CompSci 275, CONSTRAINT Networks

Rina Dechter, Fall 2022

Introduction, the constraint network model
Chapters 1-2

Class information

- Instructor: Rina Dechter
- Lectures: Monday & Wednesday
- Time: 11:00 – 12:20 pm

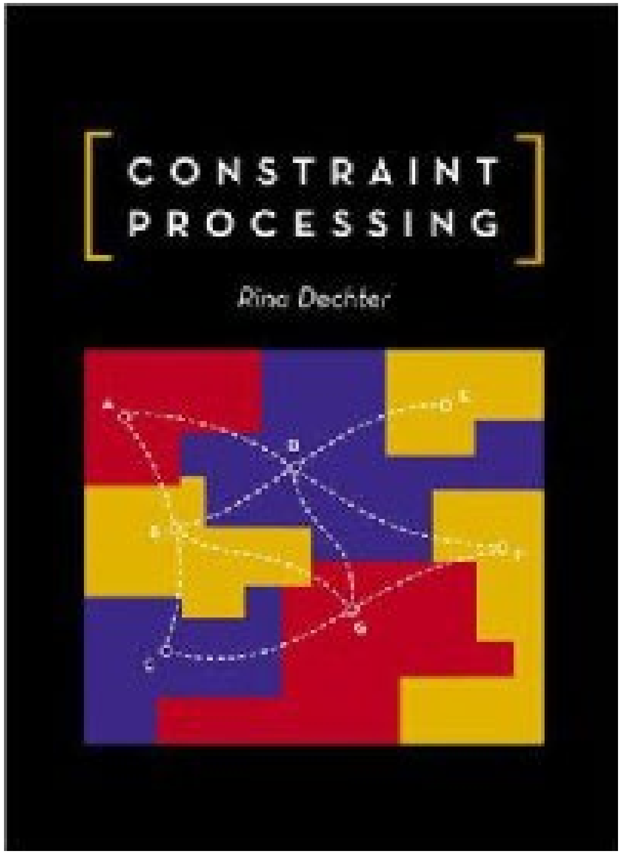
- Class page: https://www.ics.uci.edu/~dechter/courses/ics-275/fall-2022/

Text book (required)

Rina Dechter,

[Constraint Processing](#),

Morgan Kaufmann



Outline

- ✓ Motivation, applications, history
- ✓ CSP: Definition, and simple modeling examples
- ✓ Mathematical concepts (relations, graphs)
- ✓ Representing constraints
- ✓ Constraint graphs
- ✓ The binary Constraint Networks properties

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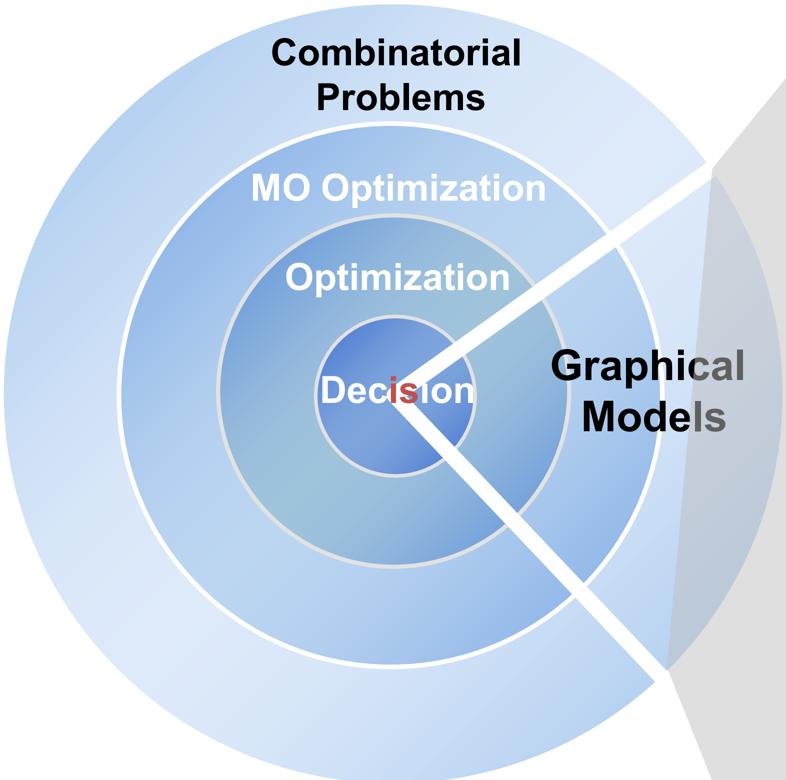
Combinatorial problems



Graphical Models

Those problems that can be expressed as:

A set of variables

Each variable takes its values from a finite set of domain values

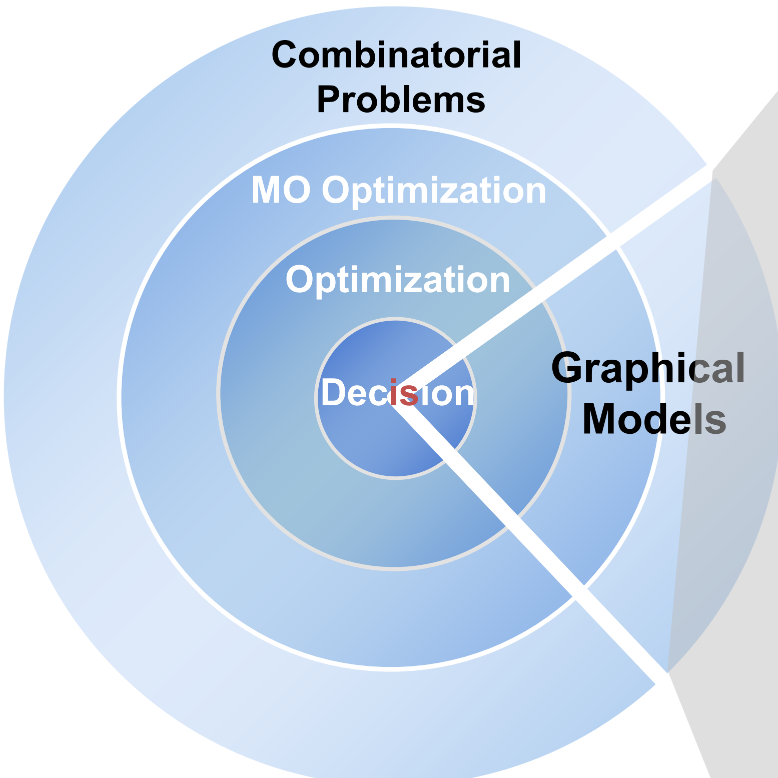
A set of local functions

Main advantage:

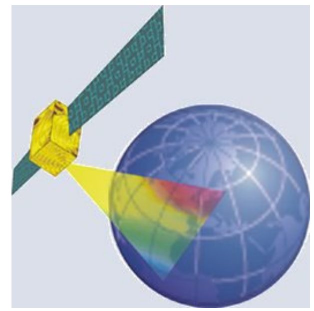
They provide unifying algorithms:

- Search
- Complete Inference
- Incomplete Inference

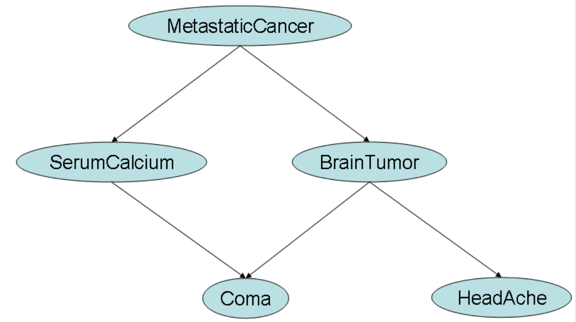
Combinatorial problems



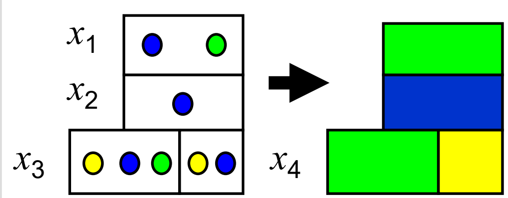
Many Examples



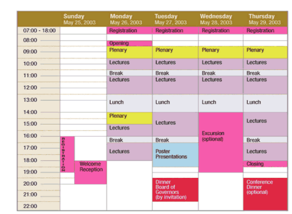
EOS Scheduling



Bayesian Networks



Graph Coloring



Timetabling

… and many others.

