Outline
(Chapter 4, continued

- Directional Arc-consistency algorithms
- Directional Path-consistency and directional i-consistency
- Greedy algorithms for induced-width
- Width and local consistency
- Adaptive-consistency and bucket-elimination

Width vs directional consistency
(Freuder 82)

Theorem 4.4.5 (Width (i-1) and directional i-consistency) *Given a general network $\mathcal{R}$, its ordered constraint graph along* **d** *has a width of* $\mathbf{i}-1$ *and if it is also strong directional* **i***-consistent, then* $\mathcal{R}$ *is backtrack-free along* **d**.

Width vs i-consistency

- DAC and width-1
- DPC and width-2
- DIC_i and width-(i-1)
- → backtrack-free representation

- If a problem has width 2, will DPC make it backtrack-free?
- **Adaptive-consistency**: applies i-consistency when i is adapted to the number of parents

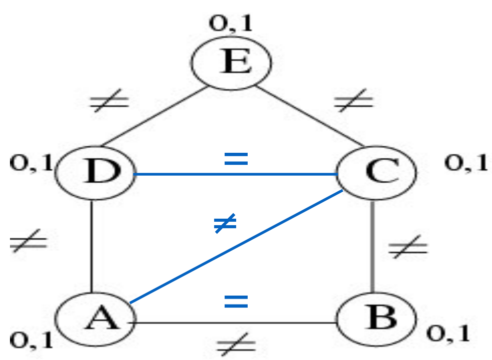
Adaptive-consistency

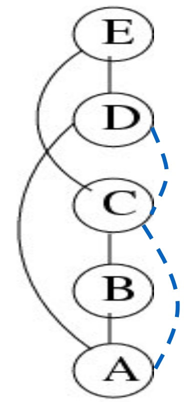
ADAPTIVE-CONSISTENCY (AC1)
Input: a constraint network $\mathcal{R} = (X, D, C)$, its constraint graph $G = (V, E)$, $d = (x_1, \ldots, x_n)$.
output: A backtrack-free network along d
Initialize: $C' \leftarrow C$, $E' \leftarrow E$
1. **for** $j = n$ to 1 do
2. Let $S \leftarrow parents(x_j)$.
3. $R_S \leftarrow Revise(S, x_j)$ (generate all partial solutions over S that can extend to x_j).
4. $C' \leftarrow C' \cup R_S$
5. $E' \leftarrow E' \cup \{(x_k, x_r) | x_k, x_r \in parents(x_j)\}$ (connect all parents of x_j)
5. **endfor**.

Figure 4.13: Algorithm adaptive-consistency– version 1

Bucket Elimination

Adaptive Consistency (Dechter & Pearl, 1987)





Bucket E: $E \neq D,\ E \neq C$

Bucket D: $D \neq A$ — $D = C$

Bucket C: $C \neq B$ — $A \neq C$

Bucket B: $B \neq A$ — $B = A$

Bucket A: — contradiction

Complexity: nk^{w^*+1}

w^* is the induced-width along the ordering

Adaptive-consistency, bucket-elimination

ADAPTIVE-CONSISTENCY (AC)

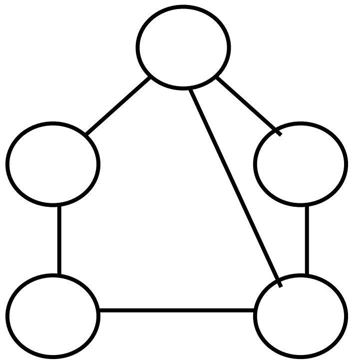
Input: a constraint network $\mathcal{R}$, an ordering $d = (x_1, \ldots, x_n)$
output: A backtrack-free network, denoted $E_d(\mathcal{R})$, along d, if the empty constraint was not generated. Else, the problem is inconsistent

1. Partition constraints into $bucket_1, \ldots, bucket_n$ as follows:
   for $i \leftarrow n$ **downto** 1, put in $bucket_i$ all unplaced constraints mentioning x_i.
2. **for** $p \leftarrow n$ **downto** 1 **do**
3.   **for** all the constraints $R_{S_1}, \ldots, R_{S_j}$ in $bucket_p$ **do**
4. $A \leftarrow \bigcup_{i=1}^{j} S_i - \{x_p\}$
5. $R_A \leftarrow \Pi_A(\bowtie_{i=1}^{j} R_{S_i})$
6.   **if** R_A is not the empty relation **then** add R_A to the bucket of the latest variable in scope A,
7.   **else** exit and return the empty network
8. **return** $E_d(\mathcal{R}) = (X, D, bucket_1 \cup bucket_2 \cup \cdots \cup bucket_n)$

Figure 4.14: Adaptive-Consistency as a bucket-elimination algorithm

Bucket Elimination

Adaptive Consistency (Dechter & Pearl, 1987)

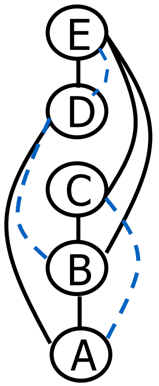


$// \ R_{DCB}$

$// \ R_{ACB}$

$// \ R_{AB}$

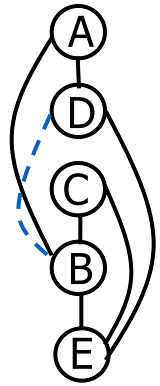
R_A



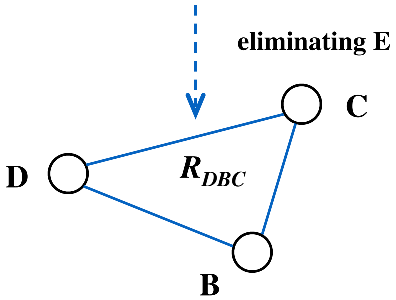
$// \ R_{DB}$

$// \ R^{D}_{BE}, R^{C}_{BE}$

$// \ R_E$



The Idea of Elimination



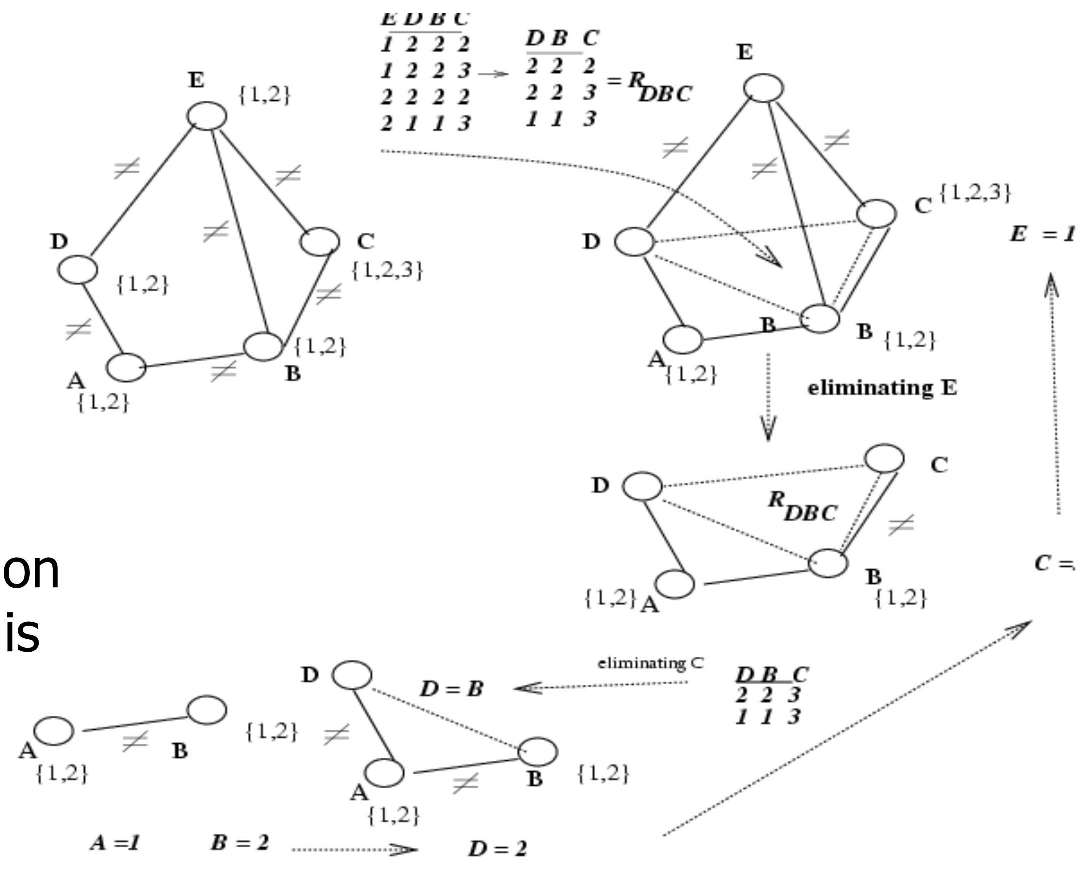
3
value assignment

⋈ ⋈

Variable Elimination

Eliminate variables one by one: "constraint propagation"

Solution generation after elimination is backtrack-free



Properties of bucket-elimination (adaptive consistency)

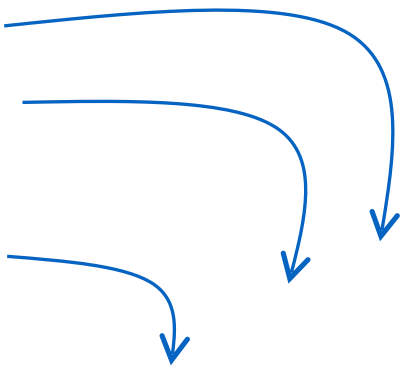
- Adaptive consistency generates a constraint network that is backtrack-free (can be solved without dead-ends).
- The time and space complexity of adaptive consistency along ordering *d* is respectively, or O(r k^{w^*+1})) when r is the number of constraints.
- Therefore, problems having bounded induced width are tractable (solved in polynomial time)
- Special cases: *trees* (w*=1), *series-parallel networks* (w*=2), and in general *k-trees* (w*=k).

Back to Induced width

- Finding minimum-w* ordering is NP-complete (Arnborg, 1985)
- Greedy ordering heuristics: *min-width, min-degree, max-cardinality* (Bertele and Briochi, 1972; Freuder 1982), Min-fill.

Solving Trees
(Mackworth and Freuder, 1985)

Adaptive consistency is linear for trees and equivalent to enforcing directional arc-consistency (recording only unary constraints)



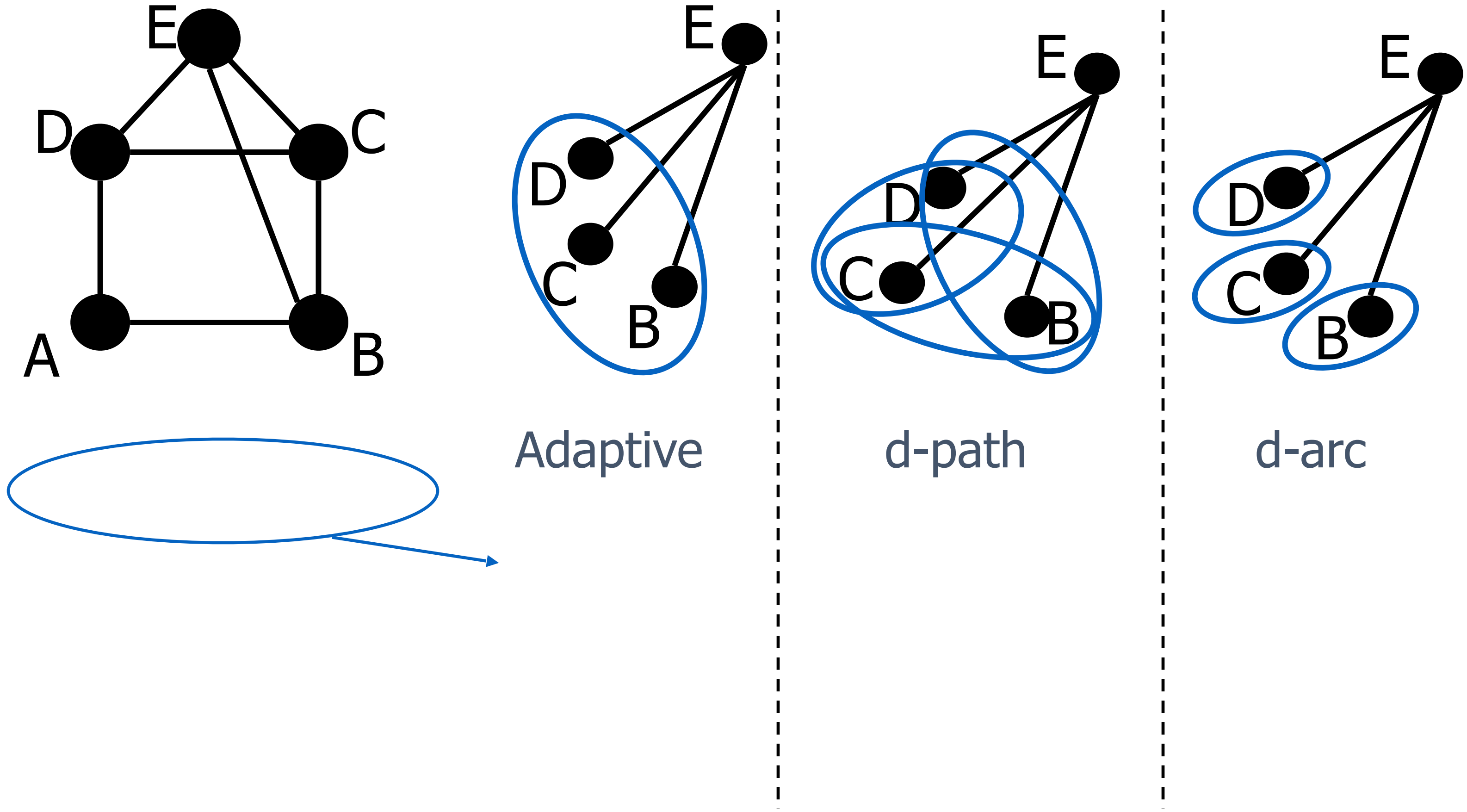
CompSci 275, Constraint Networks

Rina Dechter, Fall 2022

General Search Strategies: Look-ahead

Chapter 5

Directional i-Consistency



Outline

- The search tree for CSPs, Variable ordering an consistency level
- Look-ahead for value selection:
  - Forward checking,
  - Full-arc-consistency,
  - partial look-ahead,
  - maintaining arc-consistency
- Dynamic Variable ordering (DVO, DVFC)
- Search for Satisfiability
- Converting a CSP into a SAT problem

What if the constraint network is not backtrack-free?

