CompSci 275, Constraint Networks

Rina Dechter, Fall 2022

Boolean Satisfiability

Based in part on J. Marques-Silva tutorial, ECAI 2010
Also on Darwuche&Pipatsisawat, handbook of SAT, chapter 3.

Outline

- Review: DPLL, Resolution
- CDCL: Conflict-Directed Clause Learning
  - Implication graphs,
  - asserting clauses,
  - Unique Implication points (UIPs)
- Watch literals
- Empirical evaluation

Reading chapters 3 and 4 in Handbook on SAT)

Outline

- Review: DPLL, Resolution
- CDCL: Conflict-Directed Clause Learning
- Implication graphs, asserting clauses, Unique Implication points (UIPs)
- Watch literals

Basic Definitions

- Propositional variables can be assigned value 0 or 1
  - In some contexts variables may be unassigned

- A clause is satisfied if at least one of its literals is assigned value 1

  $(x_1 \vee \neg x_2 \vee \neg x_3)$

- A clause is unsatisfied if all of its literals are assigned value 0

  $(x_1 \vee \neg x_2 \vee \neg x_3)$

- A clause is unit if it contains one single unassigned literal and all other literals are assigned value 0

  $(x_1 \vee \neg x_2 \vee \neg x_3)$

- A formula is satisfied if all of its clauses are satisfied
- A formula is unsatisfied if at least one of its clauses is unsatisfied

Pure Literals

- A literal is pure if only occurs as a positive literal or as a negative literal in a CNF formula
  - Example:
    $\varphi = (\neg x_1 \vee x_2) \wedge (x_3 \vee \neg x_2) \wedge (x_4 \vee \neg x_5) \wedge (x_5 \vee \neg x_4)$
  - x_1 and x_3 and pure literals

- Pure literal rule:
  Clauses containing pure literals can be removed from the formula (i.e. just assign pure literals to the values that satisfy the clauses)
  - For the example above, the resulting formula becomes:
    $\varphi = (x_4 \vee \neg x_5) \wedge (x_5 \vee \neg x_4)$

- A reference technique until the mid 90s; nowadays seldom used

Unit Propagation

- Unit clause rule: [Davis&Putnam, JACM'60]
  Given a unit clause, its only unassigned literal must be assigned value 1 for the clause to be satisfied
  - Example: for unit clause $(x_1 \vee \neg x_2 \vee \neg x_3)$, x_3 must be assigned value 0
- Unit propagation
  Iterated application of the unit clause rule

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_4)$$

Unit Propagation

- Unit clause rule: [Davis&Putnam, JACM'60]
  Given a unit clause, its only unassigned literal must be assigned value 1 for the clause to be satisfied
  - Example: for unit clause $(x_1 \vee \neg x_2 \vee \neg x_3)$, x_3 must be assigned value 0
- Unit propagation
  Iterated application of the unit clause rule

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_4)$$

Unit Propagation

- Unit clause rule: [Davis&Putnam, JACM'60]
  Given a unit clause, its only unassigned literal must be assigned value 1 for the clause to be satisfied
  - Example: for unit clause $(x_1 \vee \neg x_2 \vee \neg x_3)$, x_3 must be assigned value 0
- Unit propagation
  Iterated application of the unit clause rule

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_4)$$

Unit Propagation

- Unit clause rule: [Davis&Putnam, JACM'60]
  Given a unit clause, its only unassigned literal must be assigned value 1 for the clause to be satisfied
  - Example: for unit clause $(x_1 \vee \neg x_2 \vee \neg x_3)$, x_3 must be assigned value 0
- Unit propagation
  Iterated application of the unit clause rule

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_4)$$

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4)$$

Unit Propagation

- Unit clause rule: [Davis&Putnam, JACM'60]
  Given a unit clause, its only unassigned literal must be assigned value 1 for the clause to be satisfied
  - Example: for unit clause $(x_1 \vee \neg x_2 \vee \neg x_3)$, x_3 must be assigned value 0
- Unit propagation
  Iterated application of the unit clause rule

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_4)$$

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4)$$

Unit Propagation

- Unit clause rule: [Davis&Putnam, JACM'60]
  Given a unit clause, its only unassigned literal must be assigned value 1 for the clause to be satisfied
  - Example: for unit clause $(x_1 \vee \neg x_2 \vee \neg x_3)$, x_3 must be assigned value 0
- Unit propagation
  Iterated application of the unit clause rule

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_4)$$

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4)$$

Unit Propagation

- Unit clause rule: [Davis&Putnam, JACM'60]
  Given a unit clause, its only unassigned literal must be assigned value 1 for the clause to be satisfied
  - Example: for unit clause $(x_1 \vee \neg x_2 \vee \neg x_3)$, x_3 must be assigned value 0
- Unit propagation
  Iterated application of the unit clause rule

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_4)$$

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4)$$

- Unit propagation can satisfy clauses but can also unsatisfy clauses (i.e. conflicts)

Resolution

- Resolution rule:
  - If a formula φ contains clauses $(x \vee \alpha)$ and $(\neg x \vee \beta)$, then infer $(\alpha \vee \beta)$

$$\mathrm{RES}(x \vee \alpha, \neg x \vee \beta) = (\alpha \vee \beta)$$

- Resolution forms the basis of a complete algorithm for SAT
  - Iteratively apply the following steps: [Davis&Putnam, JACM'60]
    - Select variable x
    - Apply resolution rule between every pair of clauses of the form $(x \vee \alpha)$ and $(\neg x \vee \beta)$
    - Remove all clauses containing either x or $\neg x$
    - Apply the pure literal rule and unit propagation
  - Terminate when either the empty clause or the empty formula is derived

Resolution – An Example

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

Resolution – An Example

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

$$(\neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

Resolution – An Example

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

$$(\neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

$$(x_3 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

Resolution – An Example

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

$$(\neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

$$(x_3 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

$$(x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

Resolution – An Example

$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$

$(\neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$

$(x_3 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$

$(x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$

(x_3)

- Formula is SAT

Do directional resolution in the order:
X3,x4,x2,x1

Outline

Preliminaries

Algorithms

- Local Search
- The DPLL Algorithm
- Conflict-Driven Clause Learning (CDCL)

SAT-Based Modelling

Outline

Preliminaries

Algorithms

- Local Search
- The DPLL Algorithm
- Conflict-Driven Clause Learning (CDCL)

SAT-Based Modelling

DPLL – Historical Perspective

- In 1960, M. Davis and H. Putnam proposed the DP algorithm:
  - Resolution used to eliminate 1 variable at each step
  - Applied the pure literal rule and unit propagation
- Original algorithm was inefficient
- In 1962, M. Davis, G. Logemann and D. Loveland proposed an alternative algorithm:
  - Instead of eliminating variables, the algorithm would split on a given variable at each step
  - Also applied the pure literal rule and unit propagation
- The 1962 algorithm is actually an implementation of backtrack search
- Over the years the 1962 algorithm became known as the DPLL (sometimes DLL) algorithm

The DPLL Algorithm

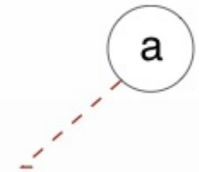
- Standard backtrack search
- At each step:
  - [DECIDE] Select decision assignment
  - [DEDUCE] Apply unit propagation and (optionally) the pure literal rule
  - [DIAGNOSE] If conflict identified, then backtrack
    - If cannot backtrack further, return UNSAT
    - Otherwise, proceed with unit propagation
  - If formula satisfied, return SAT
  - Otherwise, proceed with another decision

