# CompSci 275, CONSTRAINT Networks

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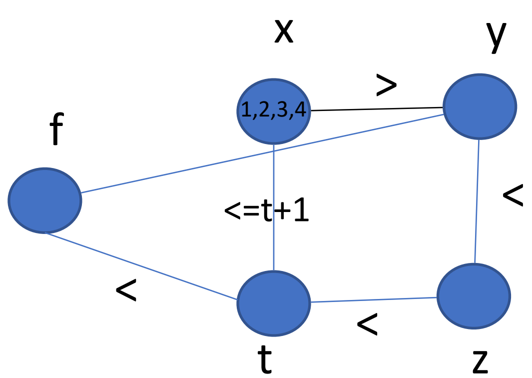
Consistency algorithms, part b
Chapter 3

# Outline

- Arc-consistency algorithms
- Path-consistency and i-consistency
- Arc-consistency, Generalized arc-consistency, relation arc-consistency
- Global and bound consistency
- Distributed (generalized) arc-consistency
- Consistency operators: join, resolution, Gausian elimination

# Exercise: make the following network arc-consistent

- Draw the network's primal and dual constraint graph
- Network =
  - Domains {1,2,3,4}
  - Constraints: y < x, z < y, t < z, f<t, x<=t+1, Y<f+2
  - What is the domain for X in an arc-consistent network?



# Arc-consistency Algorithms

- **AC-1**: brute-force, distributed
- **AC-3**, queue-based
- **AC-4**, context-based, optimal
- **AC-5,6,7**,…. Good in special cases
- **Important:** applied at every node of search
- (*n* number of variables, *e*=#constraints, *k*=domain size)
- Mackworth and Freuder (1977,1983), Mohr and Anderson, (1985) …

# Constraint tightness analysis

*t = number of tuples bounding a constraint*

- **AC-1**: brute-force,
- **AC-3**, queue-based
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- **Important:** applied at every node of search
- (n number of variables, e=#constraints, k=domain size)
- Mackworth and Freuder (1977,1983), Mohr and Anderson, (1985)…

# Is arc-consistency enough?

- Example: a triangle graph-coloring with 2 values.
  - Is it arc-consistent?
  - Is it consistent?
- It is not path, or 3-consistent.

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# Path-consistency

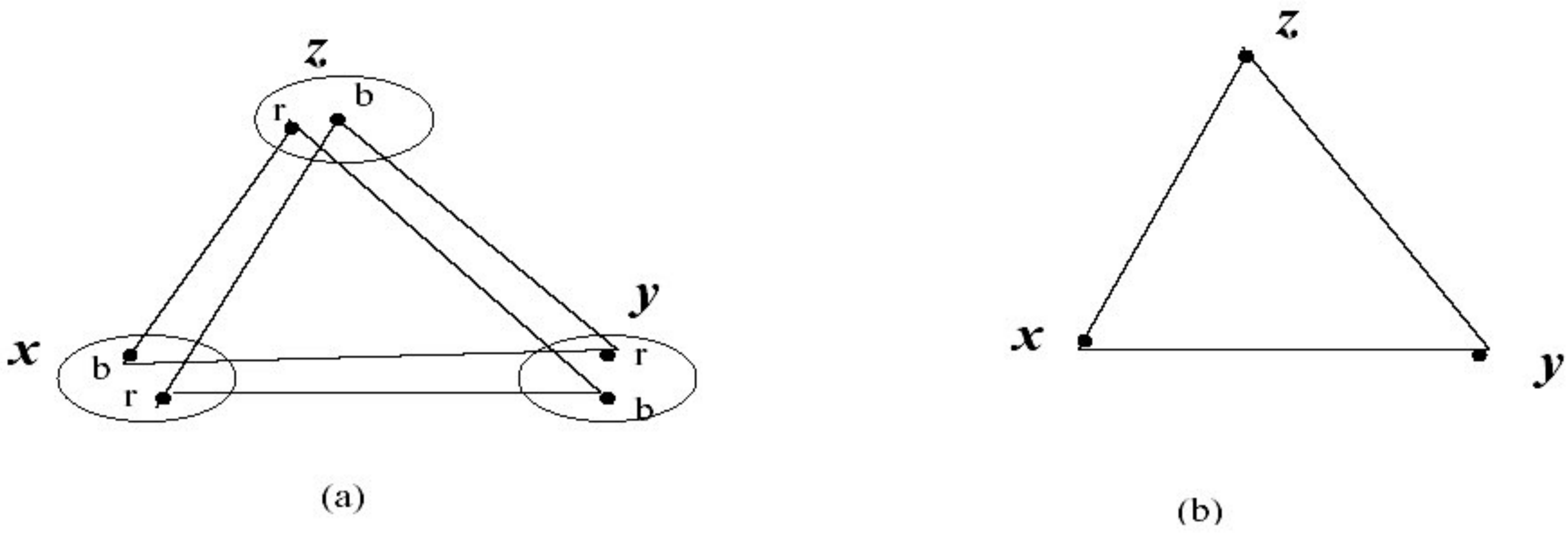



Figure 3.8: (a) The matching diagram of a 2-value graph coloring problem. (b) Graphical picture of path-consistency using the matching diagram.