Example: student course selection

- **Context**: You are a senior in college
- **Problem**: You need to register in 4 courses for the Spring semester
- **Possibilities**: Many courses offered in Math, CSE, EE, CBA, etc.
- **Constraints**: restrict the choices you can make
  - Courses have prerequisites (e.g., take 171 before 175)
  - General course restrictions https://www.ics.uci.edu/grad/Course_updates.php
  - you have/don't have Courses/instructors you like/dislike
  - Courses are scheduled at the same time
  - In CE: 4 courses from 5 tracks such that at least 3 tracks are covered
- **You have choices, but are restricted by constraints**
  - Make the right decisions!!
  - ICS Graduate program

Student course selection (continued)

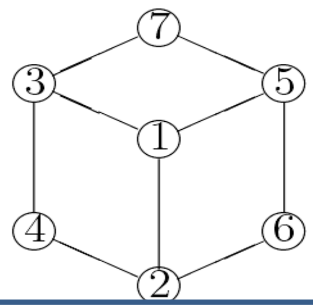
- **Given**
  - A set of variables: 4 courses at your college
  - For each variable, a set of choices (values): the available classes.
  - A set of constraints that restrict the combinations of values the variables can take at the same time
- **Questions**
  - Does a solution exist? (classical decision problem)
  - How many solutions exists? (counting)
  - How two or more solutions differ?
  - Which solution is preferable?
  - etc.

The field of constraint programming

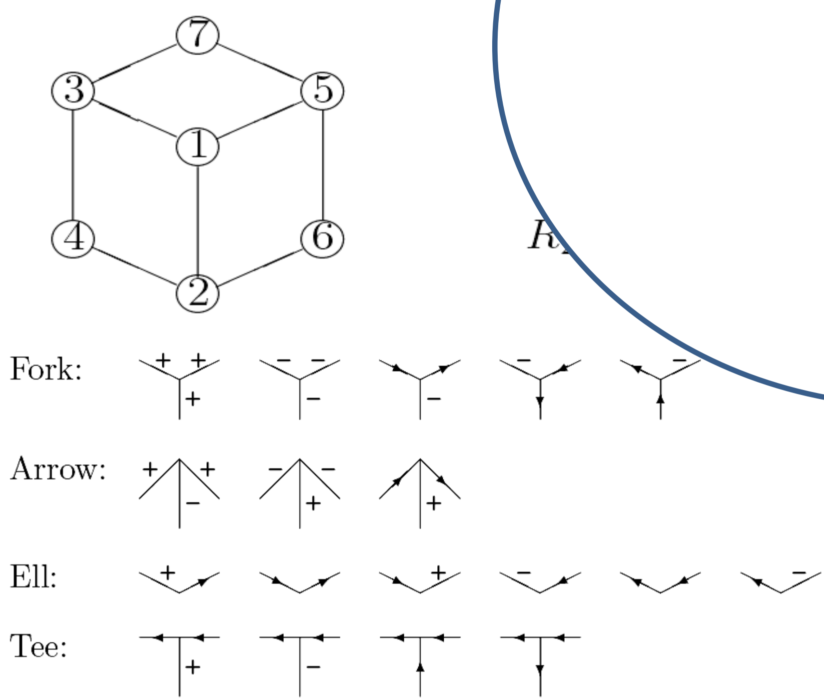
- **How did it start:**
  - Artificial Intelligence (vision)
  - Programming Languages (Logic Programming),
  - Databases (deductive, relational)
  - Logic-based languages (propositional logic)
  - SATisfiability
- **Related areas:**
  - Hardware and software verification
  - Operation Research (Integer Programming)
  - Answer set programming
- **Graphical Models; deterministic**

Scene labeling constraint network

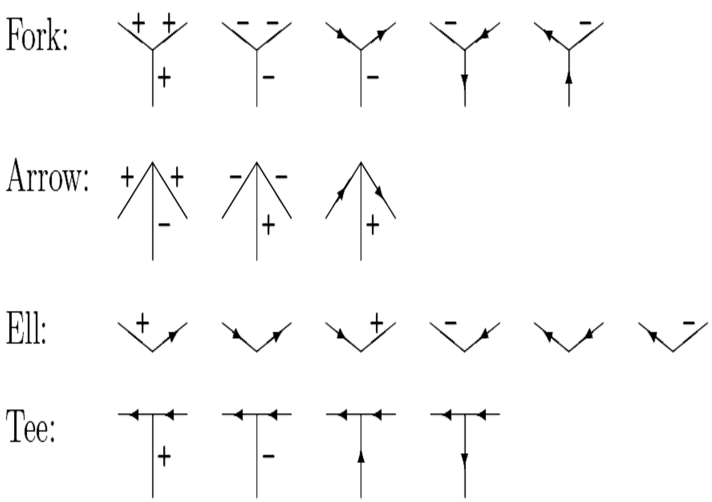
One of the earliest constraint satisfaction is three-dimensional interpretation of a two dimensional drawing (1970) . Huffman and Clowes [291] developed a labelling scheme of the arcs in a **block world picture graph**, where + for a convex, - for concave and *arrow* for occluding boundaries. Label the junction of a given drawing such that every junction type is labelled according to one of its legal labeling and that edges common to two junctions receive the same label.

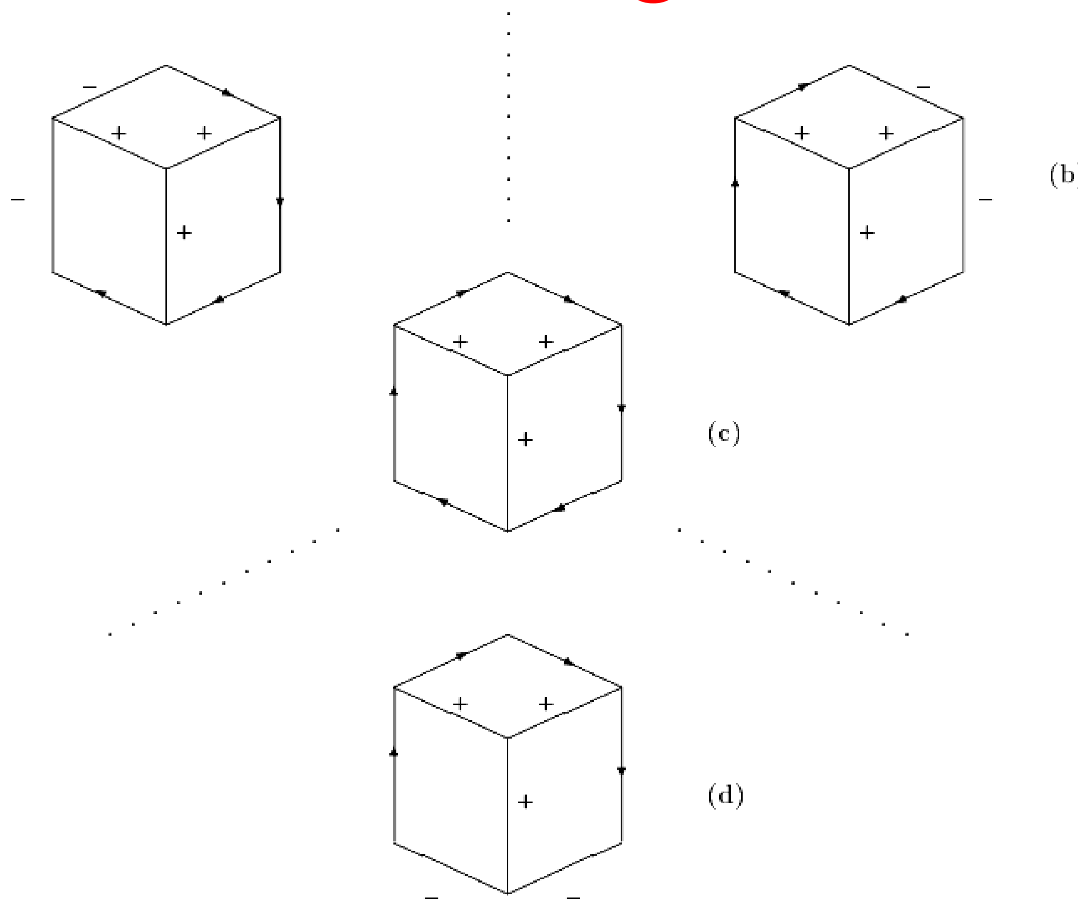


Scene labeling constraint network



3-dimentional interpretation of 2-dimentional drawings





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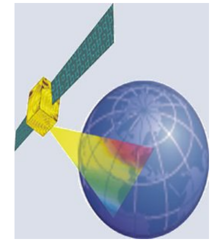
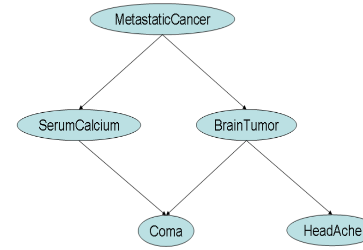
Applications