An Example of DPLL

$$
\begin{aligned}
\varphi \quad = \quad & (a \vee \neg b \vee d) \wedge (a \vee \neg b \vee e) \wedge \\
& (\neg b \vee \neg d \vee \neg e) \wedge \\
& (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \neg d) \wedge \\
& (a \vee b \vee \neg c \vee e) \wedge (a \vee b \vee \neg c \vee \neg e)
\end{aligned}
$$

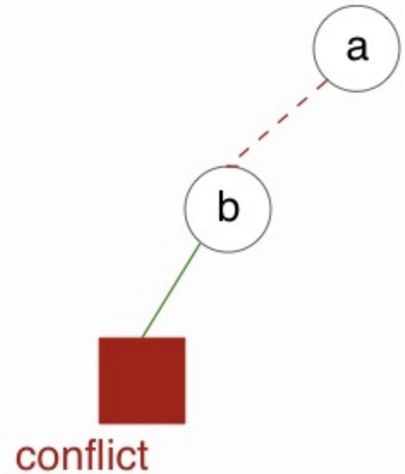
An Example of DPLL

$$\begin{aligned}\varphi \;=\;& (a \vee \neg b \vee d) \wedge (a \vee \neg b \vee e) \wedge \\ & (\neg b \vee \neg d \vee \neg e) \wedge \\ & (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \neg d) \wedge \\ & (a \vee b \vee \neg c \vee e) \wedge (a \vee b \vee \neg c \vee \neg e)\end{aligned}$$



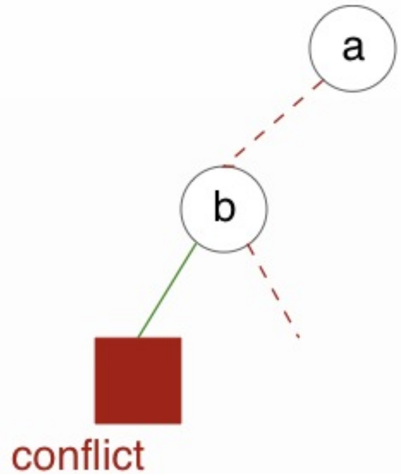
An Example of DPLL

$$
\begin{aligned}
\varphi \quad = \quad & (a \vee \neg b \vee d) \wedge (a \vee \neg b \vee e) \wedge \\
& (\neg b \vee \neg d \vee \neg e) \wedge \\
& (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \neg d) \wedge \\
& (a \vee b \vee \neg c \vee e) \wedge (a \vee b \vee \neg c \vee \neg e)
\end{aligned}
$$



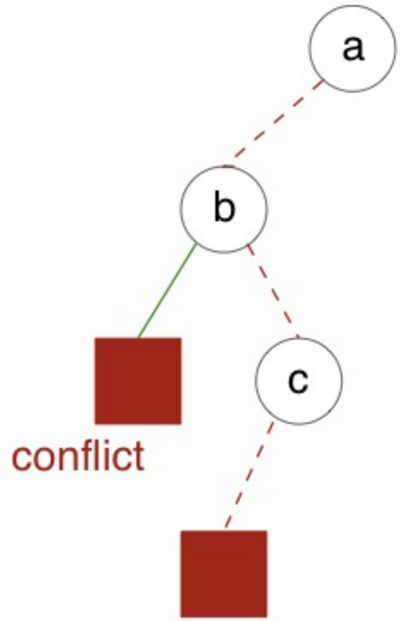
An Example of DPLL

$$\begin{aligned}\varphi &= (a \vee \neg b \vee d) \wedge (a \vee \neg b \vee e) \wedge \\ &\quad (\neg b \vee \neg d \vee \neg e) \wedge \\ &\quad (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \neg d) \wedge \\ &\quad (a \vee b \vee \neg c \vee e) \wedge (a \vee b \vee \neg c \vee \neg e)\end{aligned}$$



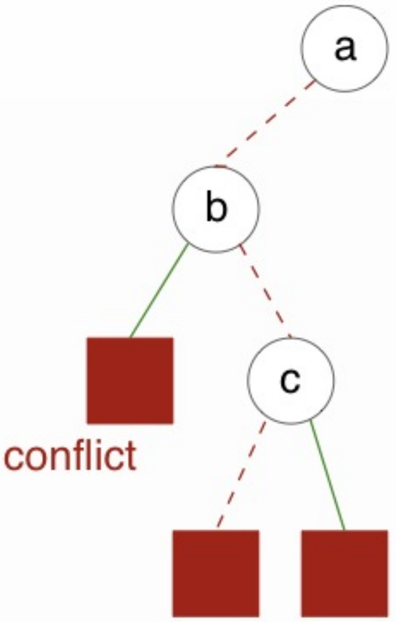
An Example of DPLL

$$
\begin{aligned}
\varphi \quad = \quad & (a \vee \neg b \vee d) \wedge (a \vee \neg b \vee e) \wedge \\
& (\neg b \vee \neg d \vee \neg e) \wedge \\
& (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \neg d) \wedge \\
& (a \vee b \vee \neg c \vee e) \wedge (a \vee b \vee \neg c \vee \neg e)
\end{aligned}
$$



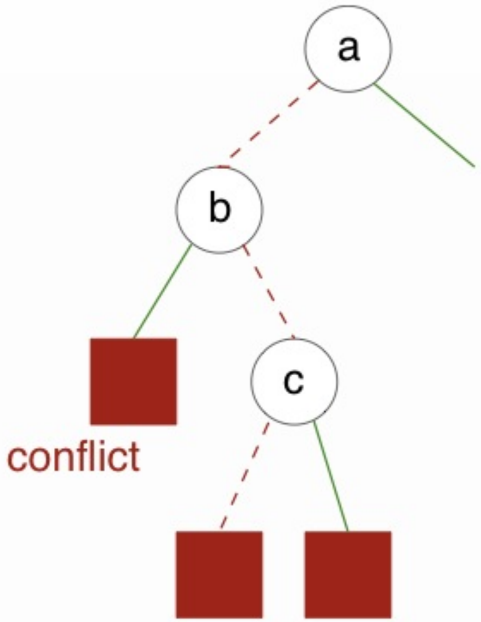
An Example of DPLL

$$
\begin{aligned}
\varphi \quad = \quad & (a \vee \neg b \vee d) \wedge (a \vee \neg b \vee e) \wedge \\
& (\neg b \vee \neg d \vee \neg e) \wedge \\
& (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \neg d) \wedge \\
& (a \vee b \vee \neg c \vee e) \wedge (a \vee b \vee \neg c \vee \neg e)
\end{aligned}
$$