# Path-consistency (3-consistency)

**Definition 3.3.2 (Path-consistency)** *Given a constraint network* $\mathcal{R} = (X, D, C)$, *a two variable set* $\{x_i, x_j\}$ *is path-consistent relative to variable* $x_k$ *if and only if for every consistent assignment* $(< x_i, a_i >, < x_j, a_j >)$ *there is a value* $a_k \in D_k$ *s.t. the assignment* $(< x_i, a_i >, < x_k, a_k >)$ *is consistent and* $(< x_k, a_k >, < x_j, a_j >)$ *is consistent. Alternatively, a binary constraint* $R_{ij}$ *is path-consistent relative to* $x_k$ *iff for every pair* $(a_i, a_j), \in R_{ij}$, *where* $a_i$ *and* $a_j$ *are from their respective domains, there is a value* $a_k \in D_k$ *s.t.* $(a_i, a_k) \in R_{ik}$ *and* $(a_k, a_j) \in R_{kj}$. *A subnetwork over three variables* $\{x_i, x_j, x_k\}$ *is path-consistent iff for any permutation of* $(i, j, k)$, $R_{ij}$ *is path consistent relative to* $x_k$. *A network is path-consistent iff for every* $R_{ij}$ *(including universal binary relations) and for every* $k \neq i, j$ $R_{ij}$ *is path-consistent relative to* $x_k$.

# Revise-3

REVISE-3$((x,y),z)$

**input**: a three-variable subnetwork over $(x,y,z)$, $R_{xy}$, $R_{yz}$, $R_{xz}$.
**output:** revised $R_{xy}$ path-consistent with $z$.

1. **for** each pair $(a,b) \in R_{xy}$
2. **if** no value $c \in D_z$ exists such that $(a,c) \in R_{xz}$ and $(b,c) \in R_{yz}$
3. **then** delete $(a,b)$ from $R_{xy}$.
4. **endif**
5. **endfor**

Figure 3.9: Revise-3

$$R_{xy} \leftarrow R_{xy} \cap \pi_{xy}(R_{xz} \bowtie D_z \bowtie R_{zy})$$

- Complexity: O($k^3$)
- Best-case: O(t)
- Worst-case O(tk)

# Revise3 = join followed by project

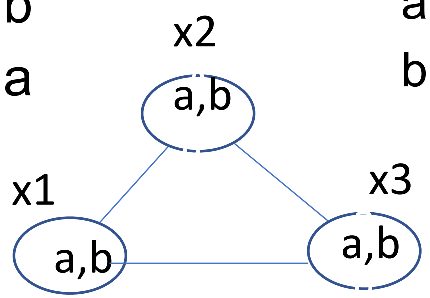
- Join :

| $x_1$ | $x_2$ |
|---|---|
| a | a |
| b | b |

$\bowtie$

| $x_2$ | $x_3$ |
|---|---|
| a | a |
| a | b |
| b | a |

| $x_1$ | $x_2$ | $x_3$ |
|---|---|---|
| a | a | a |
| a | a | b |
| b | b | a |




$\bowtie$

$R_{X_1X_3}$ = {(a,a),(a,b),(b,a)}

# PC-1

PC-1($\mathcal{R}$)

**input:** a network $\mathcal{R} = (X, D, C)$.

**output:** a path consistent network equivalent to $\mathcal{R}$.

**repeat**
    **for** $k \leftarrow 1$ to $n$
        **for** $i, j \leftarrow 1$ to $n$
            $R_{ij} \leftarrow R_{ij} \cap \pi_{ij}(R_{ik} \bowtie D_k \bowtie R_{kj})/^* \ (Revise - 3((i,j),k))$
        **endfor**
    **endfor**
**until** no constraint is changed.

Figure 3.10: Path-consistency-1 (PC-1)

- Complexity:
- O($n^3$) triplets, each take O($k^3$) steps → O($n^3k^3$)
- Max number of loops: O($n^2\ k^2$) .

