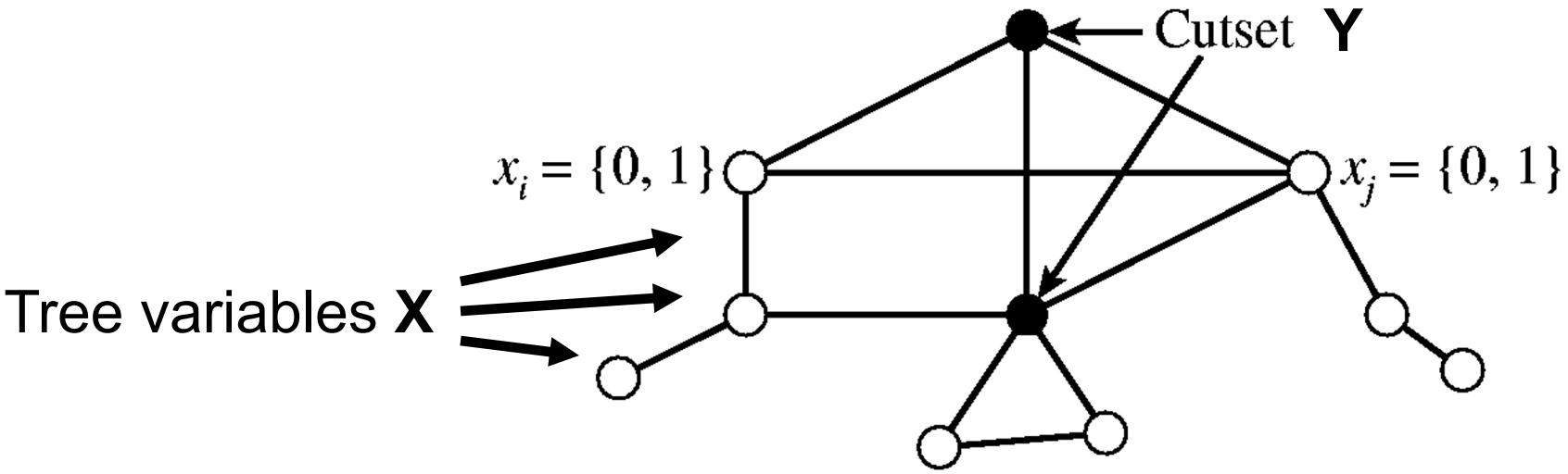
Background:

- Tree algorithm is tractable for trees.
- Networks with bounded width are tractable*.

Basic Scheme:

- Identify a cutset such that width is reduced to desired value.
- Use search with cutset conditioning.

Local search on Cycle-cutset



Tree Algorithm

Tree Algorithm : minimizing the cost of a tree-like subnetwork:
where $R_{z_i,z_j}(a_i, a_j)$ is the constraint between z_i and z_j and is either 0 if $(a_i, a_j) \in R_{z_i,z_j}$ or 1, otherwise.
Input: An arc consistent network $\mathcal{R} = (X, D, C)$. Variables X partitioned into cycle cutset Y and tree variables Z, $X = Z \cup Y$. An assignment $Y = \bar{y}$.
Output: An assignment $Z = \bar{z}$ that minimizes the number of violated constraints of the entire network when $Y = \bar{y}$.
Initialization: For any value $\bar{y}[i]$ of any cutset variable y_i, the cost $C_{y_i}(\bar{y}[i], \bar{y})$ is 0.

1. Going from leaves to root on the tree,

   (a) **for** every variable, z_i and any value $a_i \in D_{z_i}$, compute,

$$C_{z_i}(a_i, \bar{y}) = \sum_{\{z_j | z_j \ child \ of \ z_i\}} min_{a_j \in D_{z_j}} \left(C_{z_j}(a_j, \bar{y}) + R_{z_i,z_j}(a_i, a_j)\right)$$

   (b) **endfor**

Tree Algorithm (contd)

2. Compute, going from root to leaves, new assignment for every tree variable z_i:

   (a) **for** a tree variable z_i, let D_{z_i} be its consistent values with v_{p_i} the value assigned to its parent p_i, compute

$$a_i \leftarrow arg\, min_{a_i \in D_{z_i}}(C_{z_i}(a_i, \bar{y}) + R_{z_i,p_i}(a_i,\ v_{p_i}))$$

   (b) **endfor**

3. **return** $(< z_1, a_1 >, ..., < z_k, a_k >)$.