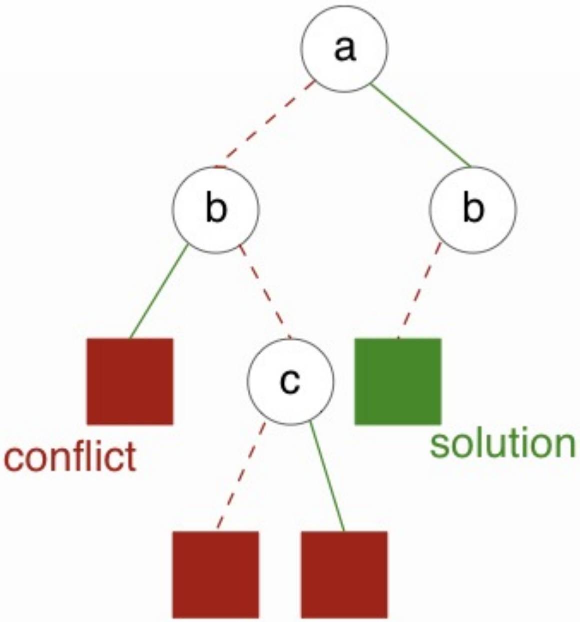
An Example of DPLL

$$\begin{aligned}\varphi \;=\; & (a \lor \neg b \lor d) \land (a \lor \neg b \lor e) \land \\ & (\neg b \lor \neg d \lor \neg e) \land \\ & (a \lor b \lor c \lor d) \land (a \lor b \lor c \lor \neg d) \land \\ & (a \lor b \lor \neg c \lor e) \land (a \lor b \lor \neg c \lor \neg e)\end{aligned}$$



An Example of DPLL

$$
\begin{aligned}
\varphi \quad = \quad & (a \vee \neg b \vee d) \wedge (a \vee \neg b \vee e) \wedge \\
& (\neg b \vee \neg d \vee \neg e) \wedge \\
& (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \neg d) \wedge \\
& (a \vee b \vee \neg c \vee e) \wedge (a \vee b \vee \neg c \vee \neg e)
\end{aligned}
$$



Comparing with CSP:

- Sat can be decided before all variables are assigned

Complexity: when is unit propagation complete?....

Think Horn clauses

Outline

- Review: DPLL, Resolution
- Conflict-Directed Clause Learning (CDCL)
  - Implication graphs,
  - asserting clauses,
  - Unique Implication points (UIPs)
- Watch literals

Conflict Analysis: Implication Graphs

- The combination of these techniques makes sure that unit resolution is empowered every time a conflict arises, and that the solver will not repeat any mistake. The identification of conflict–driven clauses is done through a process known as conflict analysis, which analyzes a trace of unit resolution known as the implication graph.

Our vanilla CSP conflict did not take arc-consistency into account
In SAT, conflict-driven analysis does.

Implication graphs

φ=

1. {A,B}
2. {B,C}
3. {¬A, ¬X, Y }
4. {¬A,X,Z}
5. {¬A, ¬Y,Z}
6. {¬A,X, ¬Z}
7. {¬A, ¬Y, ¬Z}

Each node in an implication graph has the form l/V=v, which means that variable V has been set to value v at level l. Note that a variable is set either by a decision or by an implication. A variable is set by an implication if the setting is due to the application of unit resolution. Otherwise, it is set by a decision.

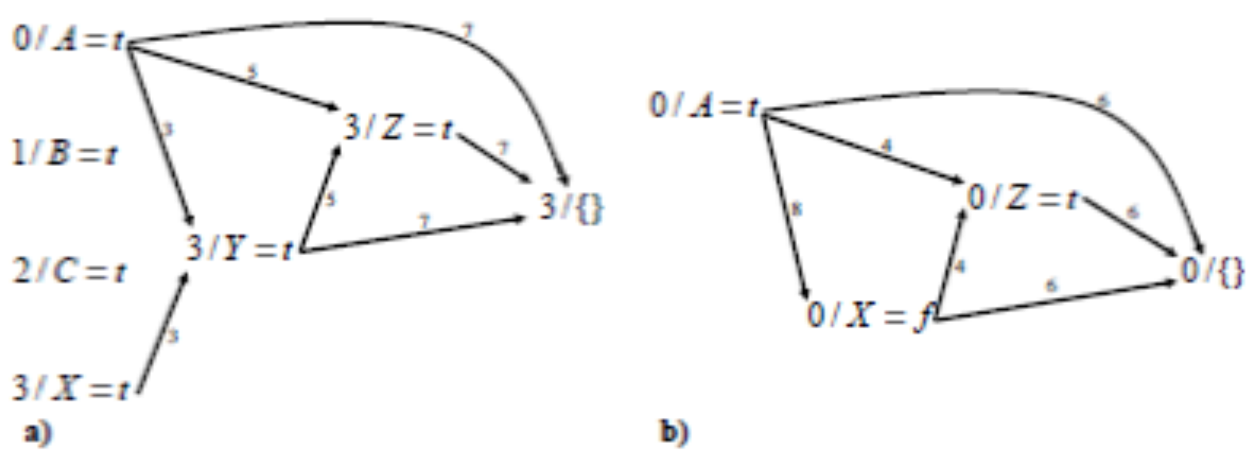


Figure 3.7. Two implication graphs.

Deriving conflict clause

- Every cut in the implication graph defines a conflict set as long as that cut seperates the decision variables (root nodes) from the contradiction (a leaf node).
- Any node (variable assignment) with an outgoing edge that cross the cut will be in the conflict set.

leading to conflicts: sets: {A=true,X=true}, {A=true, Y=true} and {A=true, Y=true, Z=true}.

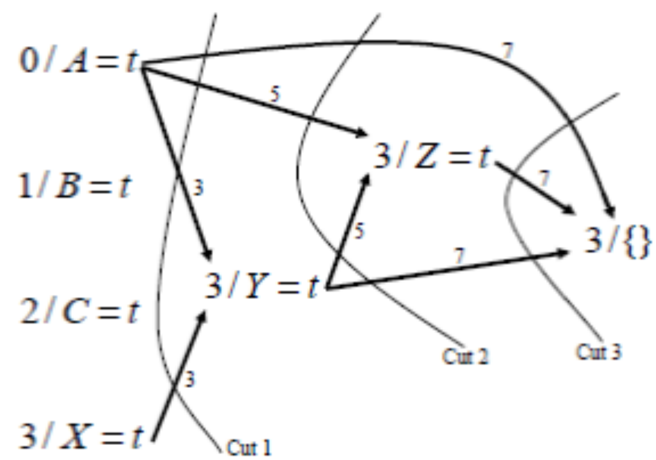


Figure 3.8. Three cuts in an implication graph, leading to three conflict sets.

φ=

1. {A,B}
2. {B,C}
3. {¬A, ¬X, Y }
4. {¬A,X,Z}
5. {¬A, ¬Y,Z}
6. {¬A,X, ¬Z}
7. {¬A, ¬Y, ¬Z}

Earliest minimal conflict?