# PC-2

PC-2($\mathcal{R}$)

**input:** a network $\mathcal{R} = (X, D, C)$.

**output:** $\mathcal{R}'$ a path consistent network equivalent to $\mathcal{R}$.

$Q \leftarrow \{(i,k,j) \mid 1 \le i < j \le n, 1 \le k \le n, k \ne i, k \ne j\}$

**while** $Q$ is not empty

  select and delete a 3-tuple $(i,k,j)$ from $Q$

  $R_{ij} \leftarrow R_{ij} \cap \pi_{ij}(R_{ik} \bowtie D_k \bowtie R_{kj})$ /* (Revise-3($(i,j),k$))

  **if** $R_{ij}$ changed then

  $Q \leftarrow Q \cup \{(l,i,j)(l,j,i) \mid 1 \le l \le n, l \ne i, l \ne j\}$

**endwhile**

- Complexity:
- Optimal PC-4:
- (each pair of values deleted may add: 2n-1 triplets, number of pairs: O($n^2\ k^2$) → size of Q is O($n^3\ k^2$), processing is O($k^3$) yielding the result)

# Example: before and after path-consistency

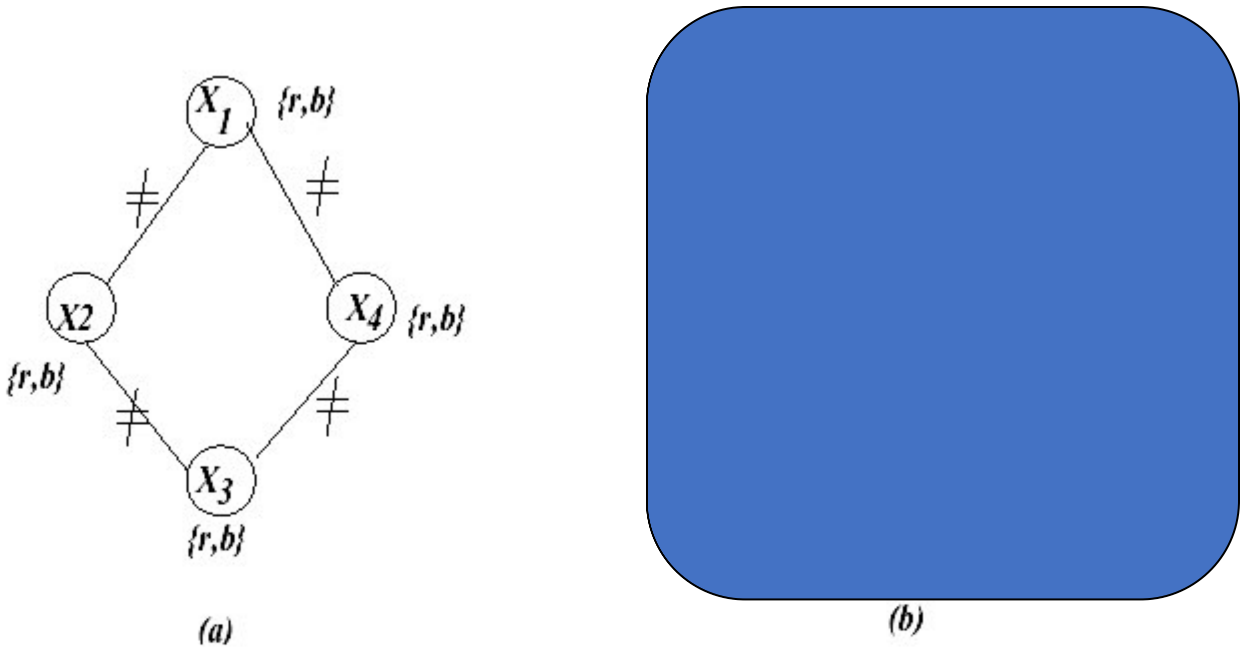



Figure 3.12: A graph-coloring graph (a) before path-consistency (b) after path-consistency

- PC-1 requires 2 processing of each arc while PC-2 may not
- Can we do path-consistency distributedly?