- Backtrack-free in general is too costly, so what to do?
- Search?
- What is the search space?
- How to search it? Breadth-first? Depth-first?

The search space for a CN

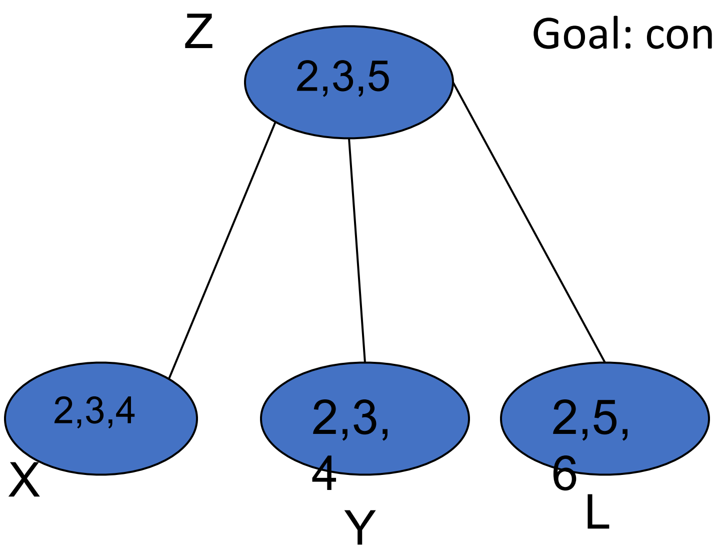
- A tree of all partial solutions
- A partial solution: $(a_1, \ldots, a_j)$ satisfying all relevant constraints
- The size of the underlying search space depends on:
  - Variable ordering
  - Level of consistency possessed by the problem

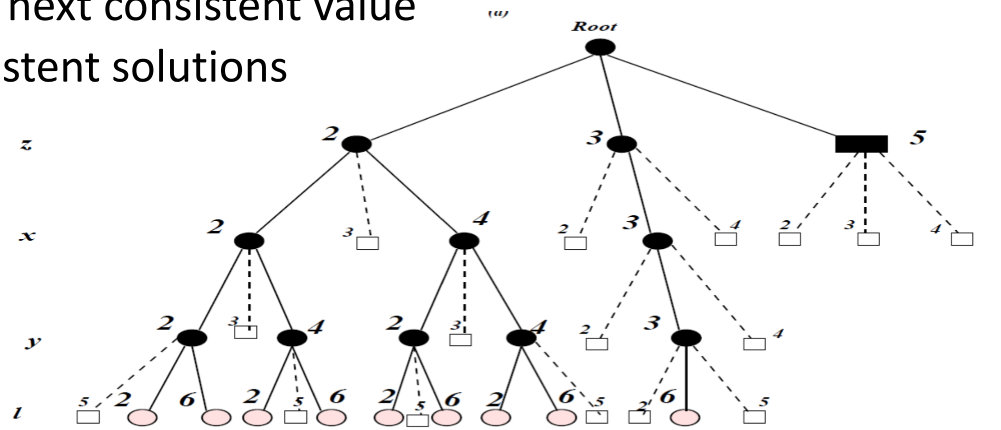
Search spaces: the effect of ordering

States = partial solutions

Operators: next consistent value

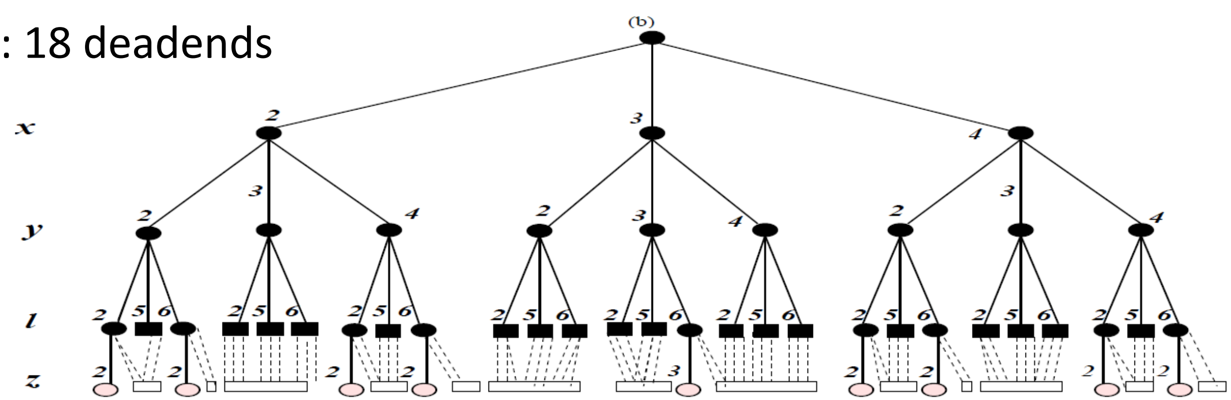
Goal: consistent solutions



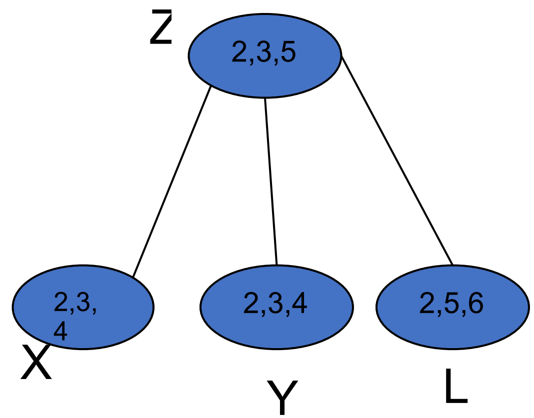


Ordering d1 = (z,x,y,l): 1 deadend

Ordering d2 = (x,y,l,z): 18 deadends



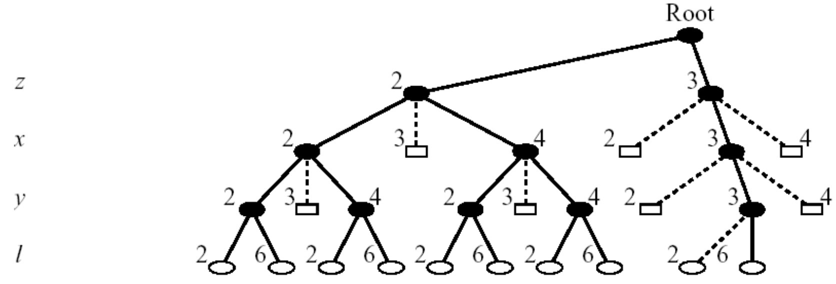
Search spaces: the effect of consistency level



- After arc-consistency z=5 and l=5 are removed

Ordering d1 = (z,x,y,l): 1 deadend

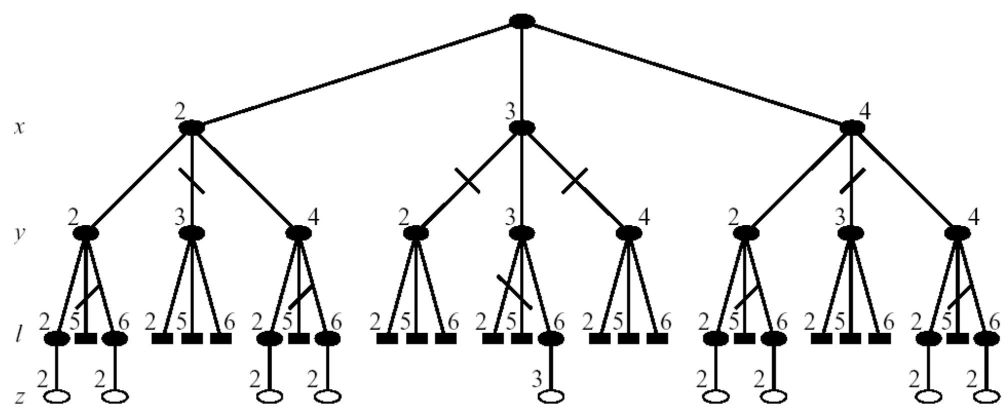
Ordering d2 = (x,y,l,z): 18 deadends



(a)

- After path-consistency

$$R'_{zx} = \{(2,2),(2,4),(3,3)\}$$
$$R'_{zy} = \{(2,2),(2,4),(3,3)\}$$
$$R'_{zl} = \{(2,2),(2,6),(3,6)\}$$
$$R'_{xy} = \{(2,2),(2,4),(4,2),(4,4),(3,3)\}$$
$$R'_{xl} = \{(2,2),(2,6),(4,2),(4,6),(3,6)\}$$
$$R'_{yl} = \{(2,2),(2,6),(4,2),(4,6),(3,6)\}.$$



(b)

The effect of consistency level on search

Theorem 5.1.3 *Let $\mathcal{R}'$ be a tighter network than $\mathcal{R}$, where both represent the same set of solutions. For any ordering d, any path appearing in the search graph derived from $\mathcal{R}'$ also appears in the search graph derived from $\mathcal{R}$.* $\square$

Cost of node's expansion;
More constraints require more consistency checks

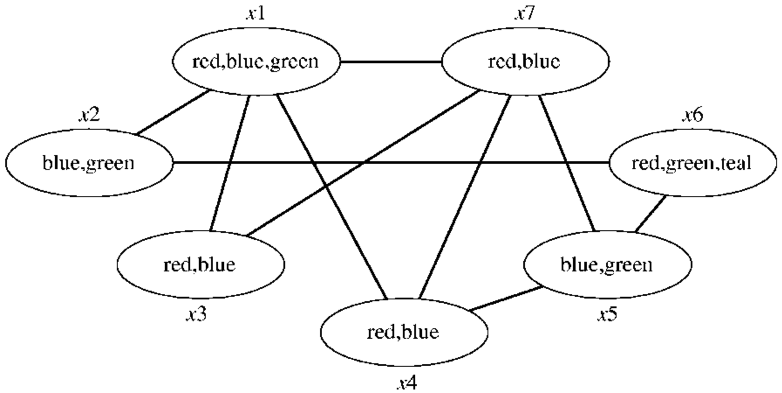
- Number of consistency checks for toy problem:
  - For d1: 19 for R (original), 43 for R' (after consistency)
  - For d2: 91 on R and 56 on R'

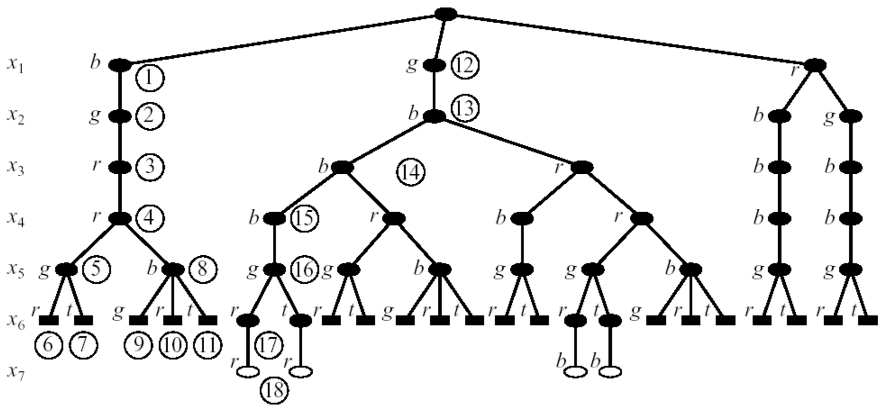
- Reminder:

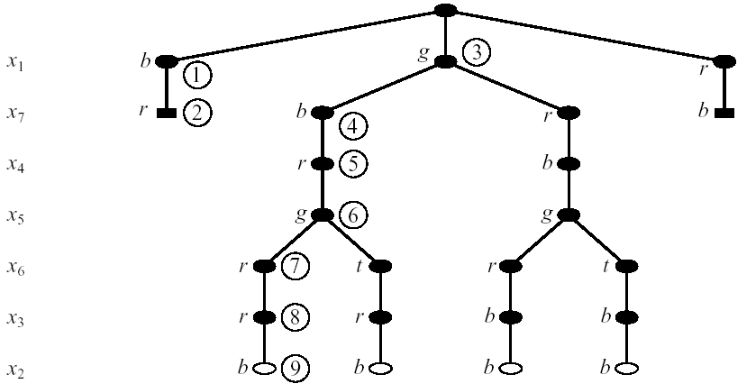
Definition 5.1.5 (backtrack-free network) *A network R is said to be* backtrack-free *along ordering d if every leaf node in the corresponding search graph is a solution.*

Backtracking search for a solution

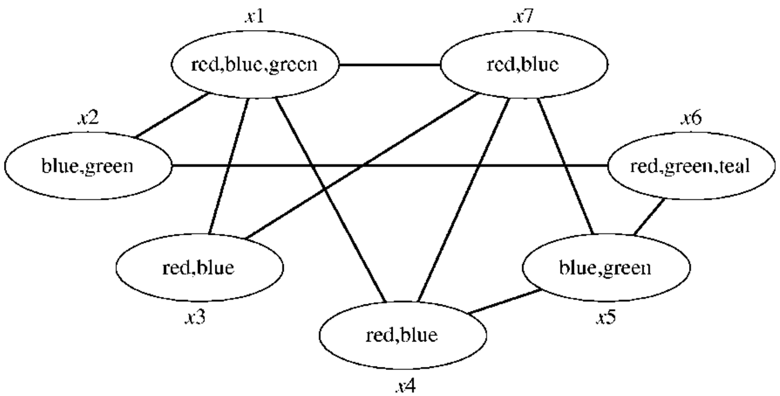
2 search spaces:
d1= (x1,x2,x3,x4,x5,x6,x7)
d2= (x1,x7,x4,x5,x6,x3,x2)

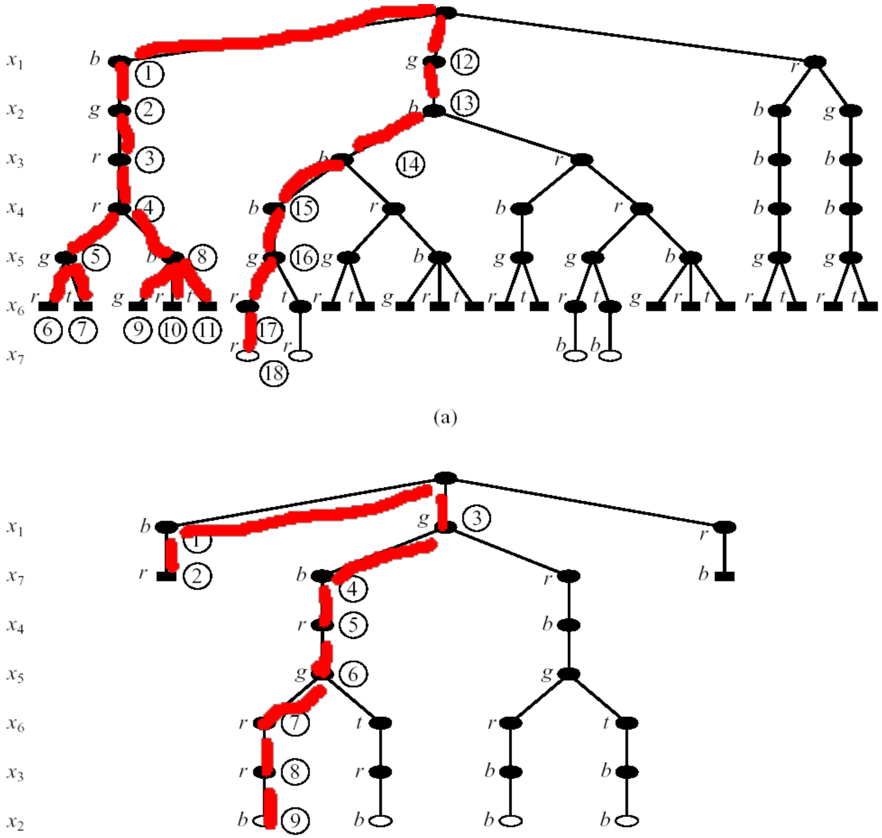



(a)


(b)

Backtracking Search for a single Solution

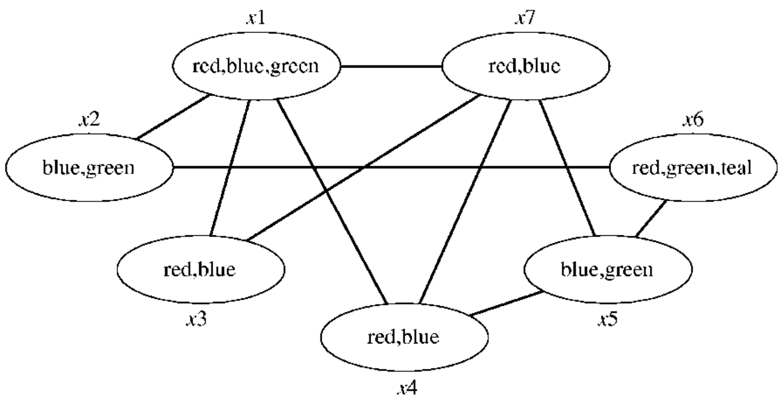


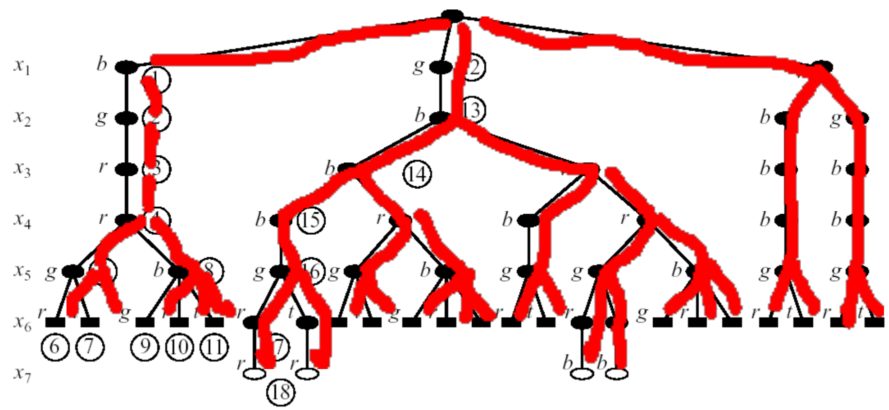


(a)

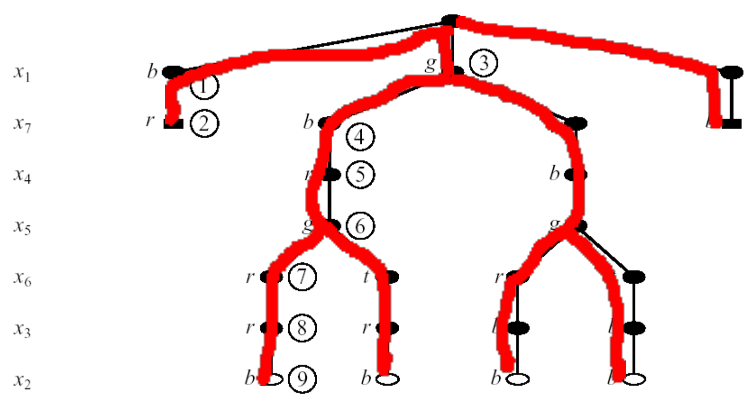
(b)

Backtracking search for *all* solutions



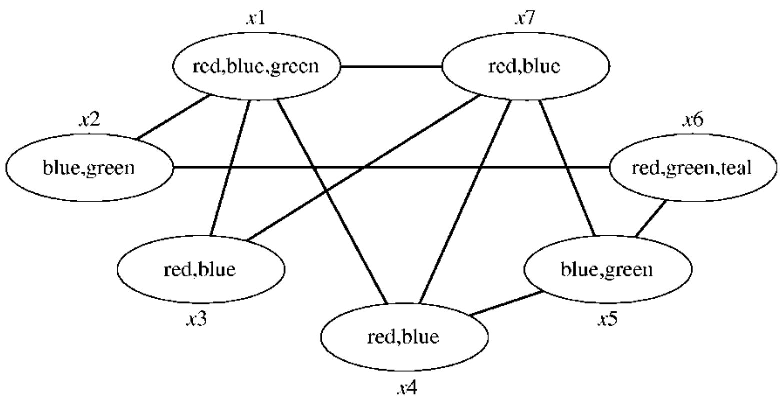


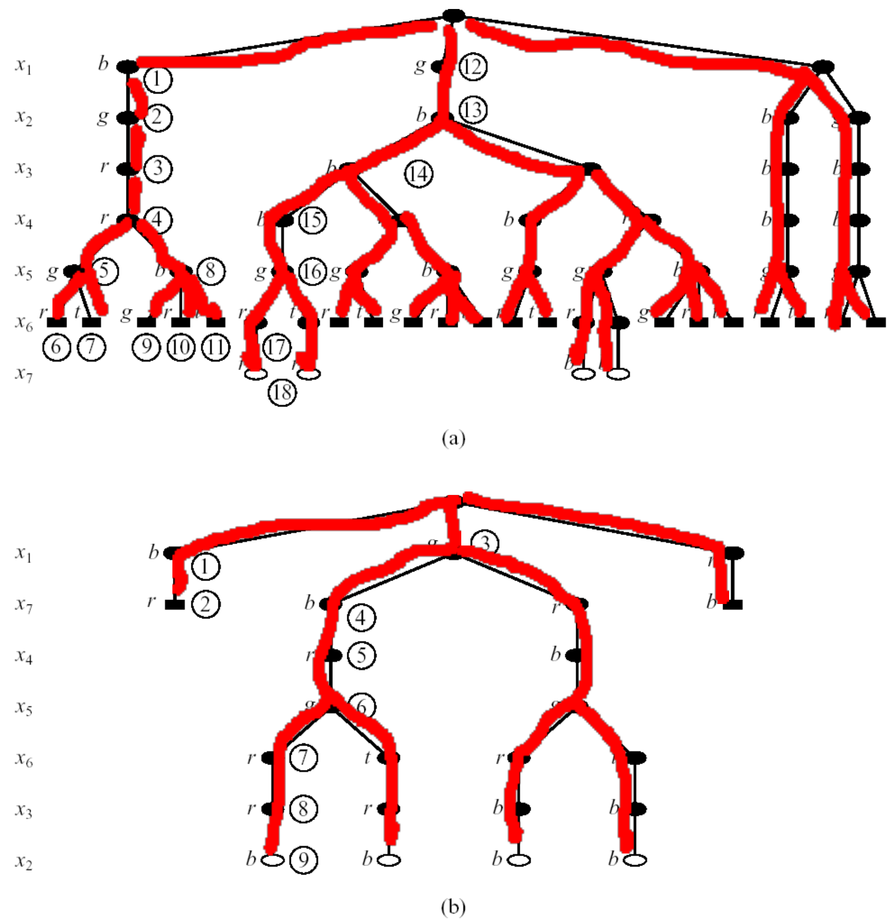
(a)



(b)

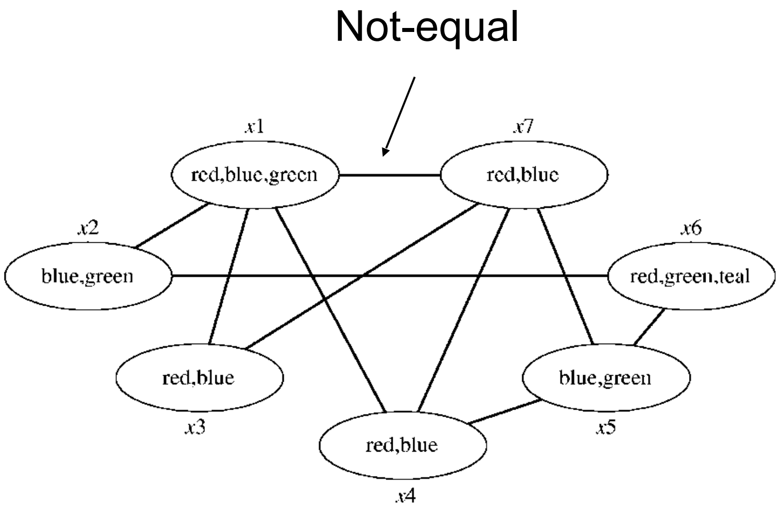
Backtracking search for *all* solutions

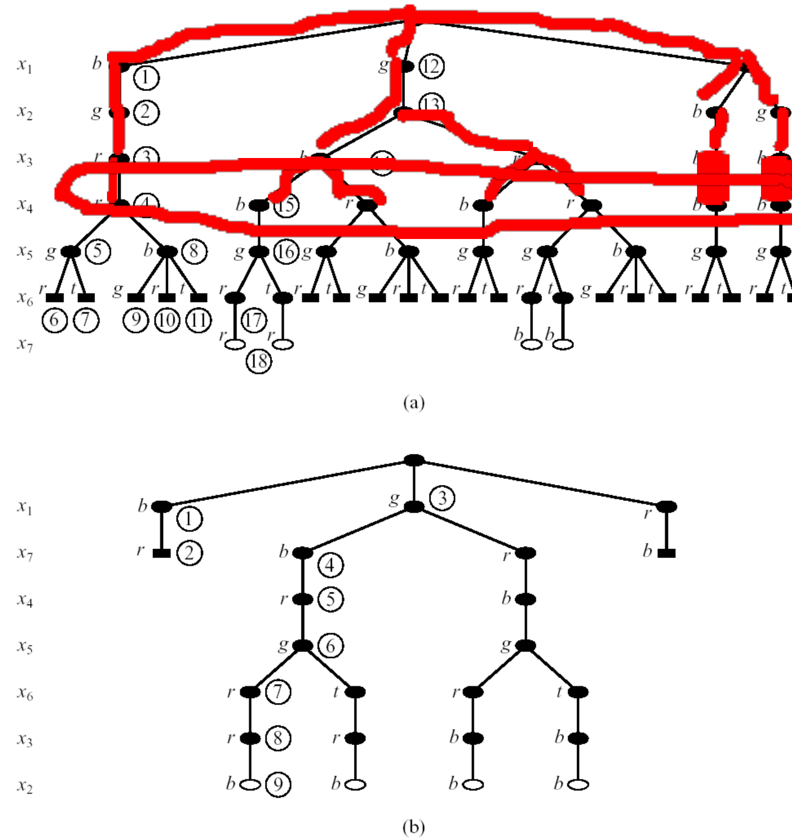




For all tasks
Time: O(exp(n))
Space: linear

Traversing breadth-first (BFS)?