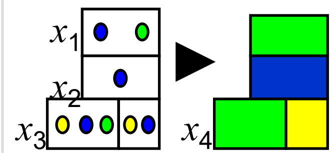
- Radio resource management (RRM)
- Databases (computing joins, view updates)
- Temporal and spatial reasoning
- Planning, scheduling, resource allocation
- Design and configuration
- Graphics, visualization, interfaces
- Hardware verification and software engineering
- HC Interaction and decision support
- Molecular biology
- Robotics, machine vision and computational linguistics
- Transportation
- Qualitative and diagnostic reasoning



EOS Scheduling



Bayesian Networks



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Constraint networks

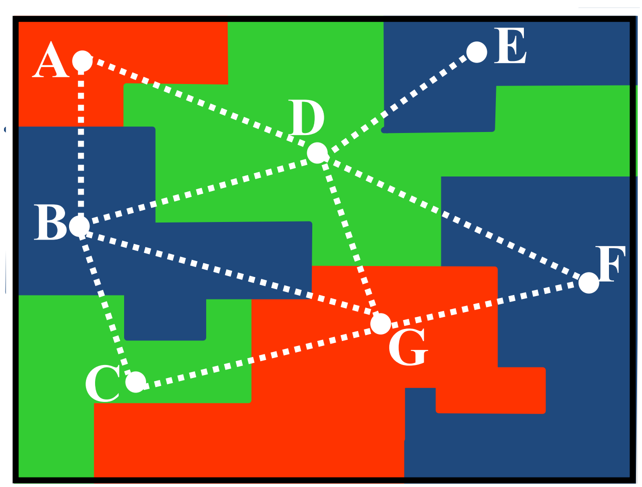
Example: map coloring

Variables - countries (A,B,C,etc.)

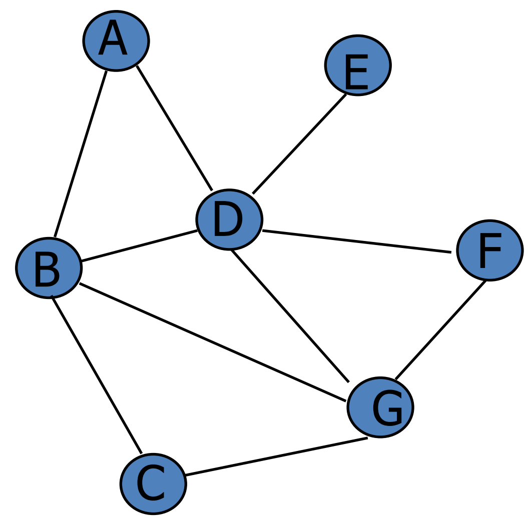
Values - colors (red, green, blue)

Constraints: $A \neq B$, $A \neq D$, $D \neq E$, *etc.*

A	B
red	green
red	yellow
green	red
green	yellow
yellow	green
yellow	red



Constraint graph



Constraint satisfaction tasks

Example: map coloring

Variables - countries (A,B,C,etc.)

Values - colors (e.g., red, green, yellow)

Constraints:

$\mathbf{A} \neq \mathbf{B},\ \mathbf{A} \neq \mathbf{D},\ \mathbf{D} \neq \mathbf{E},\ \textit{etc.}$

A	B	C	D	E…
red	green	red	green	blue
red	blue	green	green	blue
…	…	…	…	green
…	…	…	…	red
red	blue	red	green	red

Are the constraints consistent?

Find a solution, find all solutions

Count all solutions

Find a good solution

Information as constraints

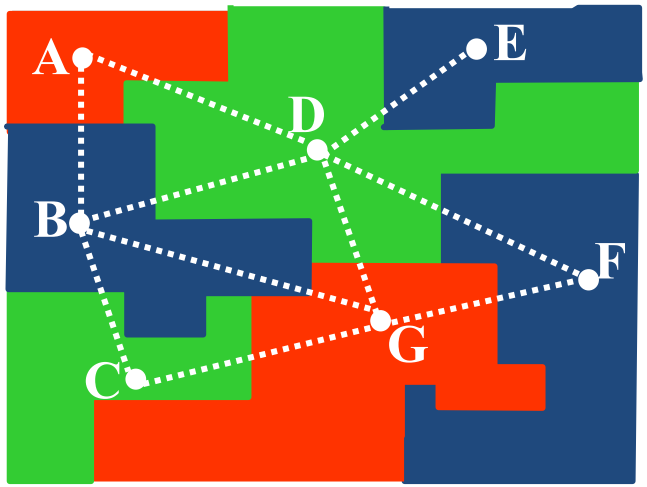
- I have to finish my class in 50 minutes
- 180 degrees in a triangle
- Memory in our computer is limited
- The four nucleotides that makes up a DNA only combine in a particular sequence
- Sentences in English must obey the rules of syntax
- Susan cannot be married to both John and Bill
- Alexander the Great died in 333 B.C.

Constraint network; definition

- A constraint network is: **R=*(X,D,C)***
  - **X variables** $X = \{X_1, \ldots, X_n\}$
  - **D domain** $D = \{D_1, \ldots, D_n\}, D_i = \{v_1, \ldots v_k\}$
  - **C constraints** $C = \{C_1, \ldots C_t\},,, C_i = (S_i, R_i)$
  - *R* expresses allowed tuples over scopes
- **A solution** is an assignment to all variables that satisfies all constraints (join of all relations).
- ***Tasks:*** *consistency?, one or all solutions, counting, optimization*

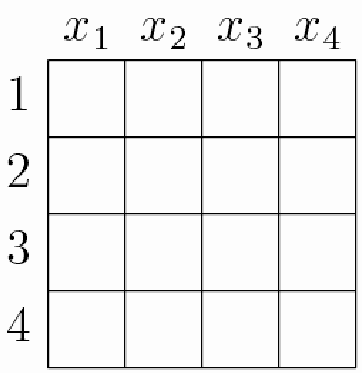
MiniZinc model

```
% Colouring book-map using nc colours
int: nc = 3;


var 1..nc: A; var 1..nc: B; var 1..nc: C; var 1..nc: D;
var 1..nc: E; var 1..nc: F; var 1..nc: G;


constraint A != D;
constraint A != B;
constraint B != D;
constraint B != G;
constraint B != C;
constraint C != G;
constraint D != G;
constraint D != F;
constraint G != F;
solve satisfy;
```



The N-queens problem

The network has four variables, all with domains D_i = {1, 2, 3, 4}.
(a) The labeled chess board. (b) The constraints between variables.



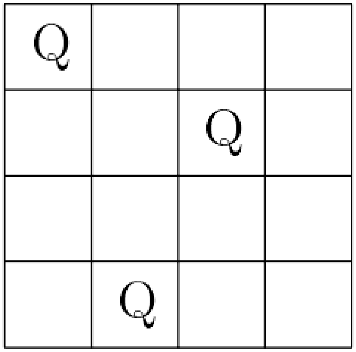
(a)

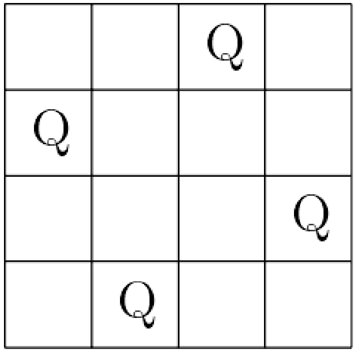
$$
\begin{aligned}
R_{12} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{13} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{14} &= \{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (2,4),\ (3,1),\ (3,2),\ (3,4) \\
&\quad (4,2),\ (4,3)\} \\
R_{23} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{24} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{34} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}
\end{aligned}
$$

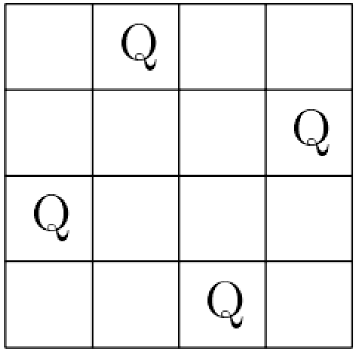
(b)

A solution and a partial consistent tuple (configuration)

Not all consistent instantiations are part of a solution:
(a) A consistent instantiation that is not part of a solution.
(b) The placement of the queens corresponding to the solution (2, 4, 1,3).
(c) The placement of the queens corresponding to the solution (3, 1, 4, 2).