$\varphi=$

1. {A,B}
2. {B,C}
3. {¬A, ¬X, Y }
4. {¬A,X,Z}
5. {¬A, ¬Y,Z}
6. {¬A,X, ¬Z}
7. {¬A, ¬Y, ¬Z}
8. {¬A, ¬X}

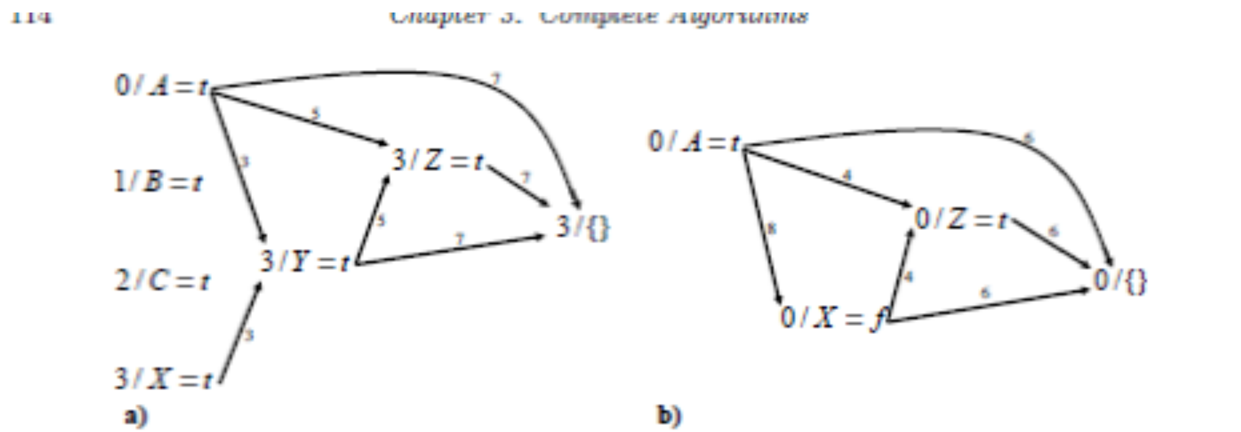


Figure 3.7. Two implication graphs.

For the graph in Figure 3.7(b), {A = true} is a conflict cut.

Conflict–driven clauses generated from cuts that contain exactly one variable assigned at the level of conflict are said to be asserting [ZMMM01]. Modern SAT solvers insist on learning only asserting clauses.

UIP: unique implication points

A UIP of a decision level in an implication graph is a variable setting at that decision level which lies on every path from the decision variable of that level to the contradiction. Intuitively, a UIP of a level is an assignment at the level that, by itself, is sufficient for implying the contradiction. In Figure 3.9, the variable setting 3/Y=true and 3/X=true would be UIPs as they lie on every path from the decision 3/X=true to the contradiction3/{}.

φ=

1. {A,B}
2. {B,C}
3. {¬A, ¬X, Y }
4. {¬A,X,Z}
5. {¬A, ¬Y,Z}
6. {¬A,X, ¬Z}
7. {¬A, ¬Y, ¬Z}
8. {¬A, ¬X}

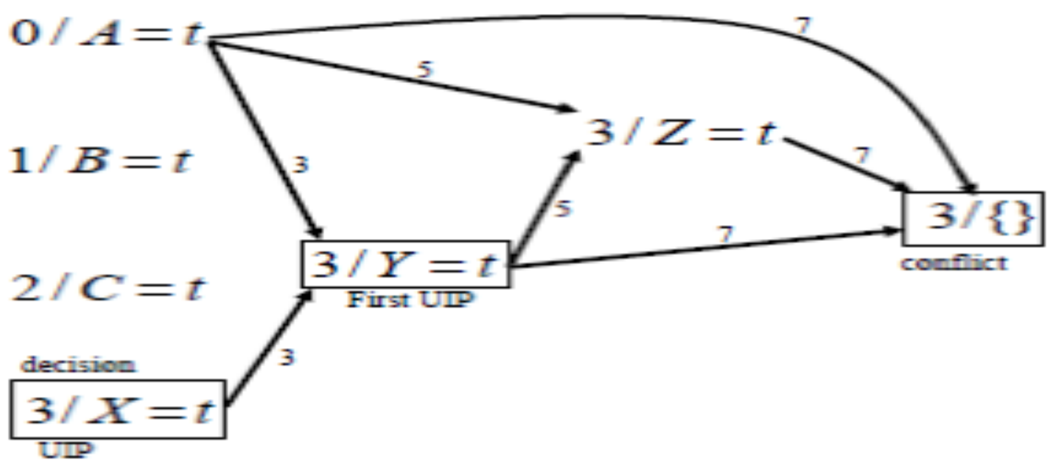


Figure 3.9. An example of a unique implication point (UIP).

Deriving asserting clauses that contain the first UIP is popular: {¬A ∨ ¬Y } will be learnt

Last slide

CDCL SAT Solvers – Basic Techniques

- Based on DPLL [Davis et al., JACM'60, CACM'62]
  - Must be able to prove unsatisfiability
- New clauses are learned from conflicts [Marques-Silva&Sakallah, ICCAD'96]
  - Backtracking can be non-chronological
- Structure of conflicts is exploited (UIPs) [Marques-Silva&Sakallah, ICCAD'96]
- Backtrack search is periodically restarted [Gomes et al., AAAI'98]
- Lazy data structures are used [Moskewicz et al, DAC'01]
  - Compact with low maintenance overhead
- Branching is guided by conflicts [Moskewicz et al, DAC'01]
  - E.g. VSIDS, etc.

CDCL SAT Solvers – Additional Techniques

- (Currently) effective techniques:
  - Unused learned clauses are discarded [Goldberg&Novikov, DATE'02]
  - Use formula preprocessing I [Een&Biere, SAT'05]
  - Minimize learned clauses [Sorensson&Biere, SAT'09]
  - Use literal progress saving [Pipatsrisawat&Darwiche, SAT'07]
  - Use dynamic restart policies [Biere, SAT'08]
  - Exploit extended implication graphs [Audemard et al., SAT'08]
  - Identify glue clauses [Audemard & Simon, IJCAI'09]

- (Currently) ineffective techniques:
  - Identify pure literals [Davis&Putnam, JACM'60]
  - Implement variable lookahead [Anbulagan&Li, IJCAI'97]
  - Use formula preprocessing II [Brafman, IJCAI'01]

Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

$$\varphi = (a \vee b) \wedge (\neg b \vee c \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee f) \ldots$$

Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

$$\varphi = (a \vee b) \wedge (\neg b \vee c \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee f) \ldots$$

  - Assume decisions $c = 0$ and $f = 0$

Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

$$\varphi = (a \vee b) \wedge (\neg b \vee c \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee f) \ldots$$

  - Assume decisions $c = 0$ and $f = 0$
  - Assign $a = 0$ and imply assignments

Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

$$\varphi = (a \vee b) \wedge (\neg b \vee c \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee f) \ldots$$

  - Assume decisions $c = 0$ and $f = 0$
  - Assign $a = 0$ and imply assignments

Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

$$\varphi = (a \vee b) \wedge (\neg b \vee c \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee f) \ldots$$

  - Assume decisions $c = 0$ and $f = 0$
  - Assign $a = 0$ and imply assignments
  - A conflict is reached: $(\neg d \vee \neg e \vee f)$ is unsatisfied

Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

$$\varphi = (\boxed{a} \vee b) \wedge (\neg b \vee \boxed{c} \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee \boxed{f}) \ldots$$

  - Assume decisions $c = 0$ and $f = 0$
  - Assign $a = 0$ and imply assignments
  - A conflict is reached: $(\neg d \vee \neg e \vee f)$ is unsatisfied
  - $(a = 0) \wedge (c = 0) \wedge (f = 0) \Rightarrow (\varphi = 0)$

Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

$$\varphi = (\boxed{a} \vee b) \wedge (\neg b \vee \boxed{c} \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee \boxed{f}) \ldots$$

  - Assume decisions $c = 0$ and $f = 0$
  - Assign $a = 0$ and imply assignments
  - A conflict is reached: $(\neg d \vee \neg e \vee f)$ is unsatisfied
  - $(a = 0) \wedge (c = 0) \wedge (f = 0) \Rightarrow (\varphi = 0)$
  - $(\varphi = 1) \Rightarrow (a = 1) \vee (c = 1) \vee (f = 1)$