BFS space is exp(n) while no Time gain → use DFS

Backtracking

```
procedure BACKTRACKING
Input: A constraint network P = (X, D, C).
Output: Either a solution, or notification that the network is inconsistent.

    i ← 1                          (initialize variable counter)
    D'_i ← D_i                     (copy domain)
    while 1 ≤ i ≤ n
       instantiate x_i ← SELECTVALUE
       if x_i is null              (no value was returned)
          i ← i − 1                (backtrack)
       else
          i ← i + 1                (step forward)
          D'_i ← D_i
    end while
    if i = 0
       return "inconsistent"
    else
       return instantiated values of {x_1, . . . , x_n}
end procedure

subprocedure SELECTVALUE   (return a value in D'_i consistent with ā_{i−1})

    while D'_i is not empty
       select an arbitrary element a ∈ D'_i, and remove a from D'_i
       if CONSISTENT(ā_{i−1}, x_i = a)
          return a
    end while
    return null                    (no consistent value)
end procedure
```

Figure 5.4: The backtracking algorithm.

- Complexity of extending a partial solution:
  - Complexity of *consistent* O(e log t), t bounds tuples, e, constraints
  - Complexity of selectValue O(e k log t)

Improving backtracking

- Before search: (reducing the search space)
  - Arc-consistency, path-consistency
  - Variable ordering (fixed)
- During search:
  - Look-ahead schemes:
    - value ordering,
    - variable ordering (if not fixed)
  - Look-back schemes:
    - Backjump
    - Constraint recording
    - Dependency-directed backtacking

Look-ahead: value orderings

- Intuition:
  - Choose value least likely to yield a dead-end
  - Approach: apply constraint propagation at each node in the search tree
- Forward-checking
  - (check each unassigned variable separately
- Maintaining arc-consistency (MAC)
  - (apply full arc-consistency)
- Full look-ahead
  - One pass of arc-consistency (AC-1)
- Partial look-ahead
  - directional-arc-consistency

Generalized look-ahead

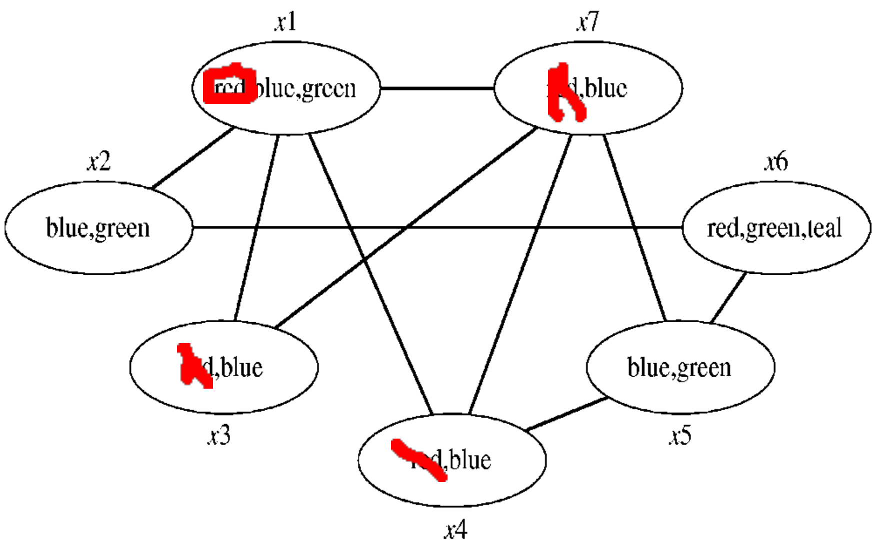
```
procedure GENERALIZED-LOOKAHEAD
Input: A constraint network P = (X, D, C)
Output: Either a solution, or notification that the network is inconsis-
tent.

    D'_i ← D_i for 1 ≤ i ≤ n          (copy all domains)
    i ← 1                             (initialize variable counter)
    while 1 ≤ i ≤ n
        instantiate x_i ← SELECTVALUE-XXX
        if x_i is null                (no value was returned)
            i ← i − 1                 (backtrack)
            reset each D'_k, k > i, to its value before x_i was last instantiated
        else
            i ← i + 1                 (step forward)
    end while
    if i = 0
        return "inconsistent"
    else
        return instantiated values of {x_1, ..., x_n}
end procedure
```

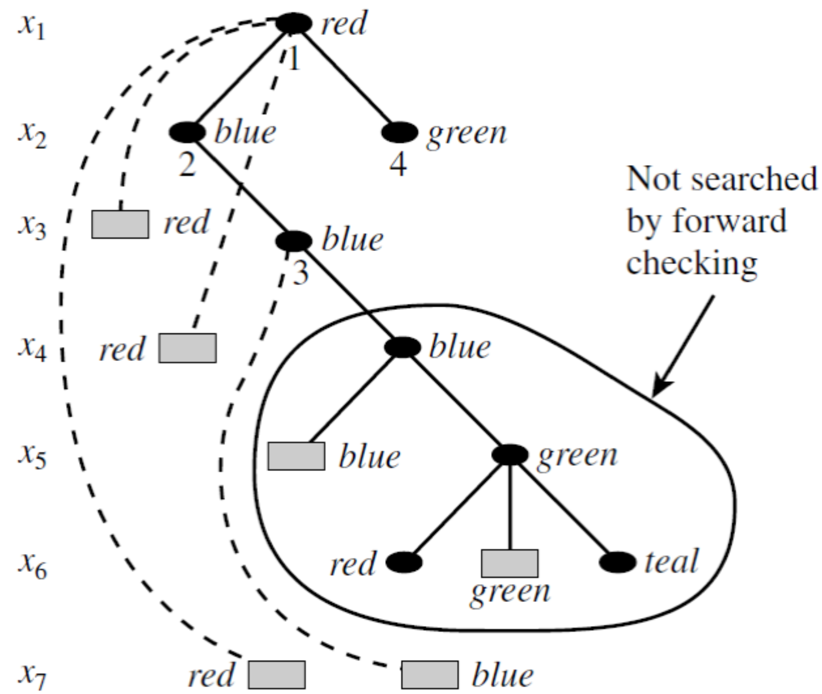
Figure 5.7: A common framework for several look-ahead based search algorithms. By replacing SELECTVALUE-XXX with SELECTVALUE-FORWARD-CHECKING, the forward checking algorithm is obtained. Similarly, using SELECTVALUE-ARC-CONSISTENCY yields an algorithm that interweaves arc-consistency and search.

Forward-checking for value rejection

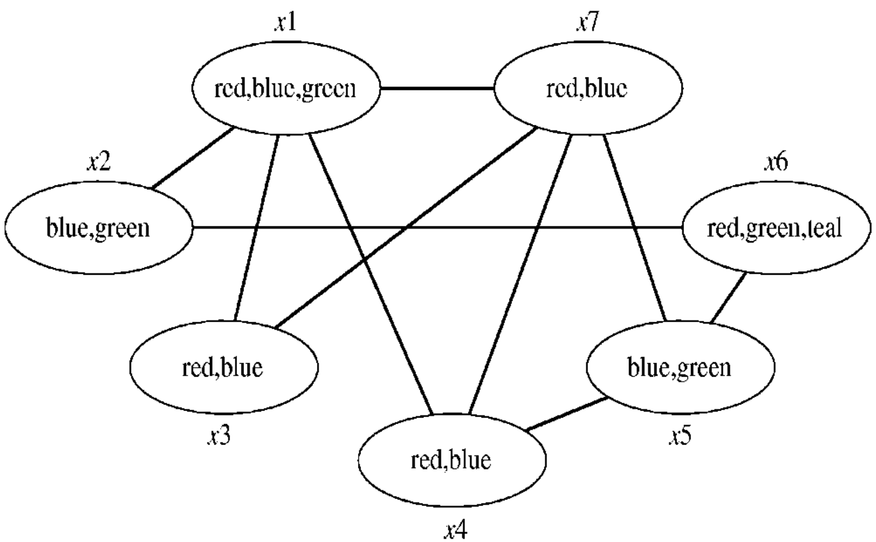
Forward-checking

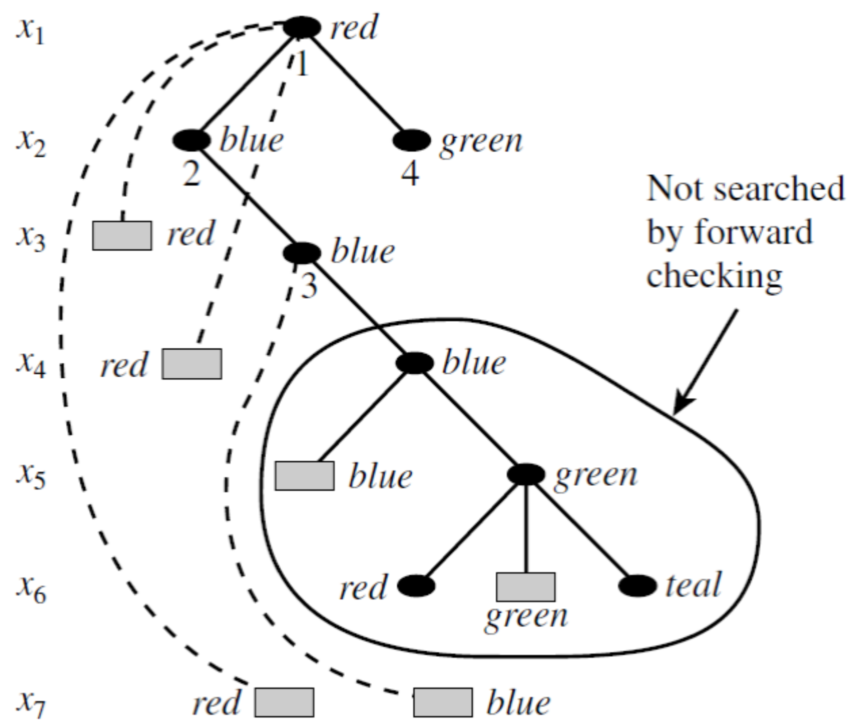
(check each unassigned variable separately





Forward-checking for value rejection





FC overhead:

For each value of a future variable e_u
Tests: O(k e_u), for all future variables O(ke)
For all current domain O(k^2 e)

Forward-checking

```
procedure SELECTVALUE-FORWARD-CHECKING
    while D'_i is not empty
        select an arbitrary element a ∈ D'_i, and remove a from D'_i
        empty-domain ← false
        for all k, i < k ≤ n
            for all values b in D'_k
                if not CONSISTENT(a⃗_{i-1}, x_i = a, x_k = b)
                    remove b from D'_k
            end for
            if D'_k is empty            (x_i = a leads to a dead-end)
                empty-domain ← true
        if empty-domain                 (don't select a)
            reset each D'_k, i < k ≤ n to value before a was selected
        else
            return a
    end while
    return null                         (no consistent value)
end procedure
```

Figure 5.8: The SELECTVALUE subprocedure for the forward checking algorithm.

Complexity of selectValue-forward-checking at each node:

Arc-consistency look-ahead

(Gacshnig, 1977)

- Applies full arc-consistency on all un-instantiated variables following each candidate value assignment to the current variable.
- Complexity:
  - If optimal arc-consistency is used:
  - What is the complexity overhead when AC-1 is used at each node?