GSAT with cycle-cutset (Kask and Dechter, 1996)

Input: a CSP, a partition of the variables into **cycle-cutset** and **tree variables**
Output: an assignment to all the variables

Within each try:
Generate a random initial asignment,
and then alternate between the two steps:

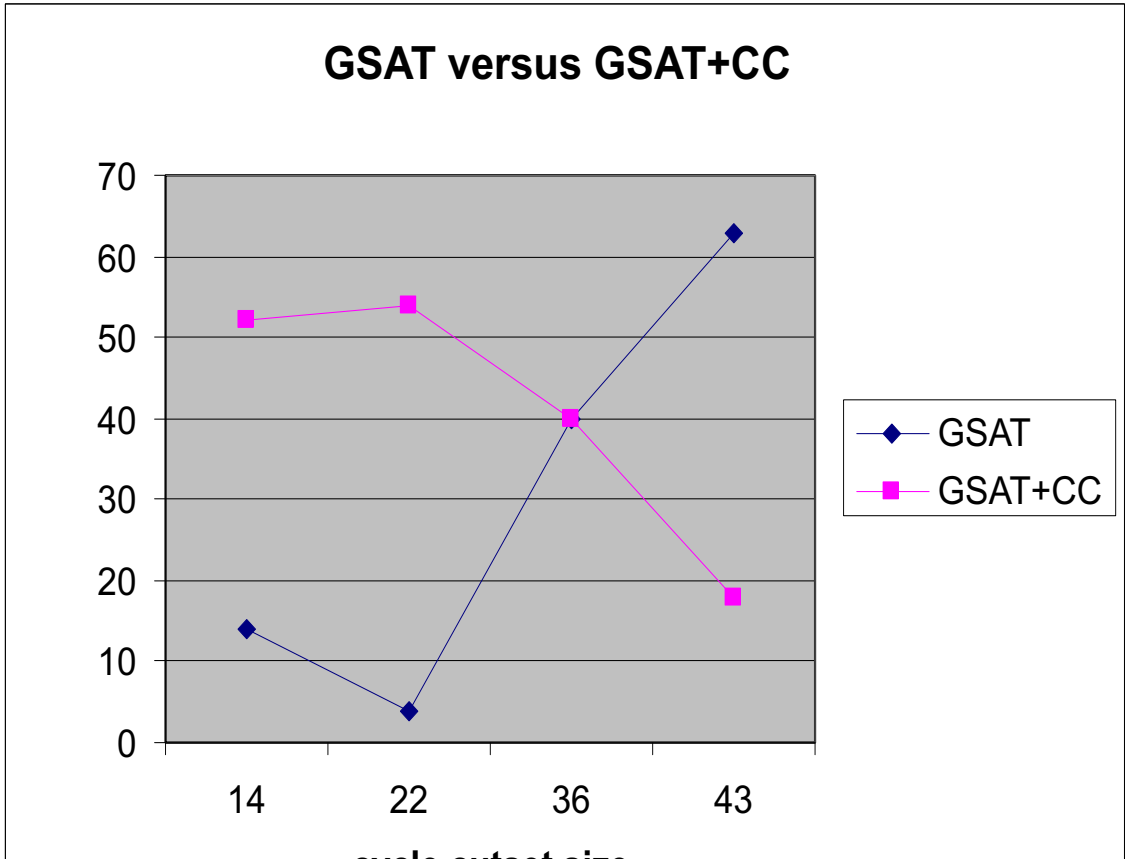
1. Run **Tree algorithm** (arc-consistency+assignment) on the problem with fixed values of cutset variables.
2. Run GSAT on the problem with fixed values of tree variables.

Theorem 7.1

Theorem 7.1 *The Tree Algorithm in Figure 7.4 is guaranteed to find an assignment that minimizes the number of violated constraints in every tree-like subnetwork, conditioned on the cutset values.*

Results GSAT with Cycle-Cutset

(Kask and Dechter, 1996)



SLS and Cutset Conditioning

Summary:

- A new combined algorithm of SLS and inference based on cutset conditioning
- Empirical evaluation on random CSPs
- SLS combined with the tree algorithm is superior to pure SLS when the cutset is small

https://www.ics.uci.edu/~dechter/publications/r24.pdf

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