Clause Learning

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

$$\varphi = (\boxed{a} \vee b) \wedge (\neg b \vee \boxed{c} \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee \boxed{f}) \ldots$$

  - Assume decisions $c = 0$ and $f = 0$
  - Assign $a = 0$ and imply assignments
  - A conflict is reached: $(\neg d \vee \neg e \vee f)$ is unsatisfied
  - $(a = 0) \wedge (c = 0) \wedge (f = 0) \Rightarrow (\varphi = 0)$
  - $(\varphi = 1) \Rightarrow (a = 1) \vee (c = 1) \vee (f = 1)$
  - Learn new clause $(a \vee c \vee f)$

Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

$$\begin{aligned}\varphi \;=\; & (a \vee b) \wedge (\neg b \vee c \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee f) \wedge \\ & (a \vee c \vee f) \wedge (\neg a \vee g) \wedge (\neg g \vee b) \wedge (\neg h \vee j) \wedge (\neg i \vee k)\end{aligned}$$

Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

$$\begin{aligned} \varphi \quad = \quad & (a \vee b) \wedge (\neg b \vee c \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee f) \wedge \\ & (a \vee c \vee f) \wedge (\neg a \vee g) \wedge (\neg g \vee b) \wedge (\neg h \vee j) \wedge (\neg i \vee k) \end{aligned}$$

  - Assume decisions $c = 0$, $f = 0$, $h = 0$ and $i = 0$

Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

$$\begin{aligned}\varphi \;=\;& (a \vee b) \wedge (\neg b \vee c \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee f) \wedge \\ & (a \vee c \vee f) \wedge (\neg a \vee g) \wedge (\neg g \vee b) \wedge (\neg h \vee j) \wedge (\neg i \vee k)\end{aligned}$$

  - Assume decisions $c = 0$, $f = 0$, $h = 0$ and $i = 0$
  - Assignment $a = 0$ caused conflict $\Rightarrow$ learnt clause $(a \vee c \vee f)$ implies $a = 1$

Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

$$\varphi = (a \vee b) \wedge (\neg b \vee c \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee f) \wedge$$
$$(a \vee c \vee f) \wedge (\neg a \vee g) \wedge (\neg g \vee b) \wedge (\neg h \vee j) \wedge (\neg i \vee k)$$

  - Assume decisions $c = 0$, $f = 0$, $h = 0$ and $i = 0$
  - Assignment $a = 0$ caused conflict $\Rightarrow$ learnt clause $(a \vee c \vee f)$ implies $a = 1$

Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

$$\varphi = (a \lor b) \land (\neg b \lor c \lor d) \land (\neg b \lor e) \land (\neg d \lor \neg e \lor f) \land$$
$$(a \lor c \lor f) \land (\neg a \lor g) \land (\neg g \lor b) \land (\neg h \lor j) \land (\neg i \lor k)$$

  - Assume decisions $c = 0$, $f = 0$, $h = 0$ and $i = 0$
  - Assignment $a = 0$ caused conflict $\Rightarrow$ learnt clause $(a \lor c \lor f)$ implies $a = 1$

Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

$$\varphi = (a \vee b) \wedge (\neg b \vee c \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee f) \wedge$$
$$(a \vee c \vee f) \wedge (\neg a \vee g) \wedge (\neg g \vee b) \wedge (\neg h \vee j) \wedge (\neg i \vee k)$$

  - Assume decisions $c = 0$, $f = 0$, $h = 0$ and $i = 0$
  - Assignment $a = 0$ caused conflict $\Rightarrow$ learnt clause $(a \vee c \vee f)$ implies $a = 1$
  - A conflict is again reached: $(\neg d \vee \neg e \vee f)$ is unsatisfied

Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

$$\begin{aligned}\varphi \;=\; & (a \vee b) \wedge (\neg b \vee \boxed{c} \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee \boxed{f}) \wedge \\ & (a \vee \boxed{c} \vee \boxed{f}) \wedge (\neg a \vee g) \wedge (\neg g \vee b) \wedge (\neg h \vee j) \wedge (\neg i \vee k)\end{aligned}$$

  - Assume decisions $c = 0$, $f = 0$, $h = 0$ and $i = 0$
  - Assignment $a = 0$ caused conflict $\Rightarrow$ learnt clause $(a \vee c \vee f)$ implies $a = 1$
  - A conflict is again reached: $(\neg d \vee \neg e \vee f)$ is unsatisfied
  - $(c = 0) \wedge (f = 0) \Rightarrow (\varphi = 0)$

Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

$$\begin{aligned} \varphi \;=\; & (a \vee b) \wedge (\neg b \vee \boxed{c} \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee \boxed{f}) \wedge \\ & (a \vee \boxed{c} \vee \boxed{f}) \wedge (\neg a \vee g) \wedge (\neg g \vee b) \wedge (\neg h \vee j) \wedge (\neg i \vee k) \end{aligned}$$

  - Assume decisions $c = 0$, $f = 0$, $h = 0$ and $i = 0$
  - Assignment $a = 0$ caused conflict $\Rightarrow$ learnt clause $(a \vee c \vee f)$ implies $a = 1$
  - A conflict is again reached: $(\neg d \vee \neg e \vee f)$ is unsatisfied
  - $(c = 0) \wedge (f = 0) \Rightarrow (\varphi = 0)$
  - $(\varphi = 1) \Rightarrow (c = 1) \vee (f = 1)$

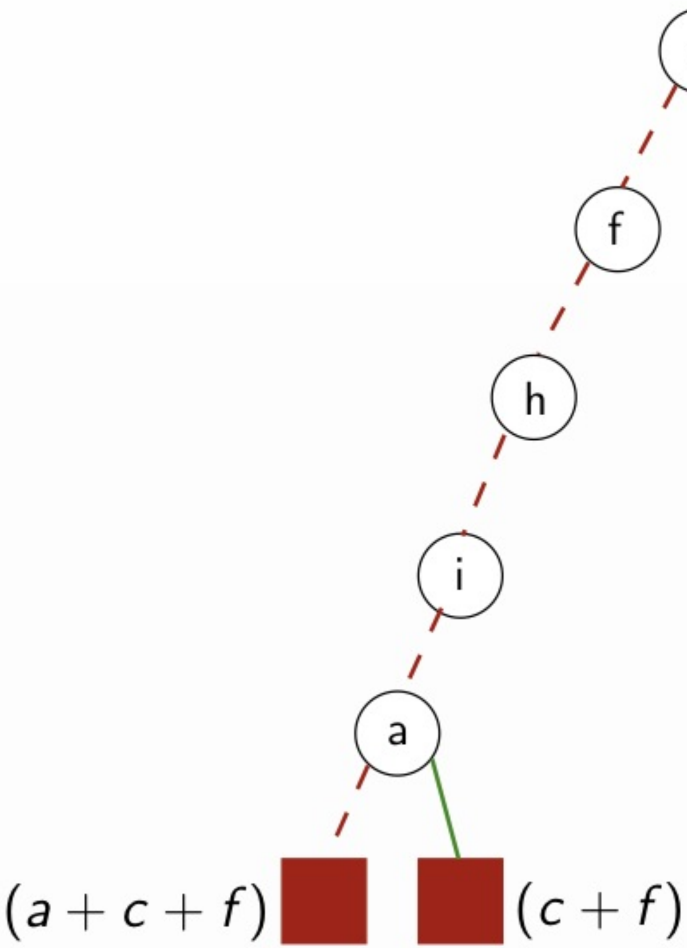
Non-Chronological Backtracking

- During backtrack search, for each conflict backtrack to one of the causes of the conflict