Forward-checking:

Full arc-consistency look-ahead
With optimal AC:

MAC: Maintaining arc-consistency (Sabin and Freuder 1994)

- Perform arc-consistency in a binary search tree: Given a domain X={1,2,3,4} the algorithm assigns X=1 (and apply arc-consistency) and if x=1 is pruned, it applies arc-consistency to X={2,3,4}
- If inconsistency is not discovered, a new variable is selected (not necessarily X)

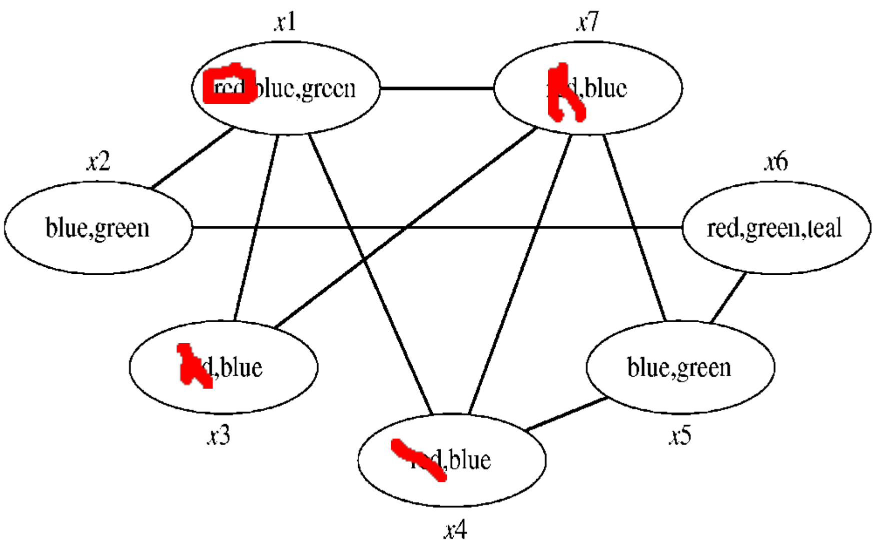
Arc-consistency look-ahead:

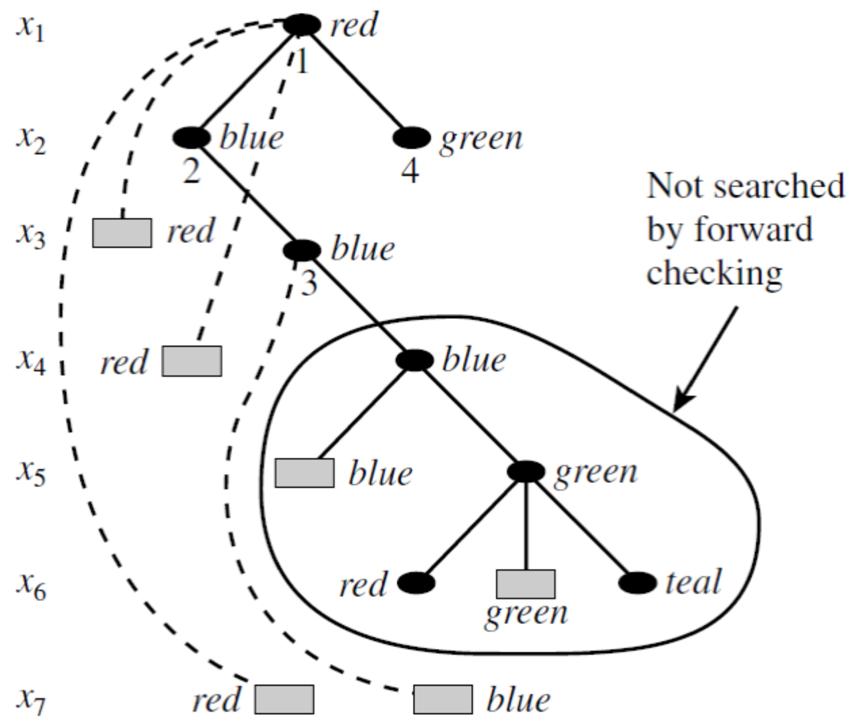
subprocedure SELECTVALUE-ARC-CONSISTENCY

while D'_i is not empty
  select an arbitrary element $a \in D'_i$, and remove a from D'_i
  repeat
  removed-value $\leftarrow$ *false*
    for all $j, i < j \leq n$
      for all $k, i < k \leq n$
        for each value b in D'_j
          if there is no value $c \in D'_k$ such that
              CONSISTENT$(\vec{a}_{i-1}, x_i = a, x_j = b, x_k = c)$
            remove b from D'_j
            removed-value $\leftarrow$ *true*
        end for
      end for
    end for
  until *removed-value* = *false*
  if any future domain is empty (don't select a)
    reset each $D'_j, i < j \leq n$, to value before a was selected
  else
    return a
end while
return null (no consistent value)
end procedure

Figure 5.10: The SELECTVALUE subprocedure for arc-consistency, based on the AC-1 algorithm.

AC for value rejection



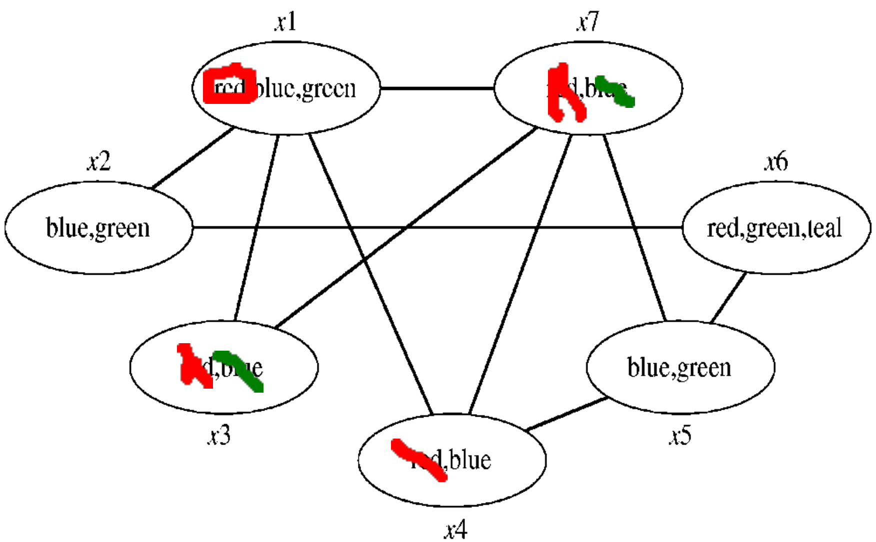


FW overhead:

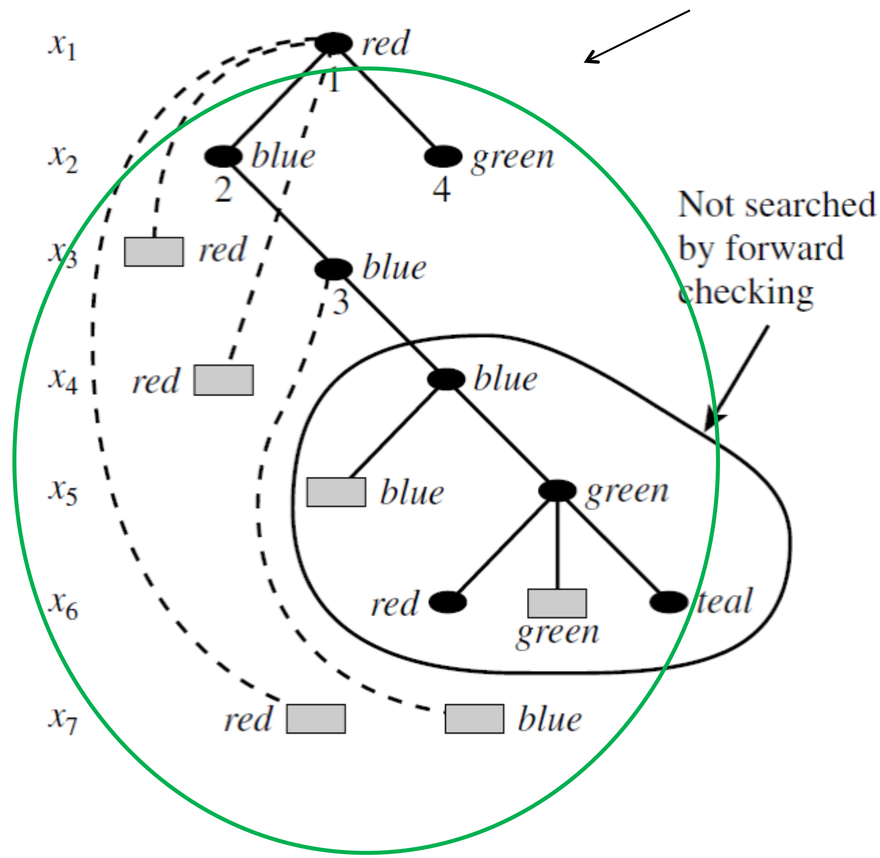
MAC overhead:

AC for value rejection

Arc-consistency prunes x1=red
Prunes the whole tree



Not searched
By MAC



FW overhead:

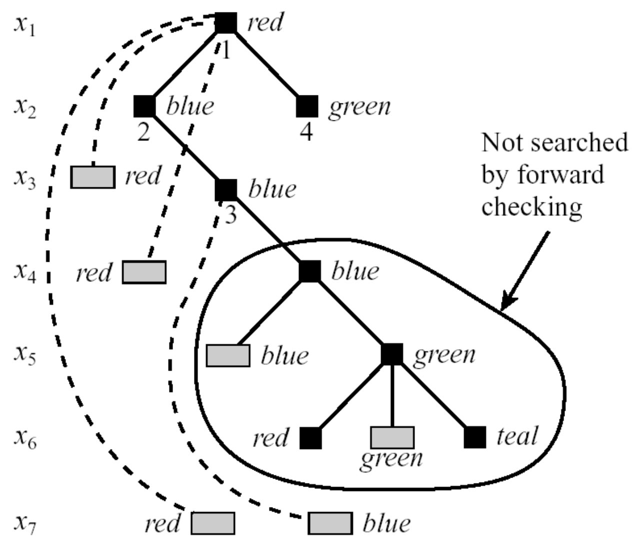
MAC overhead:

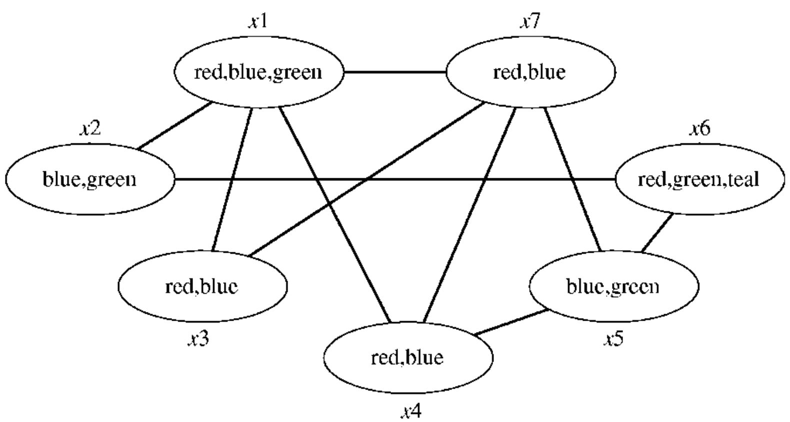
Full and partial look-ahead

- Full looking ahead:
  - Make one pass through future variables (delete, repeat-until)
- Partial look-ahead:
  - Applies (similar-to) directional arc-consistency to future variables.
  - Complexity: also
  - More efficient than MAC

Example of partial look-ahead

Example 5.3.3 Conside the problem in Figure 5.3 using the same ordering of variables and values as in Figure 5.9. Partial-look-ahead starts by considering $x_1 = red$. Applying directional arc-consistency from x_1 towards x_7 will first shrink the domains of x_3, x_4 and x_7, (when processing x_1), as was the case for forward-checking. Later, when directional arc-consistency processes x_4 (with its only value, "blue") against x_7 (with its only value, "blue"), the domain of x_4 will become empty, and the value "red" for x_1 will be rejected. Likewise, the value $x_1 = blue$ will be rejected. Therefore, the whole tree in Figure 5.9 will not be visited if either partial-look-ahead or the more extensive look-ahead schemes are used. With this level of look-ahead only the subtree below $x_1 = green$ will be expanded. □





Branching-ahead: dynamic value ordering

Rank order the promise in non-rejected values to estimate the likelihood of leading to a solution.