# Example: before and after path-consistency

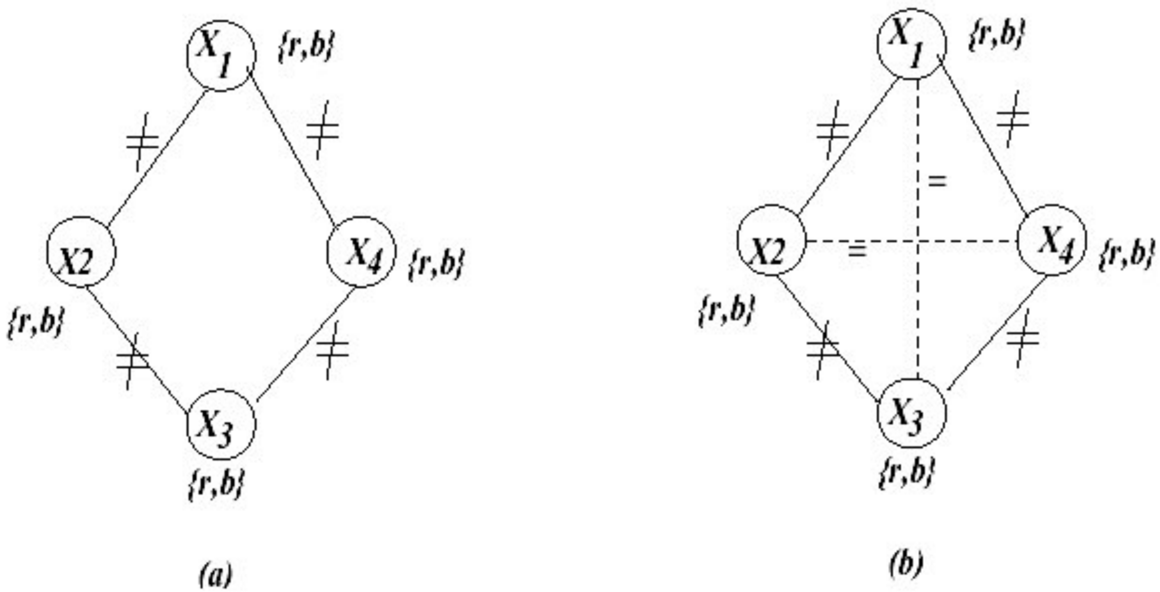



Figure 3.12: A graph-coloring graph (a) before path-consistency (b) after path-consistency

- PC-1 requires 2 processings of each arc while PC-2 may not
- Can we do path-consistency distributedly?

# Path-consistency Algorithms

- Apply Revise-3 $O(k^3)$ until no change

$$R_{ij} \leftarrow R_{ij} \cap \pi_{ij}(R_{ik} \bowtie D_k \bowtie R_{kj})$$

- Path-consistency (3-consistency) adds binary constraints.
- PC-1:
- PC-2:
- PC-4 optimal:

# I-consistency

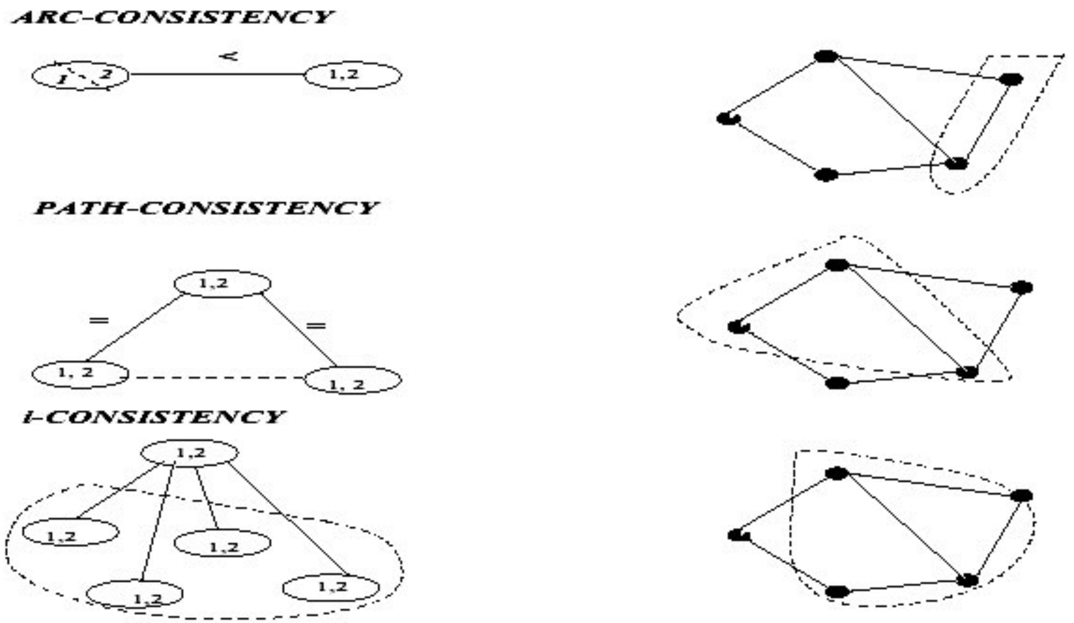



Figure 3.17: The scope of consistency enforcing: (a) arc-consistency, (b) path-consistency, (c) i-consistency

# Higher levels of consistency, global-consistency

Definition:

t. *A network is* $i$-consistent *iff given any consistent instantiation of any $i-1$ distinct variables, there exists an instantiation of any $i$th variable such that the $i$ values taken together satisfy all of the constraints among the $i$ variables. A network is* strongly $i$-consistent *iff it is $j$-consistent for all $j \leq i$. A strongly $n$-consistent network, where $n$ is the number of variables in the network, is called* globally consistent.