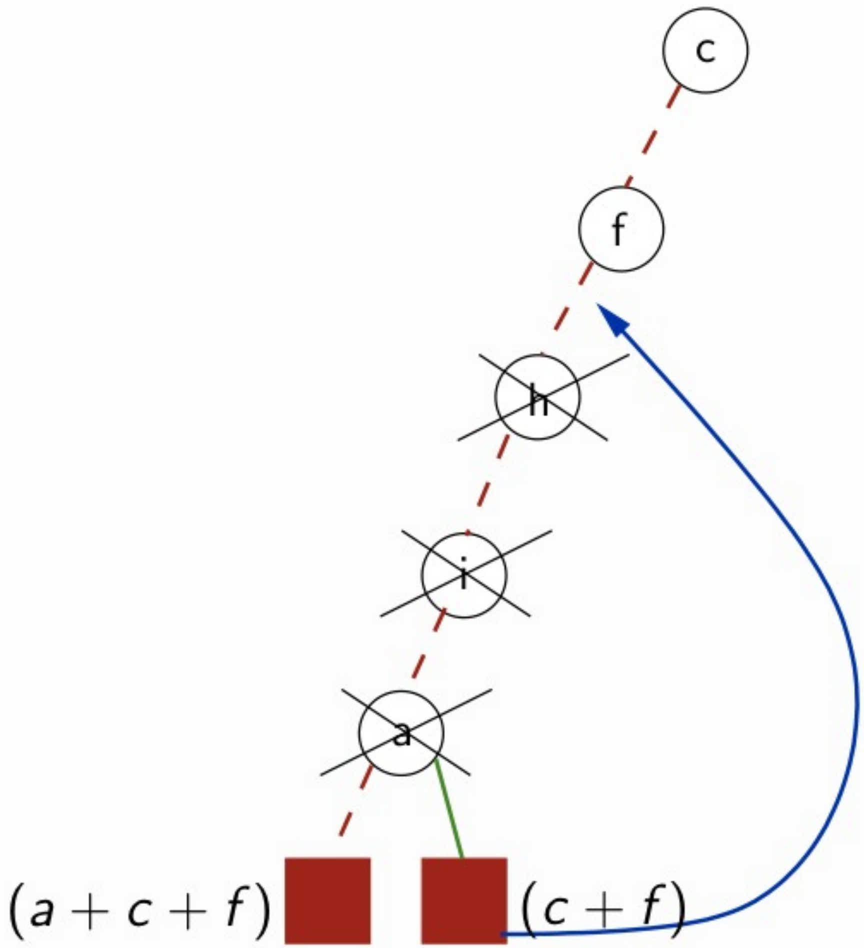
$$\begin{aligned}\varphi \;=\;& (a \vee b) \wedge (\neg b \vee \boxed{c} \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee \boxed{f}) \wedge \\ & (a \vee \boxed{c} \vee \boxed{f}) \wedge (\neg a \vee g) \wedge (\neg g \vee b) \wedge (\neg h \vee j) \wedge (\neg i \vee k)\end{aligned}$$

  - Assume decisions $c = 0$, $f = 0$, $h = 0$ and $i = 0$
  - Assignment $a = 0$ caused conflict $\Rightarrow$ learnt clause $(a \vee c \vee f)$ implies $a = 1$
  - A conflict is again reached: $(\neg d \vee \neg e \vee f)$ is unsatisfied
  - $(c = 0) \wedge (f = 0) \Rightarrow (\varphi = 0)$
  - $(\varphi = 1) \Rightarrow (c = 1) \vee (f = 1)$
  - Learn new clause $(c \vee f)$

Non-Chronological Backtracking



Non-Chronological Backtracking



- Learnt clause: $(c \vee f)$
- Need to backtrack, given new clause
- Backtrack to most recent decision: $f = 0$

- Clause learning and non-chronological backtracking are hallmarks of modern SAT solvers

Clause deletion policies

- Keep only the small clauses [Marques-Silva&Sakallah'96]
  - For each conflict record one clause
  - Keep clauses of size $\leq K$
  - Large clauses get deleted when become unresolved
- Keep only the relevant clauses [Bayardo&Schrag'97]
  - Delete unresolved clauses with $\leq M$ free literals
- Keep only the clauses that are used [Goldberg&Novikov'02]
  - Keep track of clauses activity

Outline

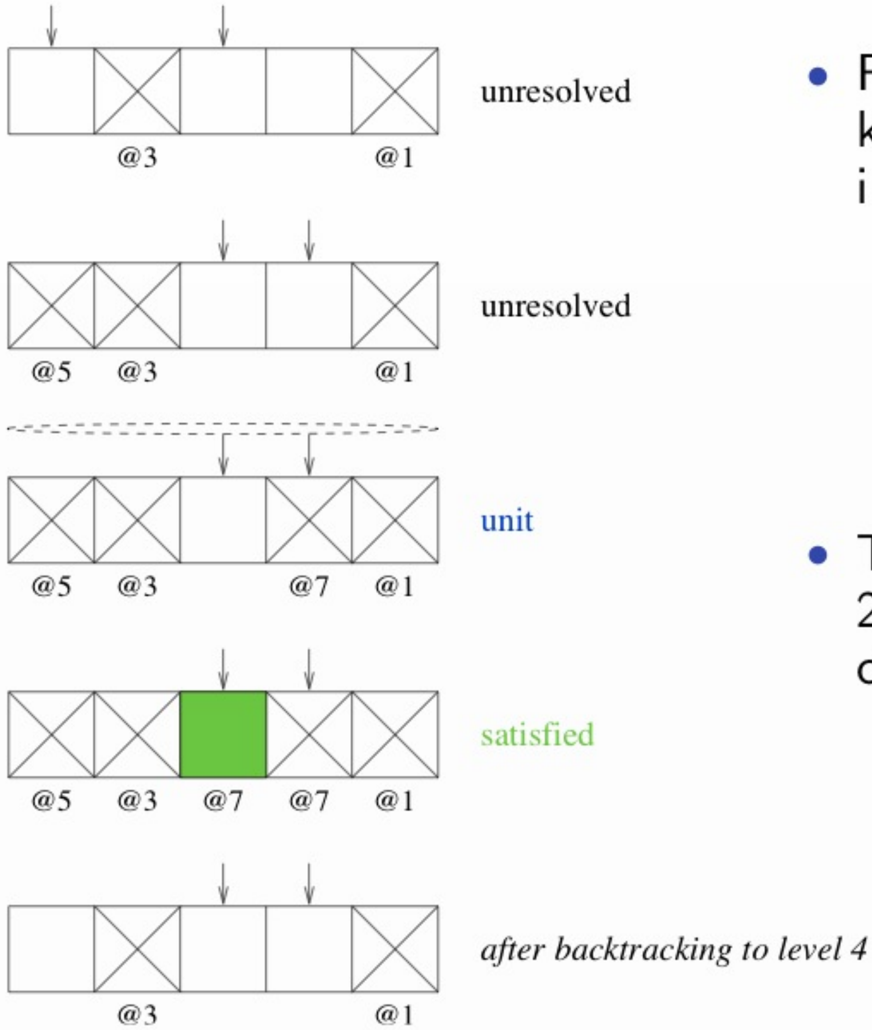
- Review: DPLL, Resolution
- Conflict-Directed Clause Learning (CDCL)
  – Implication graphs,
  – asserting clauses,
  – Unique Implication points (UIPs)
- Watch literals