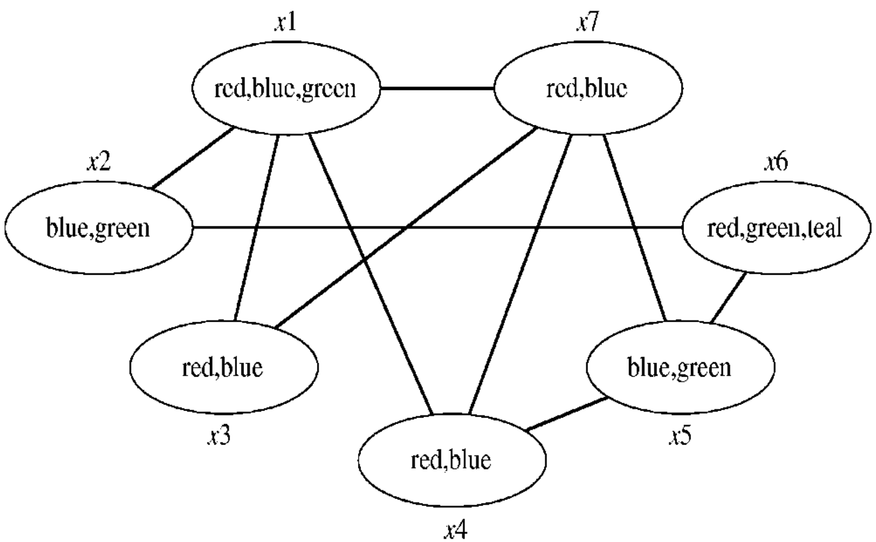
- Rank functions
  - MC (min conflict) counts the number of conflicts with each future domain that are otherwise consistent.
  - MD (min domain) score is the largest domain size of future variables.
  - ES (expected solution counts)

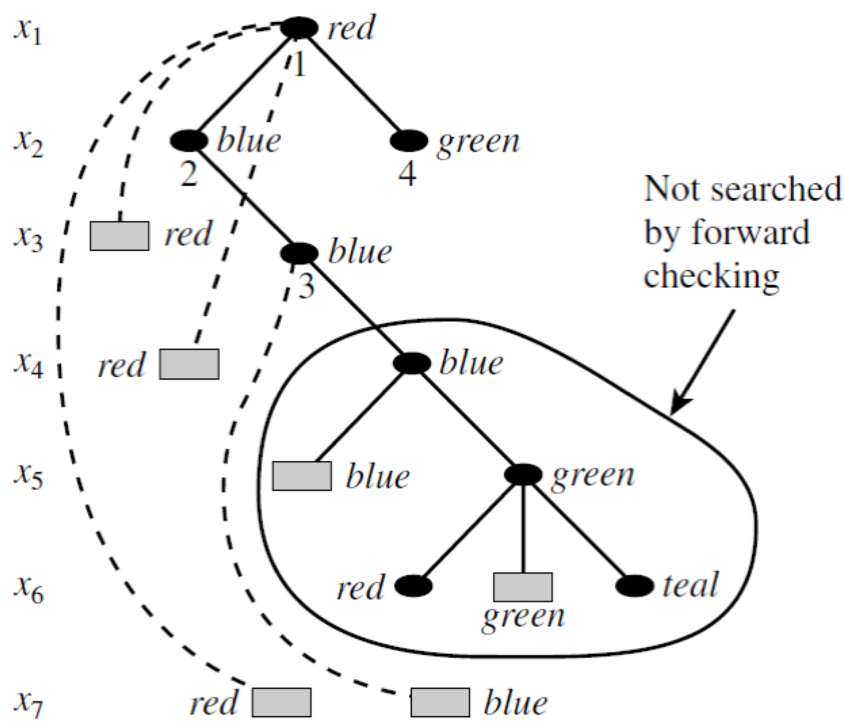
- MC results (Frost and Dechter, 1996)
- ES – showed good performance using IJGP (Kask, Dechter and Gogate, 2004)

Dynamic variable ordering (DVO)

- Following constraint propagation, choose the most constrained variable
- **Intuition:** early discovery of dead-ends
- **Highly effective**: the single most important heuristic to cut down search space
- Most popular with FC
- Dynamic search rearrangement (Bitner and Reingold, 1975) (Purdon,1983)

Forward-checking: variable ordering



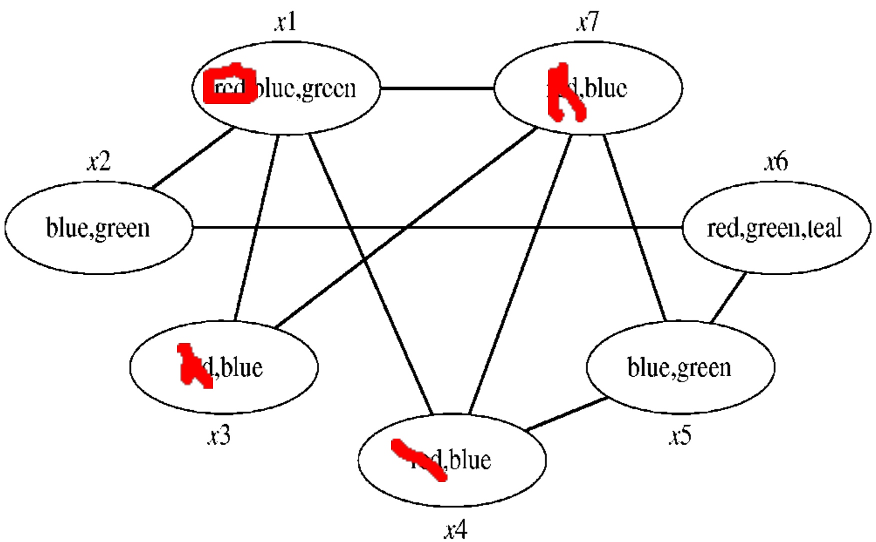


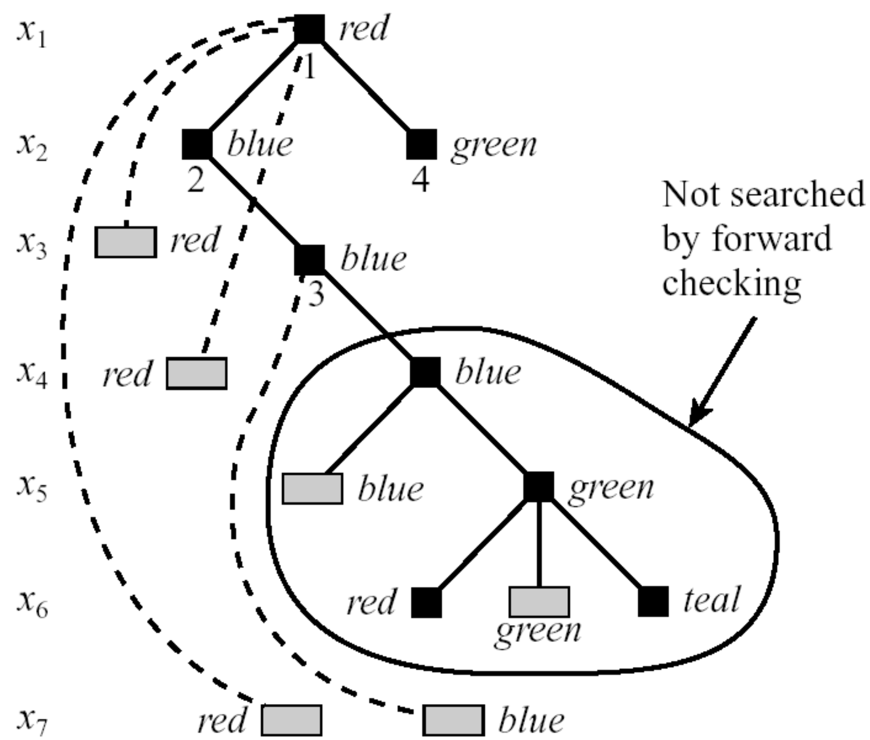
FW overhead:

MAC overhead:

Forward-checking: variable ordering

After X1 = red choose X3 and not X2



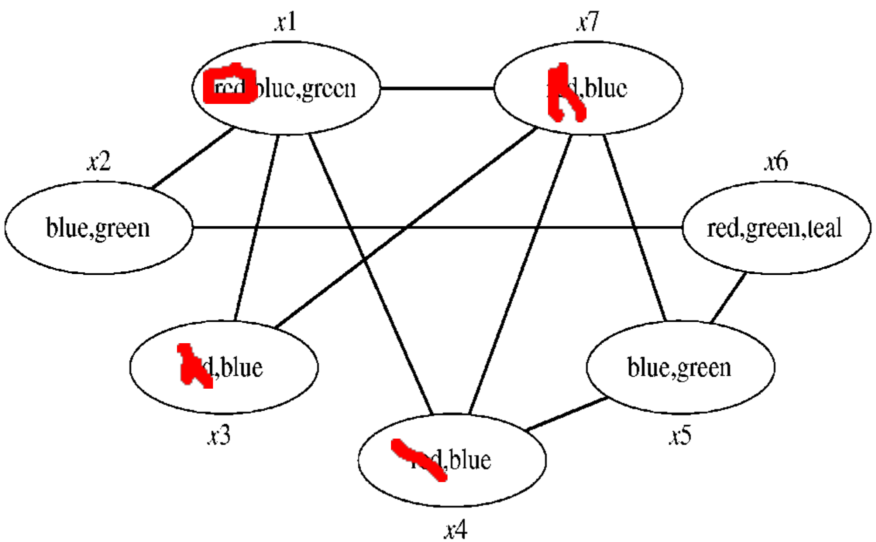


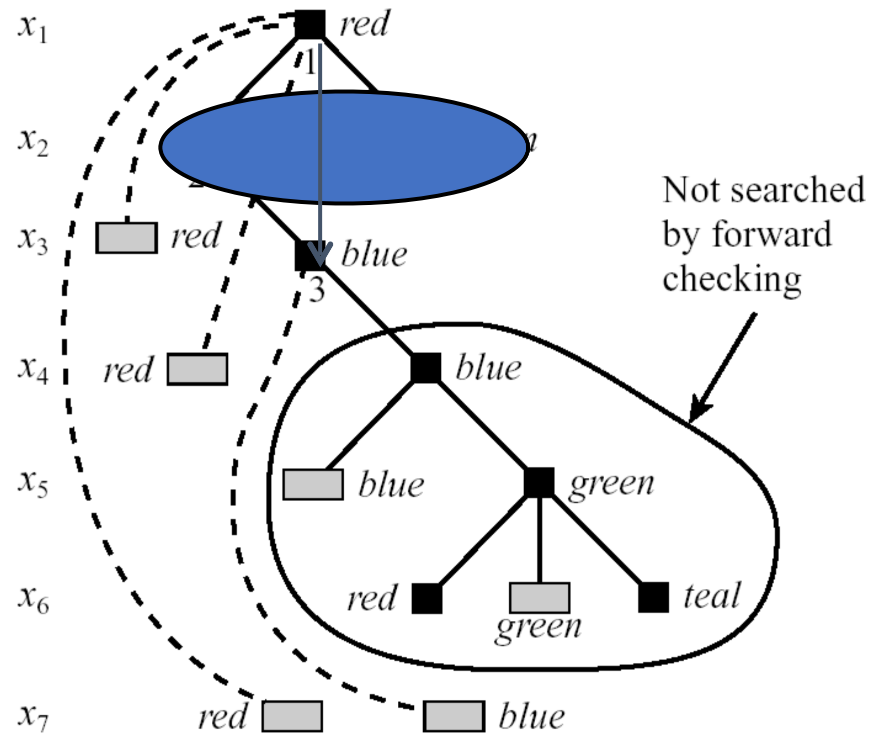
FW overhead:

MAC overhead:

Forward-checking: variable ordering

After X1 = red choose X3 and not X2



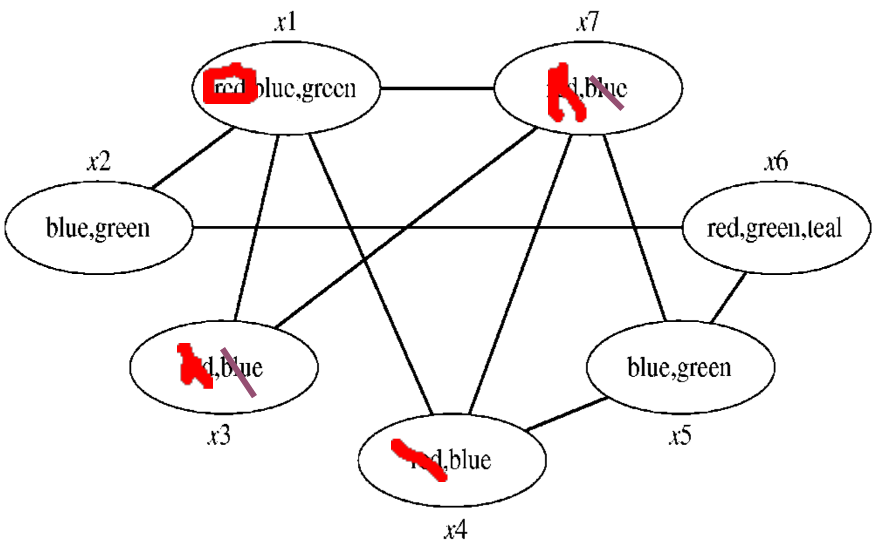


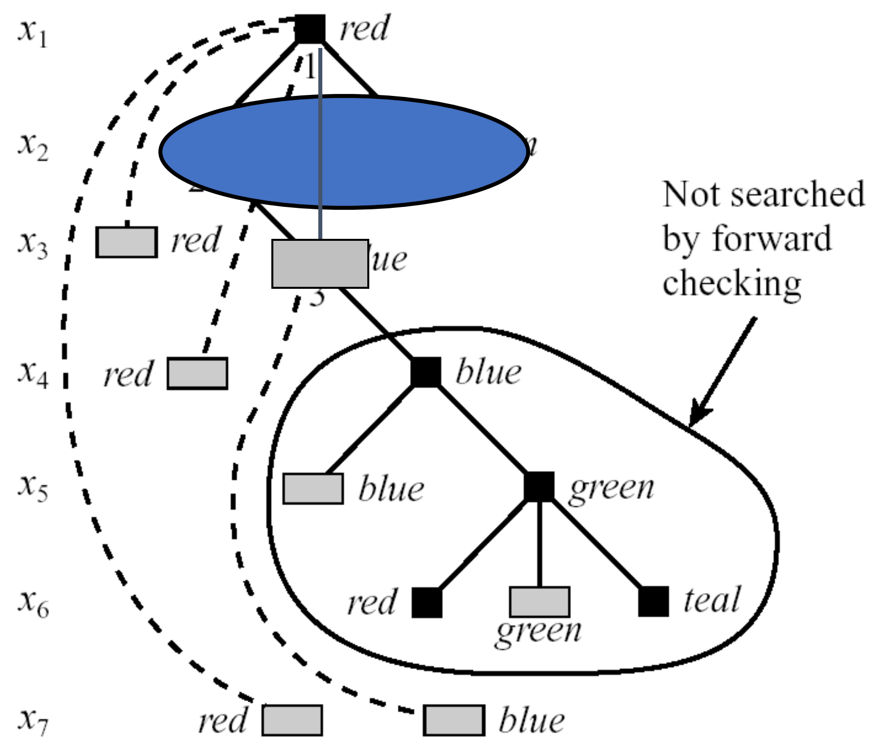
FW overhead:

MAC overhead:

Forward-checking: variable ordering

After X1 = red choose X3 and not X2

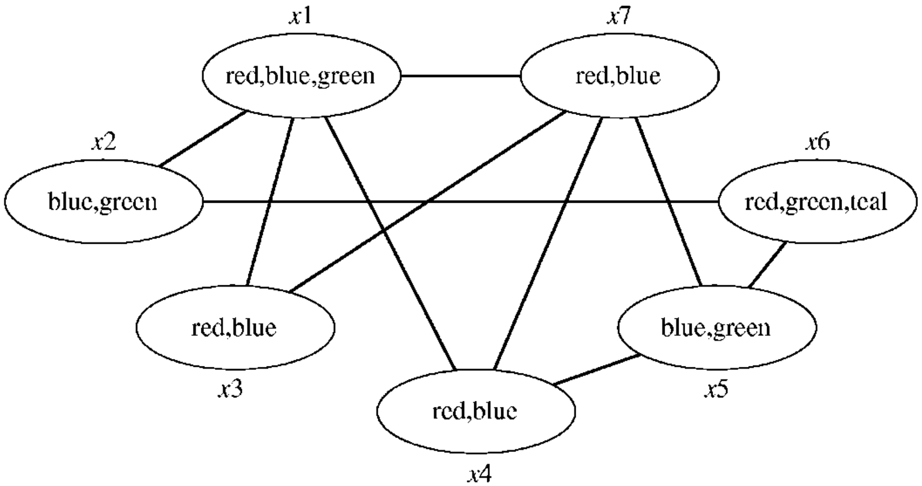




FW overhead:

MAC overhead:

Example: DVO with forward-checking (DVFC)



Example 5.3.4 Consider again the example in Figure 5.3. Initially, all variables have domain size of 2 or more. DVFC picks x_7, whose domain size is 2, and the value $< x_7, blue >$. Forward-checking propagation of this choice to each future variable restricts the domains of x_3, x_4 and x_5 to single values, and reduces the size of x_1's domain by one. DVFC selects x_3 and assigns it its only possible value, *red*. Subsequently, forward-checking causes variable x_1 to also have a singleton domain. The algorithm chooses x_1 and its only consistent value, *green*. After propagating this choice, we see that x_4 has one value, *red*; it is selected and assigned the value. Then x_2 can be selected and assigned its only consistent value, *blue*. Propagating this assignment does not further shrink any future domain. Next, x_5 can be selected and assigned *green*. The solution is then completed, without dead-ends, by assigning *red* or *teal* to x_6. □

Algorithm DVO (DVFC)

```
procedure DVFC
Input: A constraint network R = (X, D, C)
Output: Either a solution, or notification that the network is inconsistent.
    D'_i ← D_i for 1 ≤ i ≤ n          (copy all domains)
    i ← 1                             (initialize variable counter)
          s = min_{i<j≤n} |D'_j|      (find future var with smallest domain)
          x_{i+1} ← x_s (rearrange variables so that x_s follows x_i)
    while 1 ≤ i ≤ n
      instantiate x_i ← SELECTVALUE-FORWARD-CHECKING
      if x_i is null                  (no value was returned)
        reset each D' set to its value before x_i was last instantiated
        i ← i − 1                     (backtrack)
      else
        if i < n
        i ← i + 1                     (step forward to x_s)
          s = min_{i<j≤n} |D'_j|      (find future var with smallest domain)
          x_{i+1} ← x_s (rearrange variables so that x_s follows x_i)
        i ← i + 1                     (step forward to x_s)
    end while
    if i = 0
      return "inconsistent"
    else
      return instantiated values of {x_1, . . . , x_n}
end procedure
```

Figure 5.12: The DVFC algorithm. It uses the SELECTVALUE-FORWARD-CHECKING subprocedure given in Fig. 5.8.

DVO: Dynamic variable ordering, more involved heuristics

- *dom*: choose a variable with min domain
- *deg*: choose variable with max degree
- *dom+deg*: dom and break ties with max degree
- *dom/deg* (Bessiere and Ragin, 96): choose min dom/deg
- *dom/wdeg*: domain divided by weighted degree. Constraints are weighted as they get involved in more conflicts. wdeg: sum the weights of all constraints that touch x.

Implementing look-aheads

- Cost of node generation should be reduced
- Solution: keep a table of viable domains for each variable and each level in the tree.

- Space complexity
- Node generation = table updating

Outline

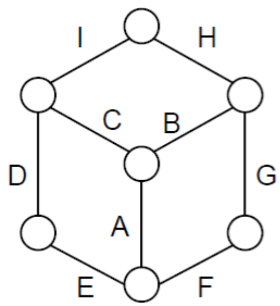
- The search tree for CSPs, Variable ordering an consistency level
- Look-ahead for value selection:
  - Forward checking,
  - Full-arc-consistency,
  - partial look-ahead,
  - maintaining arc-consistency
- Dynamic Variable ordering (DVO, DVFC)
- Search for Satisfiability
- Converting a CSP into a SAT problem

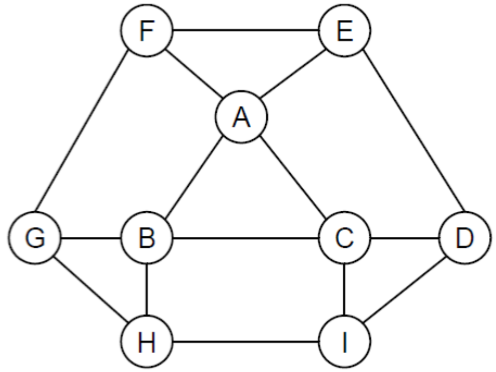
Branching strategies (selecting the search space)

(see vanBeek, chapter 4 in Handbook)

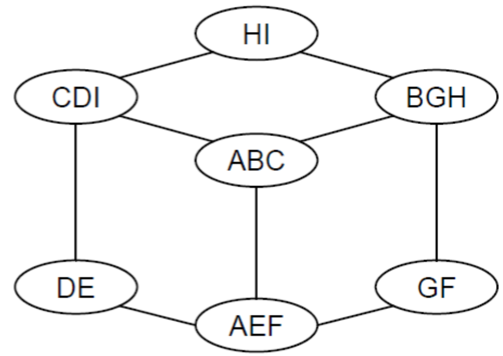
- Enumeration branching: the naïve backtracking search choice
- A branching strategy in the search tree: a set of branching constraints p($b_1, \ldots b_j$} where b_i is a branching constraint
- Branches are often ordered using a heuristic.
- To ensure completeness, the constraints that are ordered on the branches should be exclusive and exhaustive.
- Most common are unary constraints:
  - Enumeration: (x=1,x=2,x=3…)
  - Binary choices: (x=1, x != 1)
  - Domain spliting: (x>3,x<3)
- Using domain-specific formulas
  - Scheduling: one job before or after: (x_1 +d_1 < x_2, x_2+d_2 < x_1)
  - Can be simulated by auxiliary variables.
  - Searching the dual problem
  - Formula-based splitting in SAT

Branching on the dual graph

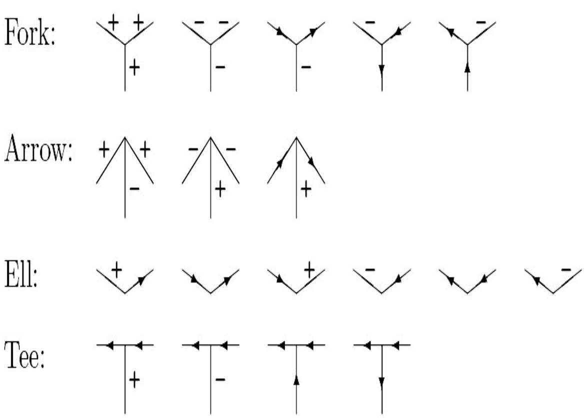


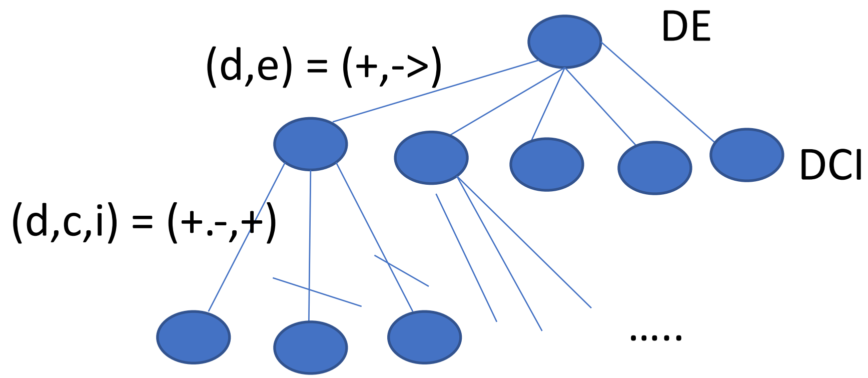


Primal graph



Dual graph





Randomization

- Randomized variable selection (for tie breaking rule)
- Randomized value selection (for tie breaking rule)
- Random restarts with increasing time-cutoff
- Capitalizing on huge performance variance
- All modern SAT solvers that are competitive use restarts.

The cycle-cutset effect
(relationship of look-ahead to some graph structure)

- A cycle-cutset is a subset of nodes in an undirected graph whose removal results in a graph with no cycles
- A constraint problem whose graph has a cycle-cutset of size c can be solved by partial look-ahead in time
- Question: what is the size of the search space when the cycle-cutset has size: 1 (cycle),2,5…

Extensions to stronger look-ahead

- Extend to path-consistency or i-consistency or generalized-arc-consistency

Definition 5.3.7 (general arc-consistency) *Given a constraint* $C = (R, S)$ *and a variable* $x \in S$, *a value* $a \in D_x$ *is supported in* C *if there is a tuple* $t \in R$ *such that* $t[x] = a$. t *is then called a support for* $< x, a >$ *in* C. C *is arc-consistent if for each variable* x, *in its scope and each of its values,* $a \in D_x$, $< x, a >$ *has a support in* C. *A CSP is arc-consistent if each of its constraints is arc-consistent.*

Search for SAT

What is SAT?

Given a sentence:

- ***Sentence***: conjunction of clauses
- ***Clause***: disjunction of literals
- ***Literal***: a term or its negation
- ***Term***: Boolean variable

Question: Find an assignment of truth values to the Boolean variables such the sentence is satisfied.

SAT (continued)
from Darwiche chapter 3

- Representation:

$$(A \vee B \vee \neg C) \wedge (\neg A \vee D) \wedge (B \vee C \vee D)$$

A convenient way to notate sentences in CNF is using sets. Specifically, a clause $l_1 \vee l_2 \vee \ldots \vee l_m$ is expressed as a set of literals $\{l_1, l_2, \ldots, l_m\}$. Moreover, a conjunctive normal form $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$ is expressed as a set of clauses $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. For example, the CNF given above would be expressed as:

$$\{\{A, B, \neg C\}, \{\neg A, D\}, \{B, C, D\}\}.$$

Resolution

1. $\{\neg P, R\}$
2. $\{\neg Q, R\}$
3. $\{\neg R\}$
4. $\{P, Q\}$

5.	$\{\neg P\}$	1, 3
6.	$\{\neg Q\}$	2, 3
7.	$\{Q\}$	4, 5
8.	$\{\}$	6, 7

1:(P → R)

2: (Q→ R)

4: (~P →Q)

The clauses before the line represent initial clauses, while clauses below the line represent resolvents, together with the identifiers of clauses used to obtain them. The above resolution trace shows that we can derive the empty clause from the initial set of Clauses (1–4). Hence, the original clauses, together, are unsatisfiable.