**A Globally consistent network is backtrack-free**

# Revise-i

REVISE-$i(\{x_1, x_2, ...., x_{i-1}\}, x_i)$

**input:** a network $\mathcal{R} = (X, D, C)$

**output:** a constraint $R_S$, $S = \{x_1, ...., x_{i-1}\}$ $i$-consistent relative to $x_i$.

1. **for** each instantiation $\bar{a}_{i-1} = (< x_1, a_1 >, < x_2, a_2 >, ..., < x_{i-1}, a_{i-1} >)$ do,
2. **if** no value of $a_i \in D_i$ exists s.t. $(\bar{a}_{i-1}, a_i)$ is consistent

   **then** delete $\bar{a}_{i-1}$ from $R_S$

   (Alternatively, let $\mathcal{S}$ be the set of all subsets of $\{x_1, ..., x_i\}$ that contain $x_i$ and appear as scopes of constraints of $\mathcal{R}$, then

   $R_S \leftarrow R_S \cap \pi_S(\bowtie_{S' \subseteq \mathcal{S}} R_{S'}))$
3. **endfor**

Figure 3.14: Revise-i

- Complexity: for binary constraints $O(k^i)$
- For arbitrary constraints:
- (because there may be $O(2^i)$ constraints to test per tuple)

# 4-queen example

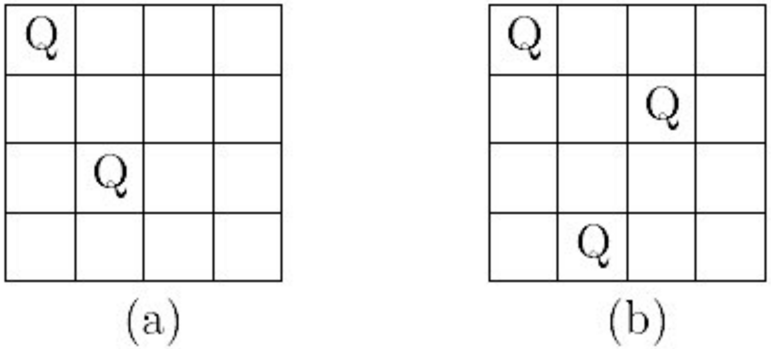



Figure 3.13: (a) Not 3-consistent; (b) Not 4-consistent

# i-consistency

I-CONSISTENCY$(\mathcal{R})$

**input:** a network $\mathcal{R}$.

**output:** an i-consistent network equivalent to $\mathcal{R}$.

1. **repeat**
2. **for** every subset $S \subseteq X$ of size $i-1$, and for every $x_i$, do
3. let $\mathcal{S}$ be the set of all subsets in of $\{x_1, ..., x_i\}$ *scheme*$(\mathcal{R})$ that contain $x_i$
4. $R_S \leftarrow R_S \cap \pi_S(\bowtie_{S' \in \mathcal{S}} R_{S'})$ ( this is Revise-i$(S, x_i)$)
6. **endfor**
7. **until** no constraint is changed.