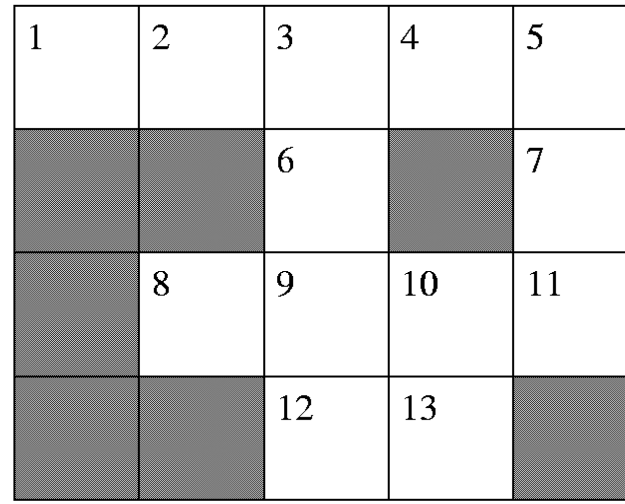

(a)


(b)


(c)

Example: crossword puzzle

- Variables: x_1, …, x_{13}
- Domains: letters
- Constraints: words from



{HOSES, LASER, SHEET, SNAIL, STEER, ALSO, EARN, HIKE, IRON, SAME, EAT, LET, RUN, SUN, TEN, YES, BE, IT, NO, US}

Example: Sudoku (constraint propagation)

Constraint propagation

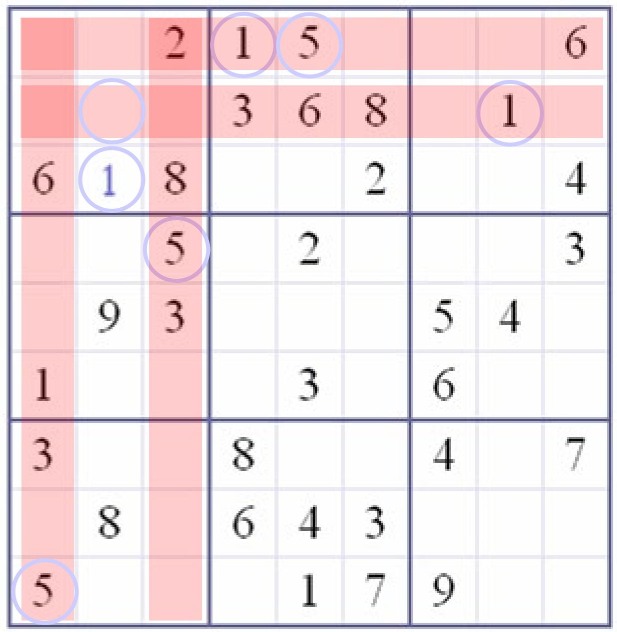
		2	4		6			
8	6	5	1			2		
	1				8	6		9
9				4		8	6	
	4	7				1	9	
	5	8		6				3
4		6	9				7	2 ~~3~~ ~~4~~ ~~6~~ 2
		9			4	5	8	1
			3		2	9		

- **Variables**: 81 slots
- **Domains** = {1,2,3,4,5,6,7,8,9}
- **Constraints**:
  - 27 not-equal

Each row, column and major block must be alldifferent

"Well posed" if it has unique solution: 27 constraints

Sudoku (inference)



Each row, column and major block must be alldifferent

"Well posed" if it has unique solution

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Mathematical background

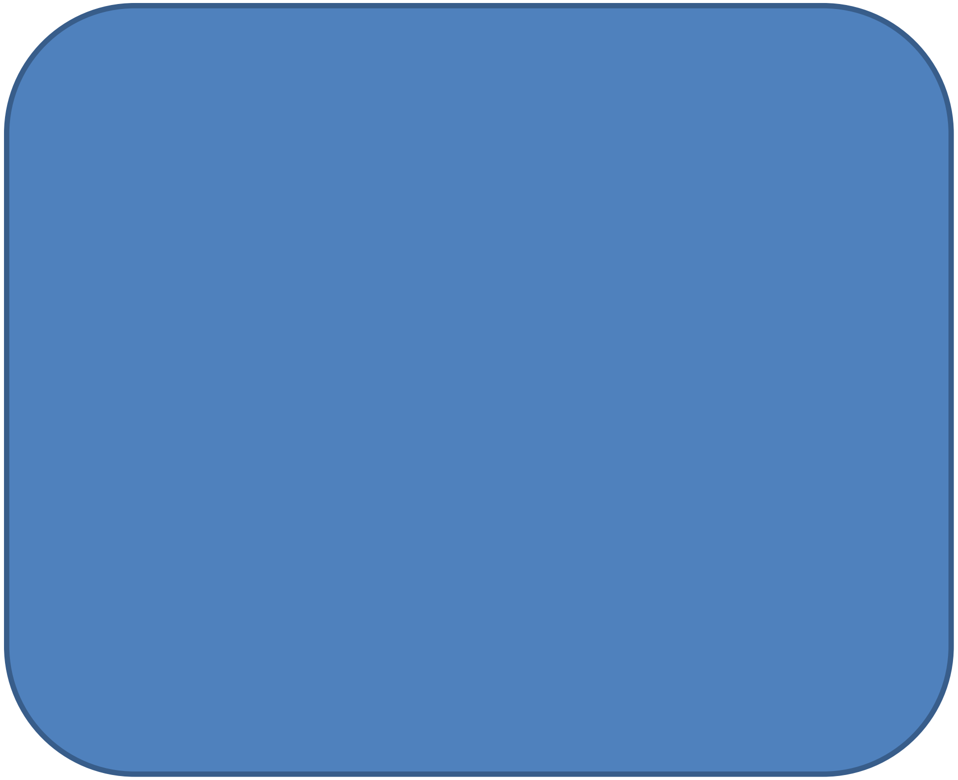
- Sets, domains, tuples
- Relations
- Operations on relations
- Graphs
- Complexity

Two Representations of a relation:
R = {(black, coffee), (black, tea), (green, tea)}.

Variables: Drink, color

x_1	x_2
black	coffee
black	tea
green	tea

(a) table



Two Representations of a relation:
R = {(black, coffee), (black, tea), (green, tea)}.

Variables: Drink, color

x_1	x_2
black	coffee
black	tea
green	tea

(a) table

x_1 (rows), x_2 (columns)

	apple juice	coffee	tea
black	0	1	1
green	0	0	1

(b) (0,1)-matrix

Examples

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s

(a) Relation R

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

(b) Relation R'

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

(c) Relation R''

Operations with relations

- Intersection
- Union
- Difference
- Selection
- Projection
- Join
- Composition

Relations are local functions

- Relations specify Local functions.

- A general function $f : \prod_{x_i \in Y} D_i \rightarrow A$

  where

  scope(f) = $Y \subseteq X$: scope of function f

  A: is a set of valuations

  In a relation A ={false,true}, telling if the tuple is in or out of the relation.

- In constraint networks: functions are boolean

x_1	x_2	f
a	a	true
a	b	false
b	a	false
b	b	true

relation →

x_1	x_2
a	a
b	b

Set operations: intersection, union, difference on relations.