DP (Davis Putnam) or directional resolution (Dechter and Rish, 1994)

The DP algorithm, also known as *directional resolution* [DR94], uses the above observation to existentially quantify all variables from a CNF, one at a time. One way to implement the DP algorithm is using a mechanism known as *bucket elimination* [Dec97], which proceeds in two stages: *constructing and filling* a set of buckets, and then *processing* them in some order. Specifically, given a variable ordering π, we construct and fill buckets as follows:

- A *bucket* is constructed for each variable P and is *labeled* with variable P.
- Buckets are sorted top to bottom by their labels according to order π.
- Each clause α in the CNF is added to the first Bucket P from the top, such that variable P appears in clause α.

$$\Delta = \{ \{\neg A, B\}, \{\neg A, C\}, \{\neg B, D\}, \{\neg C, \neg D\}, \{A, \neg C, E\} \},$$

and the variable order C, B, A, D, E. Constructing and filling buckets leads to:[3]

$$\begin{aligned} C &: \{\neg A, C\}, \{\neg C, \neg D\}, \{A, \neg C, E\} \\ B &: \{\neg A, B\}, \{\neg B, D\} \\ A &: \\ D &: \\ E &: \end{aligned}$$

DP (continued)

to Bucket A:

$$\begin{array}{ll} C: \{\neg A, C\},\ \{\neg C, \neg D\},\ \{A, \neg C, E\} & \\ B: \{\neg A, B\},\ \{\neg B, D\} & \\ A: & \{\neg A, \neg D\} \\ D: & \\ E: & \end{array}$$

The buckets below Bucket C will now contain the result of existentially quantifying variable C. Processing Bucket B adds one B–resolvent to Bucket A:

$$\begin{array}{ll} C: \{\neg A, C\},\ \{\neg C, \neg D\},\ \{A, \neg C, E\} & \\ B: \{\neg A, B\},\ \{\neg B, D\} & \\ A: & \{\neg A, \neg D\},\ \{\neg A, D\} \\ D: & \\ E: & \end{array}$$

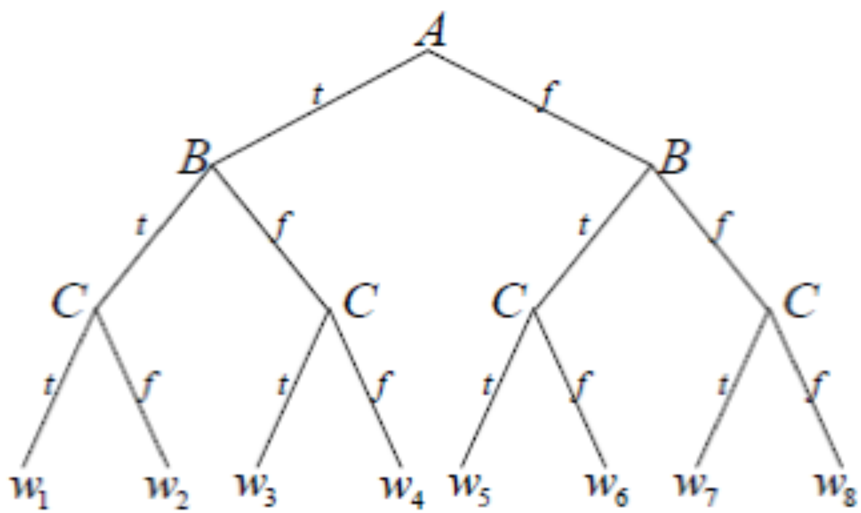


Figure 3.3. A search tree for enumerating all truth assignments over variables A, B and C.

Look-ahead for sat: DPLL

(Davis-Putnam, Logeman and Laveland, 1962)

DPLL(φ)
Input: A cnf theory φ
Output: A decision of whether φ is satisfiable.
1. Unit_propagate(φ);
2. If the empty clause is generated, return(*false*);
3. Else, if all variables are assigned, return(*true*);
4. Else
5. Q = some unassigned variable;
6. return(**DPLL**($\varphi \wedge Q$) $\vee$
DPLL($\varphi \wedge \neg Q$))

Figure 5.13: The DPLL Procedure

Boolean constraint propagation

Procedure UNIT-PROPAGATION
Input: A cnf theory, φ, $d = Q_1, ..., Q_n$.
Output: An equivalent theory such that every unit clause does not appear in any non-unit clause.

1. queue = all unit clauses.
2. **while** queue is not empty, do.
3. $\quad T \leftarrow$ next unit clause from Queue.
4. $\quad$ **for** every clause β containing T or $\neg T$
5. $\quad\quad$ **if** β contains T delete β (subsumption elimination)
6. $\quad\quad$ **else**, For each clause $\gamma = resolve(\beta, T)$.
   $\quad\quad$ **if** γ, the resolvent, is empty, the theory is unsatisfiable.
7. $\quad\quad$ **else**, add the resolvent γ to the theory and delete β.
   $\quad\quad$ **if** γ is a unit clause, add to Queue.
8. $\quad$ **endfor**.
9. **endwhile**.

Theorem 3.6.1 *Algorithm* UNIT-PROPAGATION *has a linear time complexity.*

Example of DPLL

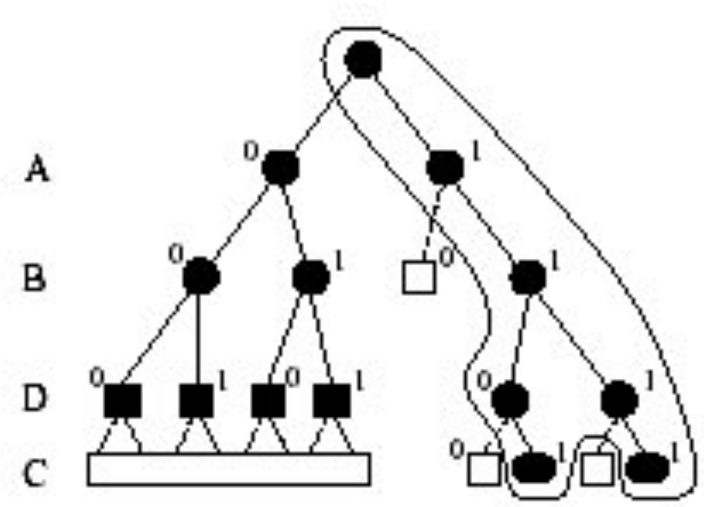


Figure 5.14: A backtracking search tree along the variables A, B, D, C for a cnf theory $\varphi = \{(\neg A \vee B), (\neg C \vee A), (A \vee B \vee D), C\}$. Hollow nodes and bars in the search tree represent illegal states, triangles represent solutions. The enclosed area corresponds to DPLL with unit-propagation.

Using Conditioned CNF at each node

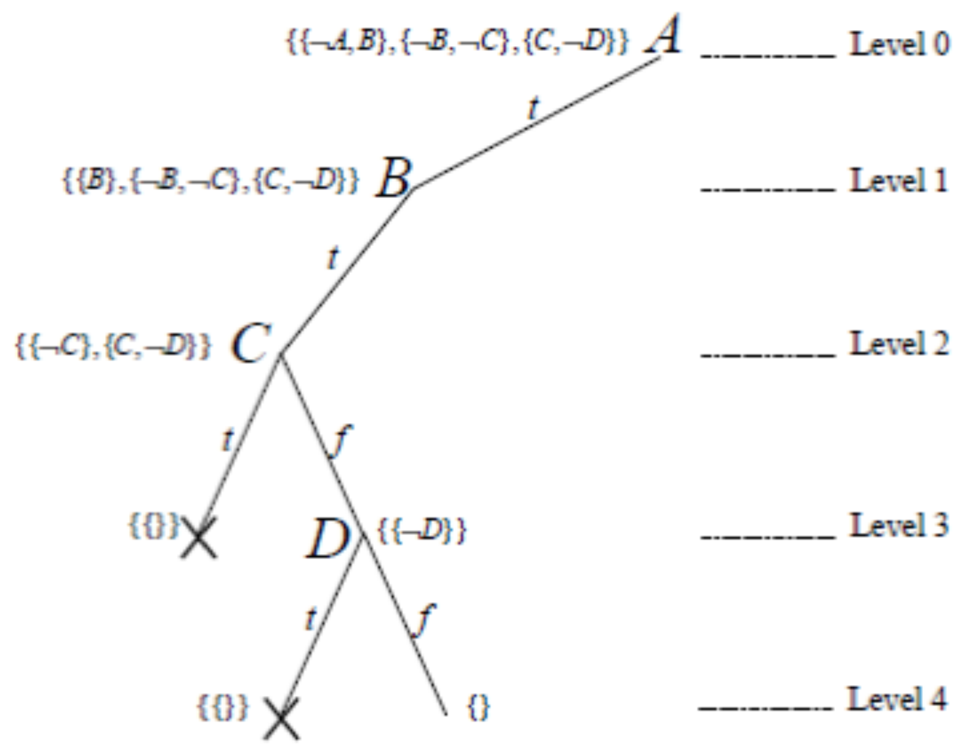


Figure 3.5. A termination tree, where each node is labelled by the corresponding CNF. The last node visited during the search is labelled with {}. The label × indicates the detection of a contradiction at the corresponding node.

On Unit Resolution

To incorporate unit resolution into our satisfiability algorithms, we will introduce a function UNIT-RESOLUTION, which applies to a CNF Δ and returns two results:

- $\mathbf{I}$: a set of literals that were either present as unit clauses in Δ, or were derived from Δ by unit resolution.
- Γ: a new CNF which results from conditioning Δ on literals $\mathbf{I}$.

For example, if the CNF

$$\Delta = \{ \{\neg A, \neg B\}, \{B, C\}, \{\neg C, D\}, \{A\} \},$$

then $\mathbf{I} = \{A, \neg B, C, D\}$ and $\Gamma = \{\}$. Moreover, if

$$\Delta = \{ \{\neg A, \neg B\}, \{B, C\}, \{\neg C, D\}, \{C\} \},$$

then $\mathbf{I} = \{C, D\}$ and $\Gamma = \{ \{\neg A, \neg B\} \}$. Unit resolution is a very important component of search-based SAT solving algorithms. Part 1, Chapter 4 discusses in details the modern implementation of unit resolution employed by many SAT solvers of this type.

Chronological Backtracking

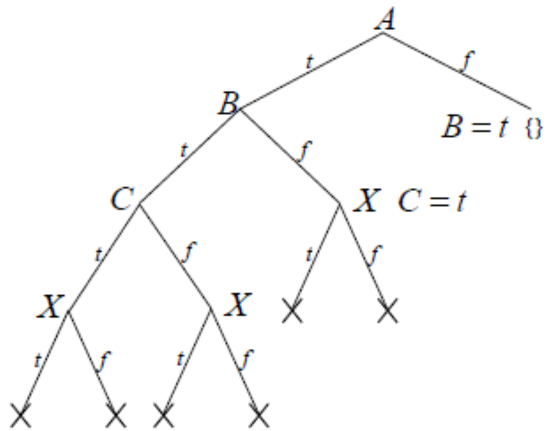


Figure 3.6. A termination tree. Assignments shown next to nodes are derived using unit resolution.

To consider a concrete example, let us look at how standard DPLL behaves on the following CNF, assuming a variable ordering of A, B, C, X, Y, Z:

$$\Delta = \begin{array}{l} 1.\ \{A, B\} \\ 2.\ \{B, C\} \\ 3.\ \{\neg A, \neg X, Y\} \\ 4.\ \{\neg A, X, Z\} \\ 5.\ \{\neg A, \neg Y, Z\} \\ 6.\ \{\neg A, X, \neg Z\} \\ 7.\ \{\neg A, \neg Y, \neg Z\} \end{array} \qquad (3.1)$$

Reduction from CSP to SAT

Example: CSP into SAT

Notation: variable-value pair = **vvp**

- vvp $\rightarrow$ term
  - V_1 = {a, b, c, d} yields $x_1 = (V_1, a)$, $x_2 = (V_1, b)$, $x_3 = (V_1, c)$, $x_4 = (V_1, d)$,
  - V_2 = {a, b, c} yields $x_5 = (V_2, a)$, $x_6 = (V_2, b)$, $x_7 = (V_2, c)$.
- The vvp's of a variable $\rightarrow$ disjunction of terms
  - V_1 = {a, b, c, d} yields
- (How do we express: "At most one VVP per variable "

CSP into SAT (cont.)

Constraint:

1. Way 1: Each inconsistent tuple → one disjunctive clause
   - For example: how many?

2. Way 2:
   a) Consistent tuple → conjunction of terms
   b) Each constraint → disjunction of these conjunctions

→ transform into conjunctive normal form (CNF)

Question: find a truth assignment of the Boolean variables such that the sentence is satisfied

Outline

- The search tree for CSPs, Variable ordering an consistency level
- Look-ahead for value selection:
  - Forward checking,
  - Full-arc-consistency,
  - partial look-ahead,
  - maintaining arc-consistency
- Dynamic Variable ordering (DVO, DVFC)
- Search for Satisfiability
- Converting a CSP into a SAT problem