*This S is different it is all subsets of size I*
*That includes xi*

Figure 3.15: i-consistency-1

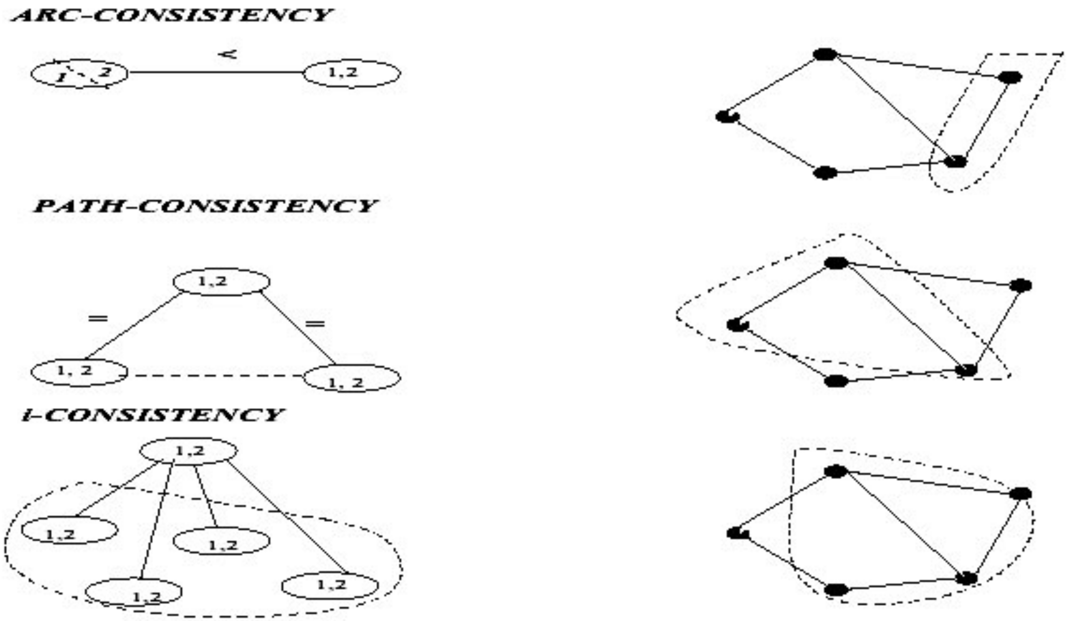
**Theorem 3.4.3 (complexity of i-consistency)** *The time and space complexity of brute-force i-consistency $O(2^i(nk)^{2i})$ and $O(n^ik^i)$, respectively. A lower bound for enforcing i-consistency is $\Omega(n^ik^i)$.* □

# Path-consistency vs 3-consistency

**Example 3.4.4** Suppose a constraint network involves three variables $\boldsymbol{x}, \boldsymbol{y}, z$ having domains $\{0, 1\}$ and a single ternary constraint $\boldsymbol{R_{xyz}} = \{(0, 0, 0)\}$. Application of the path-consistency algorithm will produce nothing since there are no binary constraints to test; the network is already path-consistent. However, the network is *not* 3-consistent. While we can assign the values $(\langle \boldsymbol{x}, 1\rangle, \langle \boldsymbol{y}, 1\rangle)$ (since there is no constraint), we cannot extend this assignment to $z$ in a way that satisfies the given ternary constraints. Indeed, if we

apply 3-consistency to this network we will add the constraint $R_{xy} = \{(< x, 0 >< y, 0 >)\}$ in addition to the constraint $R_x = \{(< x, 0 >)\}$. □

# i-consistency

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## Generalized arc-consistency (GAC) for non-binary constraints

**Definition 3.5.1 (generalized arc-consistency)** *Given a constraint network* $\mathcal{R} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$, *with* $R_S \in C$, *a variable* $x$ *is* arc-consistent *relative to* $R_S$ *if and only if for every value* $a \in D_x$ *there exists a tuple* $t \in R_S$ *such that* $t[x] = a$. $t$ *can be called a support for* $a$. *The constraint* $R_S$ *is called arc-consistent iff it is arc-consistent relative to each of the variables in its scope and a constraint network is arc-consistent if all its constraints are arc-consistent.*

$$D_x \leftarrow D_x \cap \pi_x(R_S \bowtie D_{S-x})$$

Complexity: O(t k), t bounds number of tuples.

Relational arc-consistency (different than GAC):

$$R_{S-\{x\}} \leftarrow \pi_{S-\{x\}}(R_S \bowtie D_x).$$

**Algorithm 1**: AC3 / GAC3

**function** `Revise3`(**in** $x_i$: variable; $c$: constraint)**: Boolean** ;
  **begin**
    CHANGE $\leftarrow$ **false**;
    **foreach** $v_i \in D(x_i)$ **do**
      **if** $\nexists \tau \in c \cap \pi_{X(c)}(D)$ *with* $\tau[x_i] = v_i$ **then**
        remove $v_i$ from $D(x_i)$;
        CHANGE $\leftarrow$ **true**;
    return CHANGE ;
  **end**

**function** `AC3/GAC3`(**in** $X$: set)**: Boolean** ;
  **begin**
    /* initalisation */;
    $Q \leftarrow \{(x_i, c) \mid c \in C, x_i \in X(c)\}$;
    /* propagation */;
    **while** $Q \neq \emptyset$ **do**
      select and remove $(x_i, c)$ from $Q$;
      **if** *Revise*$(x_i, c)$ **then**
        **if** $D(x_i) = \emptyset$ **then** return **false** ;
        **else** $Q \leftarrow Q \cup \{(x_j, c') \mid c' \in C \wedge c' \neq c \wedge x_i, x_j \in X(c') \wedge j \neq i\}$;
    return **true** ;
  **end**