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s

(a) Relation R

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

(b) Relation R'

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

(c) Relation R''

x_1	x_2	x_3
b	b	c
c	b	c

(a) $R \cap R'$

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s
c	n	n

(b) $R \cup R'$

x_1	x_2	x_3
a	b	c
c	b	s

(b) $R - R'$

Selection, projection, join

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s

(a) Relation R

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

(b) Relation R'

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

(c) Relation R''

x_1	x_2	x_3
b	b	c
c	b	c

(a) $\sigma_{x_3=c}(R')$

x_2	x_3
b	c
n	n

(b) $\pi_{\{x_2,x_3\}}(R')$

x_1	x_2	x_3	x_4
b	b	c	2
b	b	c	3
c	b	c	2
c	b	c	3

(c) $R' \bowtie R''$

The join and the logical "and"

- Join :

x_1	x_2
a	a
b	b

$\bowtie$

x_2	x_3
a	a
a	b
b	a

$=$

x_1	x_2	x_3
a	a	a
a	a	b
b	b	a

- Logical AND: $f \wedge g$

x_1	x_2	f
a	a	true
a	b	false
b	a	false
b	b	true

$\wedge$

x_2	x_3	g
a	a	true
a	b	true
b	a	true
b	b	false

$=$

x_1	x_2	x_3	h
a	a	a	true
a	a	b	true
a	b	a	false
a	b	b	false
b	a	a	false
b	a	b	false
b	b	a	true
b	b	b	false

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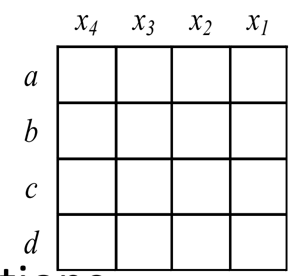
Modeling; Representing a problems

- If a CSP M = <X,D,C> represents a real problem P , then every solution of M corresponds to a solution of P and every solution of P can be derived from at least one solution of M



- The variables and values of M represent entities in P
- The constraints of M ensure the correspondence between solutions
- The aim is to find a model M that can be solved as quickly as possible
- **goal of modelling: choose a set of variables and values that allows the constraints to be expressed easily and concisely**

Example: satisfiability

Given a proposition theory $\varphi = \{(A \vee B), (C \vee \neg B)\}$ does it have a model?

Can it be encoded as a constraint network?

Variables: {A, B, C}

Domains: $D_A = D_B = D_C = \{0, 1\}$

Relations:

A	B
0	1
1	0
1	1

B	C
0	0
0	1
1	1

Constraint's representations

- Relation: allowed tuples

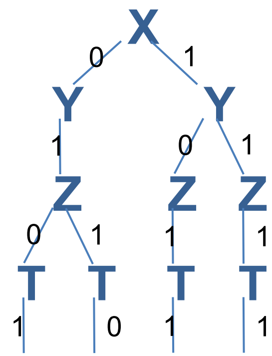
X	Y	Z
1	3	2
2	1	3

- Algebraic expression: $X + Y^2 \leq 10, X \neq Y$
- Propositional formula: $(a \vee b) \rightarrow \neg c$
- A decision tree, a procedure
- Semantics: by a relation

A decision tree or a neural network

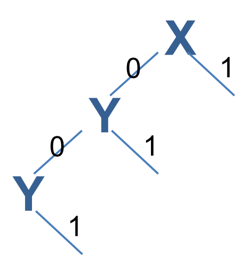
Decision tree representations

X	Y	Z	T
0	1	1	0
1	0	1	1
1	1	1	1
0	1	0	1



X v Y v Z

X	Y	Z
1	0	0
1	1	0
1	0	1
1	1	1
0	1	0
0	1	1
0	0	1



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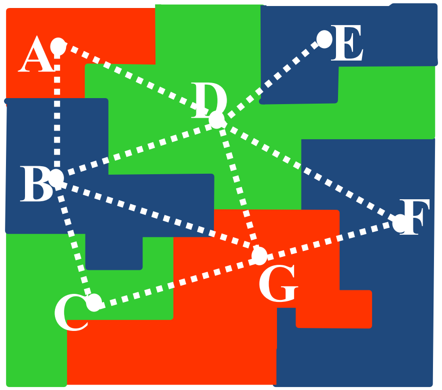
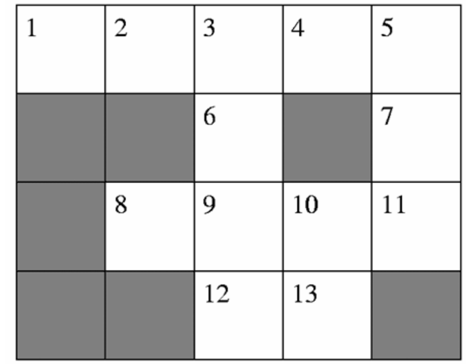
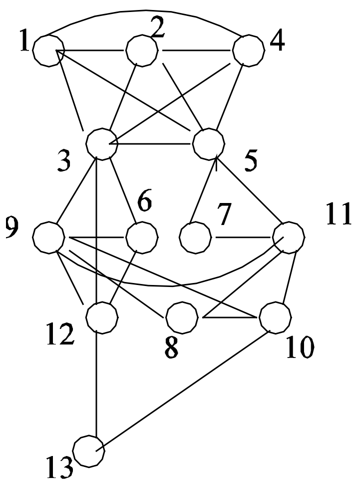


Constraint graphs:

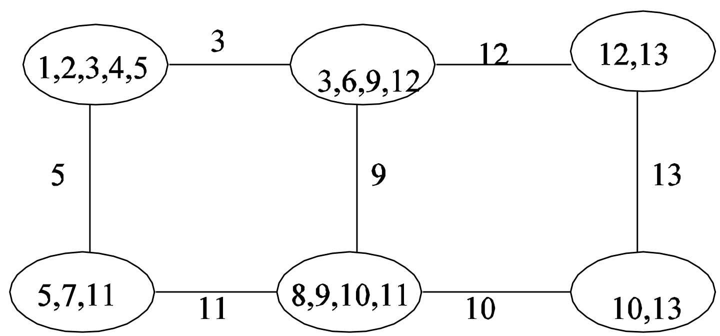
Primal, Dual and Hypergraphs



- A (primal) constraint graph: a node per variable, arcs connect constrained variables.
- A dual constraint graph: a node per constraint's scope, an arc connect nodes sharing variables =hypergraph



(a)



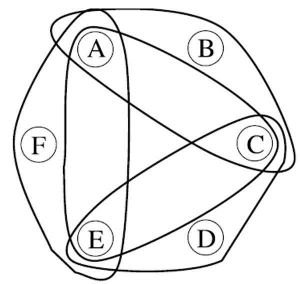
(b)

R{1,2,3,4,5}= {(H,O,S,E,S), (L,A,S,E,R), (S,H,E,E,T), (S,N,A,I,L), (S,T,E,E,R)}
R{3,6,9,12} = {(A,L,S,O), (E,A,R,N),(H,I,K,E), (I,R,O,N), (S,A,M,E)}
R{5,7,11} = {(E,A,T), (L,E,T), (R,U,N), (S,U,N), (T,E,N), (Y,E,S)}

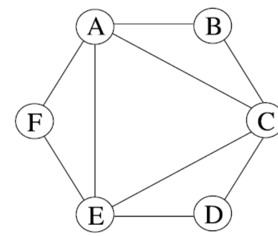
Graph concepts Reviews:

Hyper Graphs and Dual Graphs

A hypergraph

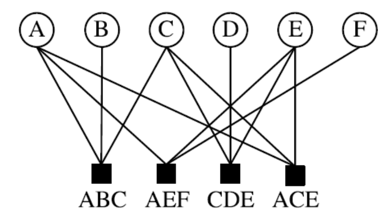


(a)



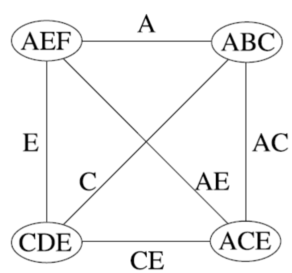
(b)

Primal graphs



(e)

Dual graph



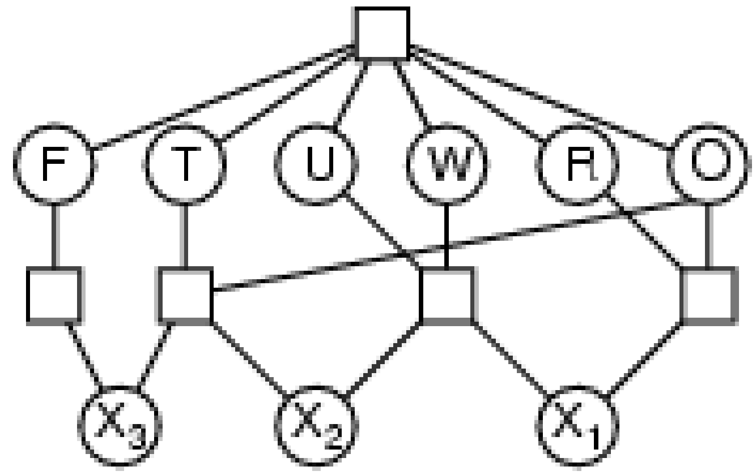
(c)

Factor graphs

Definition 2.1.1 (hypergraph) *A hypergraph is a structure H = (V, S) that consists of vertices V = {v_1,…, v_n} and a set of subsets of these vertices S = {S_1,…;S_l}*