Data Structures

- Key point: only unit and unsatisfied clauses *must* be detected during search
  - Formula is unsatisfied when at least one clause is unsatisfied
  - Formula is satisfied when all the variables are assigned and there are no unsatisfied clauses
- In practice: unit and unsatisfied clauses may be identified using only two references

- Standard data structures (adjacency lists):
  - Each variable x keeps a reference to all clauses containing a literal in x
- Lazy data structures (watched literals):
  - For each clause, only two variables keep a reference to the clause, i.e. only 2 literals are watched

Lazy Data Structures (watched literals)



- For each clause, only two variables keep a reference to the clause, i.e. only 2 literals are watched
  - If variable x is assigned, only the clauses where literals in x are watched need to be evaluated
  - If search backtracks, then nothing needs to be done
- Total number of references is $2 \times C$, where C is the number of clauses
  - In general $L \gg 2 \times C$, in particular if clauses are learnt

BCP Algorithm (1.1/8)

- Big Invariants
  - Each clause has two watched literals.
  - If a clause can become unit via any sequence of assignments, then this sequence will include an assignment of one of the watched literals to F.
    - Example again: (v1 + v2 + v3 + v4 + v5)
    - (**v1=X** + **v2=X** + v3=? + v4=? + v5=?)
- BCP consists of identifying unit (and conflict) clauses (and the associated implications) while maintaining the "Big Invariants"

BCP Algorithm (2/8)

- Let’s illustrate this with an example:

```
v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'
v1 + v2'
v1'+ v4
v1'
```

BCP Algorithm (2.1/8)

- Let's illustrate this with an example:

watched literals

```
v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'
v1 + v2'
v1'+ v4
v1'
```

One literal clause breaks invariants: handled as a special case (ignored hereafter)

- Initially, we identify any two literals in each clause as the watched ones
- Clauses of size one are a special case

BCP Algorithm (3/8)

- We begin by processing the assignment v1 = F (which is implied by the size one clause)

```
State:(v1=F)

Pending:
```

```
v2 + v3 + v1 + v4 + v5

v1 + v2 + v3'

v1 + v2'

v1'+ v4
```

BCP Algorithm (3.1/8)

- We begin by processing the assignment v1 = F (which is implied by the size one clause)

```
State:(v1=F)          v2 + v3 + v1 + v4 + v5
Pending:          =>  v1 + v2 + v3'
                  =>  v1 + v2'
                      v1'+ v4
```

- To maintain our invariants, we must examine each clause where the assignment being processed has set a watched literal to F.

BCP Algorithm (3.2/8)

- We begin by processing the assignment v1 = F (which is implied by the size one clause)

```
State:(v1=F)

Pending:
```

```
  v2 + v3 + v1 + v4 + v5

  v1 + v2 + v3'

  v1 + v2'

⇨ v1'+ v4
```

- To maintain our invariants, we must examine each clause where the assignment being processed has set a watched literal to F.
- We need not process clauses where a watched literal has been set to T, because the clause is now satisfied and so can not become unit.

BCP Algorithm (3.3/8)

- We begin by processing the assignment v1 = F (which is implied by the size one clause)

```
State:(v1=F)

Pending:
```

```
⇨ v2 + v3 + v1 + v4 + v5
  v1 + v2 + v3'
  v1 + v2'
  v1'+ v4
```

- To maintain our invariants, we must examine each clause where the assignment being processed has set a watched literal to F.
- We need not process clauses where a watched literal has been set to T, because the clause is now satisfied and so can not become unit.
- We *certainly* need not process any clauses where neither watched literal changes state (in this example, where v1 is not watched).

BCP Algorithm (4/8)

- Now let's actually process the second and third clauses:

```
v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'
v1 + v2'
v1'+ v4
```

```
State:(v1=F)

Pending:
```

BCP Algorithm (4.1/8)

- Now let's actually process the second and third clauses:

```
v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'
v1 + v2'
v1'+ v4
```

```
State: (v1=F)
Pending:
```

```
v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'
v1 + v2'
v1'+ v4
```

```
State: (v1=F)
Pending:
```

- For the second clause, we replace v1 with v3' as a new watched literal. Since v3' is not assigned to F, this maintains our invariants.

BCP Algorithm (4.2/8)

- Now let's actually process the second and third clauses:

`v2 + v3 + v1 + v4 + v5`	`v2 + v3 + v1 + v4 + v5`
`v1 + v2 + v3'`	`v1 + v2 + v3'`
`v1 + v2'`	`v1 + v2'`
`v1' + v4`	`v1' + v4`
State: (v1=F) Pending:	State: (v1=F) Pending: (v2=F)

- For the second clause, we replace v1 with v3' as a new watched literal. Since v3' is not assigned to F, this maintains our invariants.
- The third clause is unit. We record the new implication of v2', and add it to the queue of assignments to process. Since the clause cannot again become unit, our invariants are maintained.

BCP Algorithm (5/8)

- Next, we process v2’. We only examine the first 2 clauses.

```
v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'
v1 + v2'
v1'+ v4
```

```
State:(v1=F, v2=F)
Pending:
```

```
v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'
v1 + v2'
v1'+ v4
```

```
State:(v1=F, v2=F)
Pending:(v3=F)
```

- For the first clause, we replace v2 with v4 as a new watched literal. Since v4 is not assigned to F, this maintains our invariants.
- The second clause is unit. We record the new implication of v3’, and add it to the queue of assignments to process. Since the clause cannot again become unit, our invariants are maintained.

BCP Algorithm (6/8)

- Next, we process v3'. We only examine the first clause.

```
v2 + v3 + v1 + v4 + v5          v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'                   v1 + v2 + v3'
v1 + v2'                        v1 + v2'
v1'+ v4                         v1'+ v4
```

```
State:(v1=F, v2=F, v3=F)        State:(v1=F, v2=F, v3=F)
Pending:                        Pending:
```

- For the first clause, we replace v3 with v5 as a new watched literal. Since v5 is not assigned to F, this maintains our invariants.
- Since there are no pending assignments, and no conflict, BCP terminates and we make a decision. Both v4 and v5 are unassigned. Let's say we decide to assign v4=T and proceed.

BCP Algorithm (7/8)

- Next, we process v4. We do nothing at all.

```
v2 + v3 + v1 + v4 + v5          v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'           =>      v1 + v2 + v3'
v1 + v2'                        v1 + v2'
v1'+ v4                         v1'+ v4

State:(v1=F, v2=F, v3=F,        State:(v1=F, v2=F, v3=F,
v4=T)                           v4=T)
```

- Since there are no pending assignments, and no conflict, BCP terminates and we make a decision. Only v5 is unassigned. Let's say we decide to assign v5=F and proceed.