# Generalized arc-consistency

**Proposition 27 (GAC3).** *GAC3 is a sound and complete algorithm for achieving arc consistency that runs in* $O(er^3d^{r+1})$ *time and* $O(er)$ *space, where* $r$ *is the greatest arity among constraints.*

# Examples of generalized AC and relational AC

- $x+y+z \leq 15$ and $z \geq 13$ implies

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$x \leq 1, y \leq 1$

- Example of relational arc-consistency

Here given the 2 top Boolean constraints we infer the 3rd.

# Examples: of generalized AC

- $x+y+z \leq 15$ and $z \geq 13$ implies

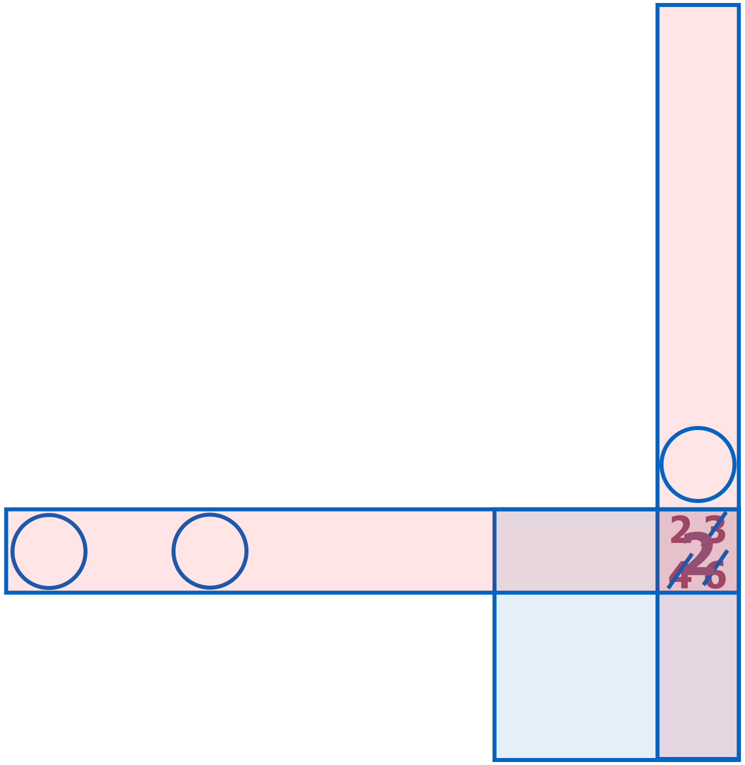
$x \leq 2$, $y \leq 2$

- Example of relational arc-consistency

Here given the 2 top Boolean constraints we infer the 3rd.

# Sudoku

- Constraint Propagation
- Inference




- Variables: empty slots
- Domains = {1,2,3,4,5,6,7,8,9}
- Constraints: 27 all-different

Each row, column and major block must be alldifferent

"Well posed" if it has unique solution: 27 constraints

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# More arc-based consistency

- Global constraints: e.g., all-different constraints
  - Special semantic constraints that appears often in practice and a specialized constraint propagation. Used in constraint programming.
- Bounds-consistency: pruning the boundaries of domains

# Global constraints

Constraints of arbitrary scope length defined by expression, a Boolean function

*Global* constraints are classes of constraints defined by a formula of arbitrary arity (see Section 9.2).

**Example 2.** The constraint **alldifferent**$(x_1, x_2, x_3) \equiv (v_i \neq v_j \wedge v_i \neq v_k \wedge v_j \neq v_k)$ allows the infinite set of 3-tuples in $\mathbb{Z}^3$ such that all values are different. The constraint $c(x_1, x_2, x_3) = \{(2,2,3), (2,3,2), (2,3,3), (3,2,2), (3,2,3), (3,3,2)\}$ allows the finite set of 3-tuples containing both values 2 and 3 and only them.

# Global constraints

**Example 86.** The `alldifferent`$(x_1, \ldots, x_n)$ global constraint is the class of constraints that are defined on any sequence of $n$ variables, $n \geq 2$, such that $x_i \neq x_j$ for all $i, j, 1 \leq i, j \leq n, i \neq j$. The `NValue`$(y, [x_1, \ldots, x_n])$ global constraint is the class of constraints that are defined on any sequence of $n+1$ variables, $n \geq 1$, such that $|\{x_i \mid 1 \leq i \leq n\}| = y$ [100, 8].