Example: cryptarithmetic

```
   T W O
 + T W O
 -------
 F O U R
```



Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

Domains: $\{0,1,2,3,4,5,6,7,8,9\}$

Constraints: *Alldiff (F,T,U,W,R,O)*

$O + O = R + 10 \cdot X_1$

$X_1 + W + W = U + 10 \cdot X_2$

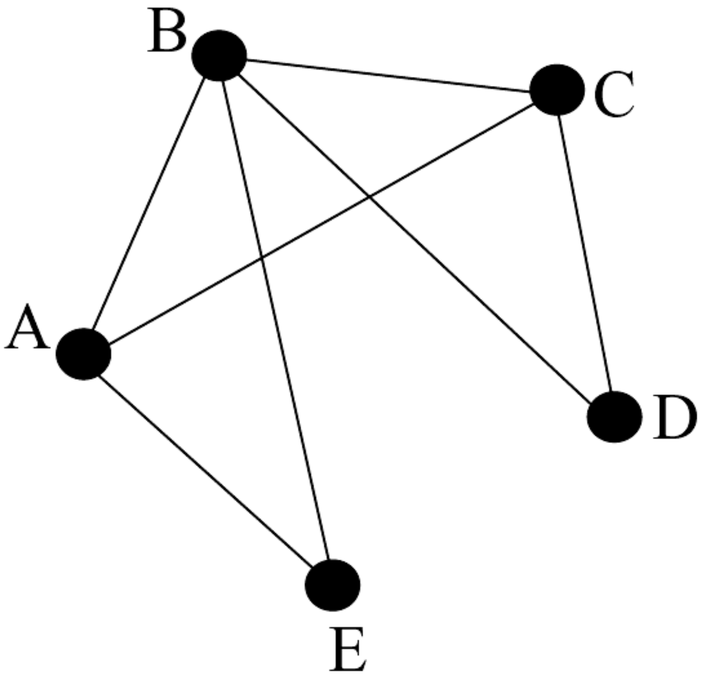
$X_2 + T + T = O + 10 \cdot X_3$

$X_3 = F,\ T \neq 0,\ F \neq 0$

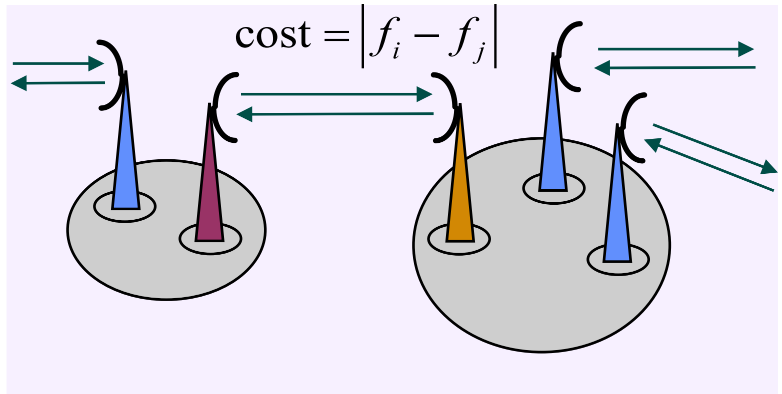
What is the primal graph?
What is the dual graph?

Propositional satisfiability

φ = {(¬*C*), (*A* v *B* v *C*), (¬*A* v *B* v *E*), (¬*B* v *C* v *D*)}.



Example: radio link assignment



Given a telecommunication network (where each communication link has various antenas) , assign a frequency to each antenna in such a way that all antennas may operate together without noticeable interference.

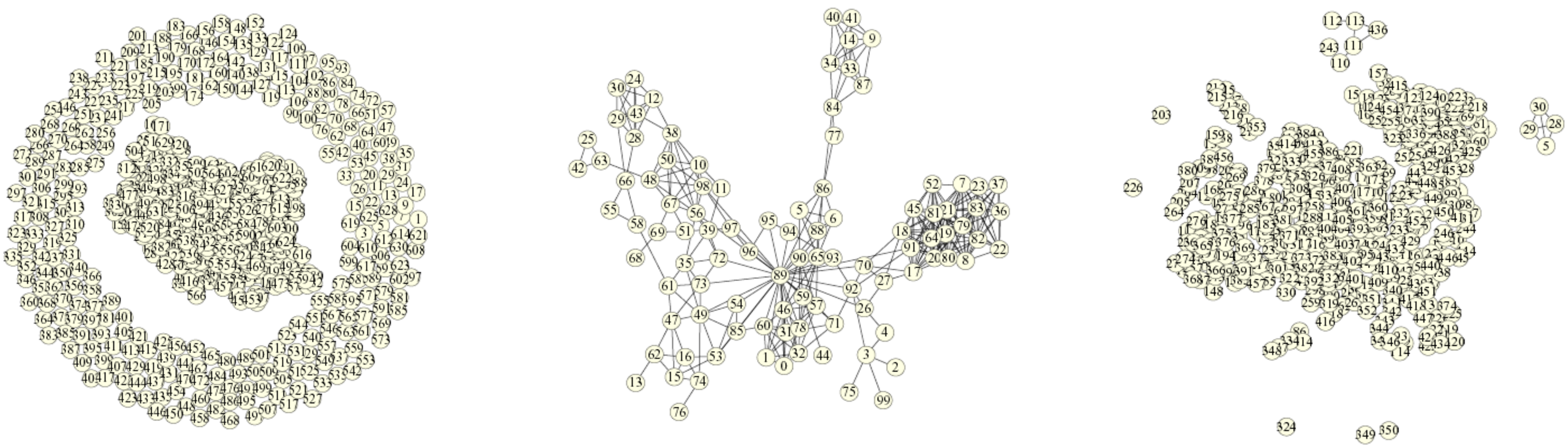
Encoding?

Variables: one for each antenna

Domains: the set of available frequencies

Constraints: the ones referring to the antennas in the same communication link

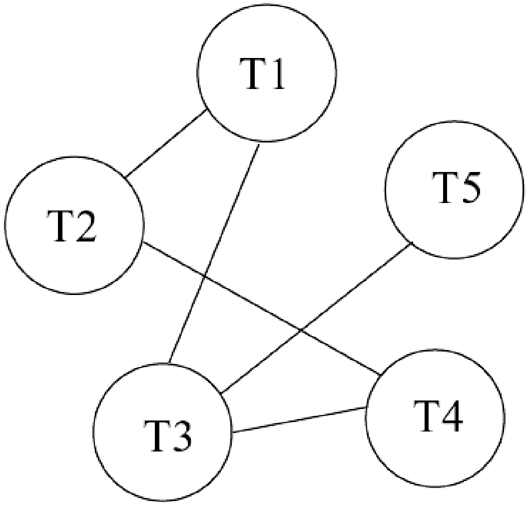
Constraint graphs, 3 instances of radio frequency assignment in CELAR’s benchmark



Example: a scheduling problem

Five tasks: T1, T2, T3, T4, T5
Each one takes one hour to complete
The tasks may start at 1:00, 2:00 or 3:00
Requirements:

- T1 must start after T3
- T3 must start before T4 and after T5
- T2 cannot execute at the same time as T1 or T4
- T4 cannot start at 2:00

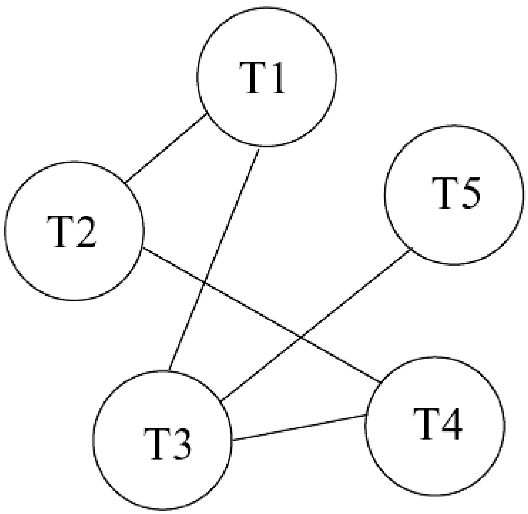


Variables: one for each task

Domains: $D_{T1} = D_{T2} = D_{T3} = D_{T3} = \{1{:}00, 2{:}00, 3{:}00\}$

T4
1:00
3:00

The constraint graph and relations of scheduling problem



Unary constraint

$D_{T4} = \{1{:}00,\ 3{:}00\}$

Binary constraints

$R_{\{T1,T2\}}$: $\{(1{:}00,2{:}00), (1{:}00,3{:}00), (2{:}00,1{:}00), (2{:}00,3{:}00), (3{:}00,1{:}00), (3{:}00,2{:}00)\}$