BCP Algorithm (8/8)

- Next, we process v5=F. We examine the first clause.

v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'
v1 + v2'
v1' + v4

State: (v1=F, v2=F, v3=F, v4=T, v5=F)

v2 + v3 + v1 + v4 + v5
v1 + v2 + v3'
v1 + v2'
v1' + v4

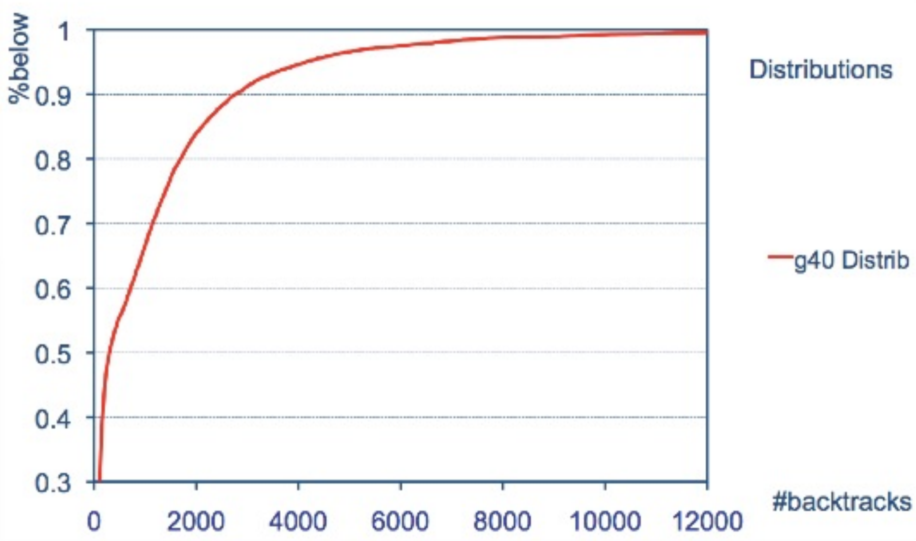
State: (v1=F, v2=F, v3=F, v4=T, v5=F)

- The first clause is already satisfied by v4 so we ignore it.
- Since there are no pending assignments, and no conflict, BCP terminates and we make a decision. No variables are unassigned, so the instance is SAT, and we are done.

Outline

- Review: DPLL, Resolution
- Conflict-Directed Clause Learning (CDCL)
  - Implication graphs,
  - asserting clauses,
  - Unique Implication points (UIPs)
- Watch literals
- Restarts, Empirical evaluation

Restarts I

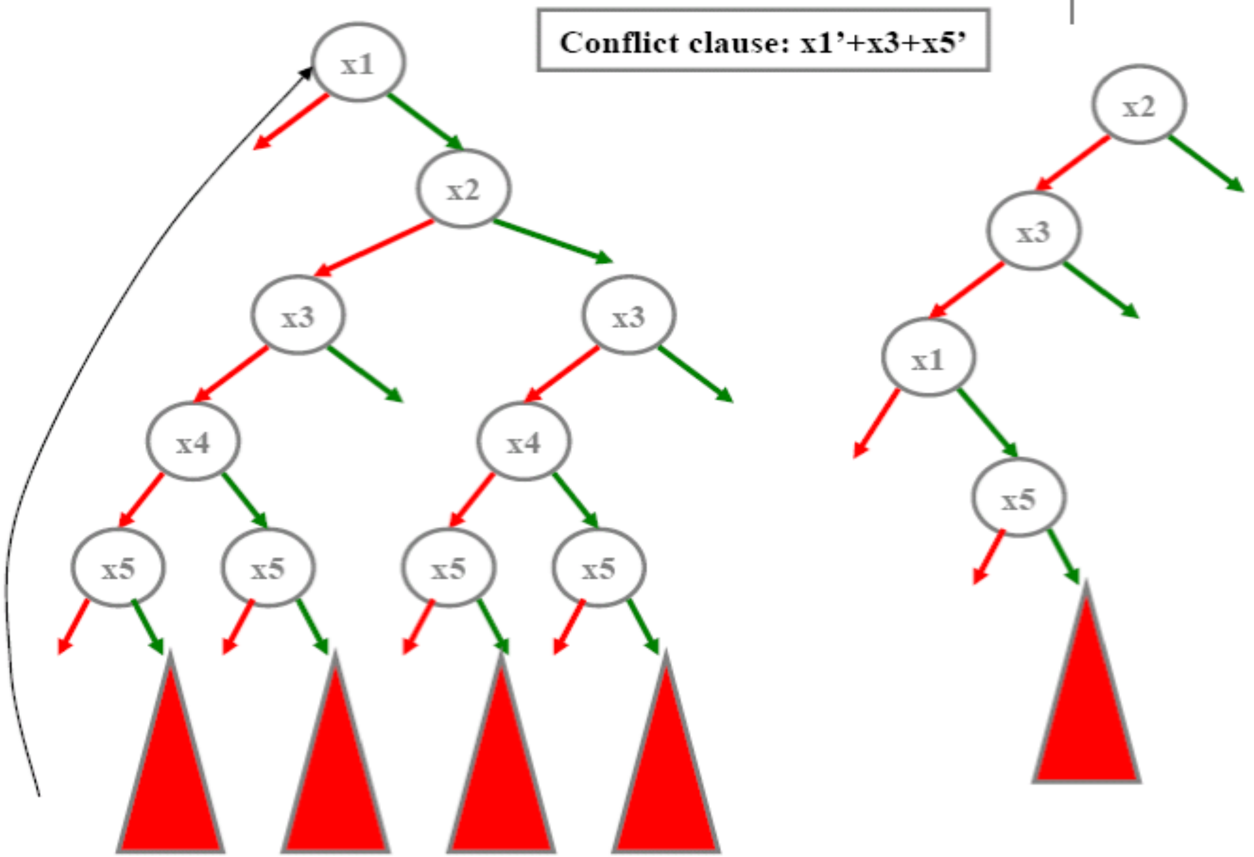


- Plot for processor verification instance with branching randomization and 10000 runs
  - More than 50% of the runs require less than 1000 backtracks
  - A small percentage requires more than 10000 backtracks
- Run times of backtrack search SAT solvers characterized by heavy-tail distributions

[Gomes et al.'98]

Restart

- Abandon the current search tree and reconstruct a new one
- Helps reduce variance - adds to robustness in the solver
- The clauses learned prior to the restart are *still there* after the restart and can help pruning the search space



Evolution of SAT Solvers

Instance	Posit'94	Grasp'96	Chaff'03	Minisat'03	Picosat'08
ssa2670-136	13.57	0.22	0.02	0.00	0.01
bf1355-638	310.93	0.02	0.02	0.00	0.03
design_3	> 1800	3.93	0.18	0.17	0.93
design_1	> 1800	34.55	0.35	0.11	0.68
4pipe_4_ooo	> 1800	> 1800	17.47	110.97	44.95
fifo8_300	> 1800	> 1800	348.50	53.66	39.31
w08_15	> 1800	> 1800	> 1800	99.10	71.89
9pipe_9_ooo	> 1800	> 1800	> 1800	> 1800	> 1800
c6288	> 1800	> 1800	> 1800	> 1800	> 1800

- Modern SAT algorithms can solve instances with hundreds of thousands of variables and tens of millions of clauses

Benchmarks

- Random
- Crafted
- Industrial

Outline

- Review: DPLL, Resolution
- Conflict-Directed Clause Learning (CDCL)
  - Implication graphs,
  - asserting clauses,
  - Unique Implication points (UIPs)
- Watch literals
- Restarts, Empirical evaluation