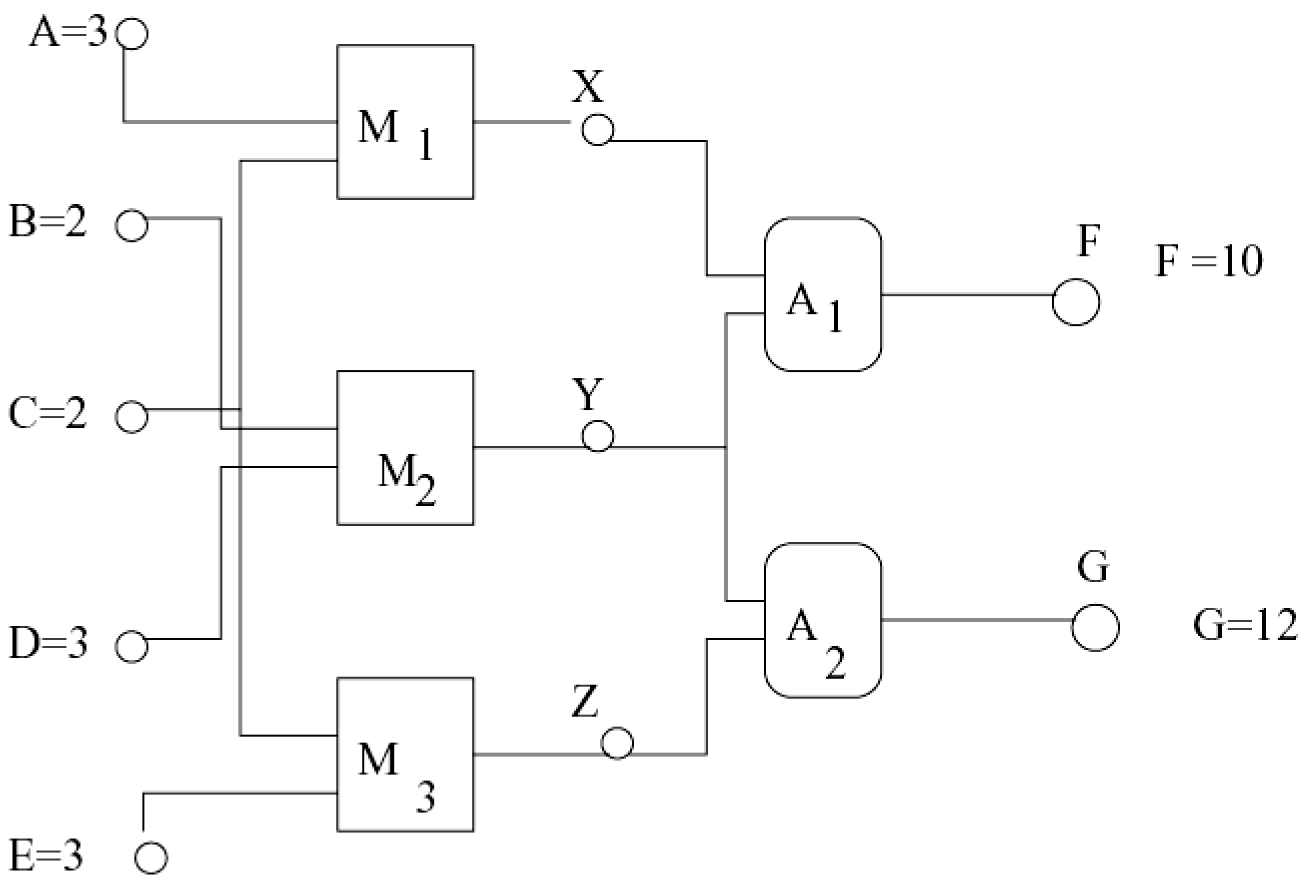
$R_{\{T1,T3\}}$: $\{(2{:}00,1{:}00), (3{:}00,1{:}00), (3{:}00,2{:}00)\}$

$R_{\{T2,T4\}}$: $\{(1{:}00,2{:}00), (1{:}00,3{:}00), (2{:}00,1{:}00), (2{:}00,3{:}00), (3{:}00,1{:}00), (3{:}00,2{:}00)\}$

$R_{\{T3,T4\}}$: $\{(1{:}00,2{:}00), (1{:}00,3{:}00), (2{:}00,3{:}00)\}$

$R_{\{T3,T5\}}$: $\{(2{:}00,1{:}00), (3{:}00,1{:}00), (3{:}00,2{:}00)\}$

A combinatorial circuit
M a multiplier, *A* is an adder



Outline

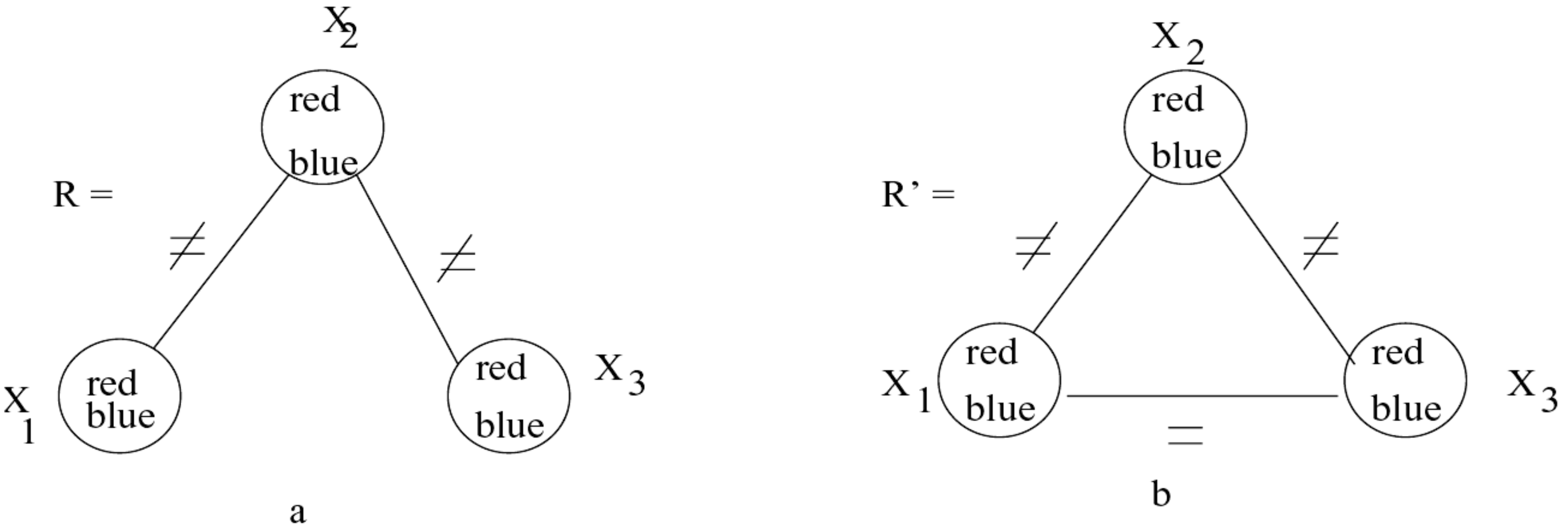
- ✓ Motivation, applications, history
- ✓ CSP: Definition, and simple modeling examples
- ✓ Mathematical concepts (relations, graphs)
- ✓ Representing constraints
- ✓ Constraint graphs
- ✓ The binary Constraint Networks properties

Outline

- ✓ Motivation, applications, history
- ✓ CSP: Definition, representation and simple modeling examples
- ✓ Mathematical concepts (relations, graphs)
- ✓ Representing constraints
- ✓ Constraint graphs
- ✓ **The binary Constraint Networks properties**

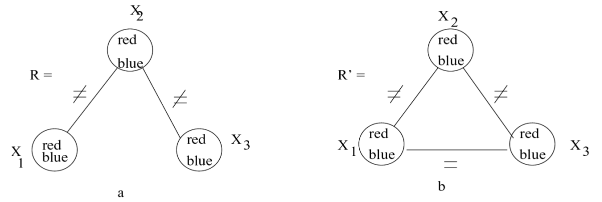
Binary constraint networks

A graph $\Re$ to be colored by two colors,
an equivalent representation $\Re'$ having a newly inferred constraint between *x*1 and *x*3.



Equivalence and deduction with constraints (composition)

Composition of relations (*Montanari'74*)



Definition 2.3.2 (composition) *Given two binary or unary constraints R_{xy} and R_{yz}, the composition $R_{xy} \cdot R_{yz}$ generates the binary relation R_{xz} defined by:*

$$R_{xz} = \{(a,b) | a \in D_x,\ b \in D_z\ , \exists c \in D_y\ s.t.\ (a,c) \in R_{xy}\ and\ (c,b) \in R_{yz}\}$$

An operational definition of composition by Join-project

$$R_{xz} = R_{xy} \cdot R_{yz} = \pi_{\{x,z\}}(R_{xy} \bowtie R_{yz}).$$

In Figure 2.12a, we deduced that $R'_{13} = \pi_{\{x_1,x_3\}}(R_{12} \bowtie R_{23}) = \{(red, red), (blue, blue)\}$,

Composition of relations (*Montanari'74*)

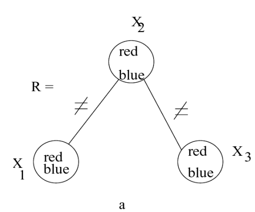
Bit-matrix operation: matrix multiplication

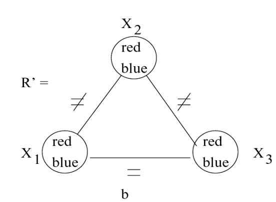
Example 2.3.3 Continuing with our simple graph-coloring example, the two inequality constraints can be expressed as 2×2 matrices having zeros along the main diagonal.

$$R_{12} = \left(\begin{array}{c|cc} & red & blue \\ \hline red & 0 & 1 \\ blue & 1 & 0 \end{array}\right) \qquad R_{23} = \left(\begin{array}{c|cc} & red & blue \\ \hline red & 0 & 1 \\ blue & 1 & 0 \end{array}\right)$$

Multiplying two such matrices yields the following two-dimensional identity matrix:

$$R_{12} \cdot R_{23} = R_{13} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{c|cc} & red & blue \\ \hline red & 1 & 0 \\ blue & 0 & 1 \end{array}\right)$$





Equivalence, redundancy, composition

- Equivalence: Two constraint networks are equivalent if they have the same set of solutions.
- Composition in matrix notation

$$R_{xz} = R_{xy} \cdot R_{yz}$$

- Composition in relational operation

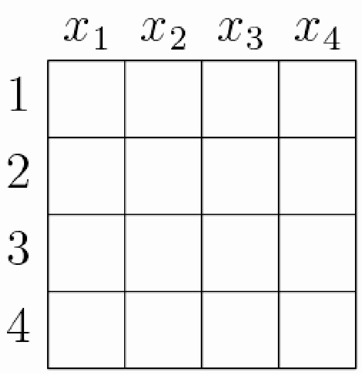
$$R_{xz} = \pi_{xz}(R_{xy} \bowtie R_{yz})$$

Relations vs networks

- Can we represent by binary constraint networks the relations
- R(x_1, x_2, x_3) = {(0,0,0)(0,1,1)(1,0,1)(1,1,0)}
- R(x_1, x_2, x_3, x_4) = {(1,0,0,0)(0,1,0,0) (0,0,1,0)(0,0,0,1)}
- Number of relations: 2^{k^n}
- Number of networks: $2^{n^2 k^2}$
- Most relations cannot be represented by binary constraint networks
- Most relations cannot be expressed by binary networks but they can be approximated by them.

The N-queens constraint network is there a tighter network?

The network has four variables, all with domains *Di* = {1, 2, 3, 4}.
(a) The labeled chess board. (b) The constraints between variables.



(a)

$$
\begin{aligned}
R_{12} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{13} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{14} &= \{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (2,4),\ (3,1),\ (3,2),\ (3,4) \\
&\quad (4,2),\ (4,3)\} \\
R_{23} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{24} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{34} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}
\end{aligned}
$$

(b)

Solutions are: (2,4,1,3) (3,1,4,2)

The projection networks

- The projection network of a relation is obtained by projecting it onto each pair of its variables (yielding a binary network).
- *Relation = {(1,1,2)(1,2,2)(1,2,1)}*
  - *What is the projection network?*
- What is the relationship between a relation and its projection network?
- R = {(1,1,2)(1,2,2)(2,1,3)(2,2,2)}
- What are the solutions of its projection network?

x_1	x_2	x_3
1	1	2
1	2	2
2	1	3
2	2	2

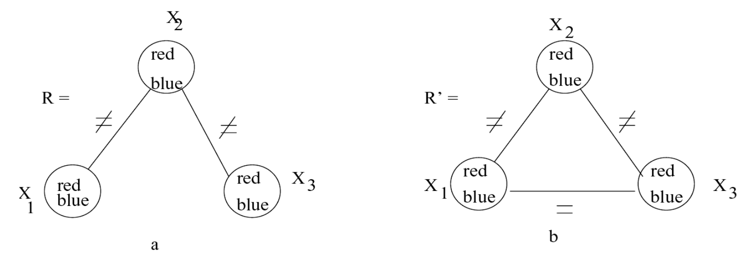
x_1	x_2	x_3
1	1	2
1	2	2
2	1	2
2	1	3
2	2	2

Projection network (continued)

- **Theorem**: *Every relation is included in the set of solutions of its projection network.*
- **Theorem**: *The projection network is the tightest upper bound binary networks representation of the relation.*

Therefore, If a network cannot be represented by its projection network it has no binary network representation

Partial order between networks, The minimal network



Definition 2.3.10 *Given two binary networks, $\mathcal{R}'$ and $\mathcal{R}$, on the same set of variables $x_1, ..., x_n$, $\mathcal{R}'$ is* at least as tight as *$\mathcal{R}$ iff for every i and j, $R'_{ij} \subseteq R_{ij}$.*

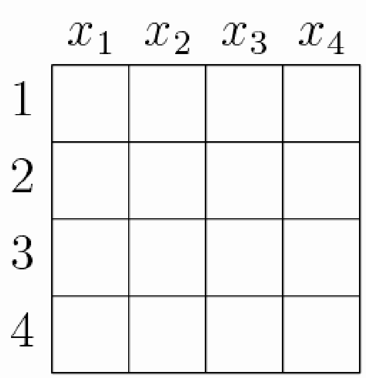
- An intersection of two networks is tighter (as tight) than both
- An intersection of two equivalent networks is equivalent to both

Definition: The minimal network is the intersection of all equivalent networks

Theorem: The projection network is identical to the minimal network

The N-queens constraint network

The network has four variables, all with domains *Di* = {1, 2, 3, 4}.
(a) The labeled chess board. (b) The constraints between variables.

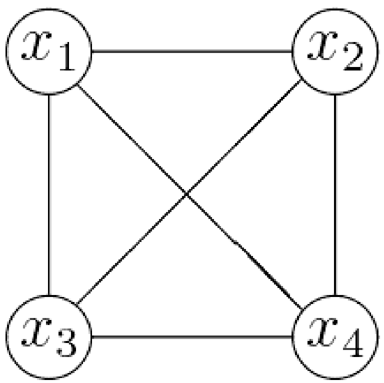


(a)

$$
\begin{aligned}
R_{12} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{13} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{14} &= \{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (2,4),\ (3,1),\ (3,2),\ (3,4) \\
&\qquad (4,2),\ (4,3)\} \\
R_{23} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{24} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{34} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}
\end{aligned}
$$

(b)

The 4-queens constraint network:
(a) The constraint graph. (b) The minimal binary constraints.
(c) The minimal unary constraints (the domains).



(a)

$$
\begin{aligned}
M_{12} &= \{(2,4),\ (3,1)\} \\
M_{13} &= \{(2,1),\ (3,4)\} \\
M_{14} &= \{(2,3),\ (3,2)\} \\
M_{23} &= \{(1,4),\ (4,1)\} \\
M_{24} &= \{(1,2),\ (4,3)\} \\
M_{34} &= \{(1,3),\ (4,2)\}
\end{aligned}
$$

(b)

$$
\begin{aligned}
D_1 &= \{2,3\} \\
D_2 &= \{1,4\} \\
D_3 &= \{1,4\} \\
D_4 &= \{2,3\}
\end{aligned}
$$

(c)

Solutions are: (2,4,1,3) (3,1,4,2)

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MiniZinc Tutorial

CS 275: Constraint Networks

Annie Raichev Fall 2022

MiniZinc References & More Examples

- The minizinc handbook: https://www.minizinc.org/doc-2.5.5/en/index.html
- Lecture 1: https://www.ics.uci.edu/~dechter/courses/ics-275/fall-2022/