We need specialized procedures for generalize Arc-consistency because it is too expensive to try and apply the general algorithm (see Bessiere, section 9.2)

We can decompose a global constraint, or use various specialized representation

# Example for alldiff

- A = {3,4,5,6}
- B = {3,4}
- C= {2,3,4,5}
- D= {2,3,4}
- E = {3,4}
- F= {1,2,3,4,5,6}
- Alldiff (A,B,C,D,E)
- Arc-consistency does nothing
- Apply GAC to sol(A,B,C,D,E,F)?
- → A = {6}, F = {1}….
- Alg: bipartite matching kn^1.5
- (Lopez-Ortiz, et. Al, IJCAI-03 pp 245 (A fast and simple algorithm for bounds consistency of alldifferent constraint)

# Global constraints

- Alldifferent
- Sum constraint (variable equal the sum of others)
- Global cardinality constraint (a value can be assigned a bounded number of times to a set of variables)
- The cummulative constraint (related to scheduling tasks)

*In summary, a global constraint $C = \{C(i)\}$ is a family of scope-parameterized constraints, (normally $i \geq 2$), where $C(i)$ is a constraint whose relation is often defined implicitly by either a natural language statement, or as a set of solutions to a subproblem defined by lower arity explicit constraints (e.g., alldifferent). It is associated with one or more specialized propagation algorithms trying to achieve generalized arc-consistency relative to $C(i)$ (or an approximation of it) in a way that is more efficient than a brute-force approach.*

# Bounds consistency

**Definition 3.5.4 (bounds consistency)** *Given a constraint $C$ over a scope $S$ and domain constraints, a variable $x \in S$ is bounds-consistent relative to $C$ if the value $\mathbf{min}\{D_x\}$ (respectively, $\mathbf{max}\{D_x\}$) can be extended to a full tuple $t$ of $C$. We say that $t$ supports $\mathbf{min}\{D_x\}$. A constraint $C$ is* bounds-consistent *if each of its variables is bounds-consistent.*

# Bounds consistency

**Example 3.5.5** Consider the constraint problem with variables $x_1, ...x_6$, each with domains $1, ..., 6$, and constraints:

$$C_1 : x_4 \geq x_1 + 3, \quad C_2 : x_4 \geq x_2 + 3, \quad C_3 : x_5 \geq x_3 + 3, \quad C_4 : x_5 \geq x_4 + 1,$$

$$C_5 : alldifferent\{x_1, x_2, x_3, x_4, x_5\}$$

The constraints are not bounds consistent. For example, the minimum value 1 in the domain of $x_4$ does not have support in constraint $C_1$ as there is no corresponding value for $x_1$ that satisfies the constraint. Enforcing bounds consistency using constraints $C_1$ through $C_4$ reduces the domains of the variables as follows: $D_1 = \{1, 2\}$, $D_2 = \{1, 2\}$, $D_3 = \{1, 2, 3\}$ $D_4 = \{4, 5\}$ and $D_5 = \{5, 6\}$. Subsequently, enforcing bounds consistency using constraints $C_5$ further reduces the domain of $C$ to $D_3 = \{3\}$.Now constraint $C_3$ is no longer bound consistent. Reestablishing bounds consistency causes the domain of $x_5$ to be reduced to $\{6\}$. Is the resulting problem already arc-consistent? $\square$

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# Boolean constraint propagation

- $(A \vee \neg B)$ and $(B)$
  - B is arc-consistent relative to A but not vice-versa
- Arc-consistency by resolution:

  $res((A \vee \neg B),B) = A$

Given also $(B \vee C)$, path-consistency:

$res((A \vee \neg B),(B \vee C) = (A \vee C)$

Relational arc-consistency rule = unit-resolution

# Boolean constraint propagation

**Procedure** UNIT-PROPAGATION
**Input:** A cnf theory, $\varphi$, $d = Q_1, ..., Q_n$.
**Output:** An equivalent theory such that every unit clause does not appear in any non-unit clause.

```
1. queue = all unit clauses.
2. while queue is not empty, do.
3.     T ← next unit clause from Queue.
4.     for every clause β containing T or ¬T
5.         if β contains T delete β (subsumption elimination)
6.         else, For each clause γ = resolve(β, T).
           if γ, the resolvent, is empty, the theory is unsatisfiable.
7.         else, add the resolvent γ to the theory and delete β.
           if γ is a unit clause, add to Queue.
8.     endfor.
9. endwhile.
```

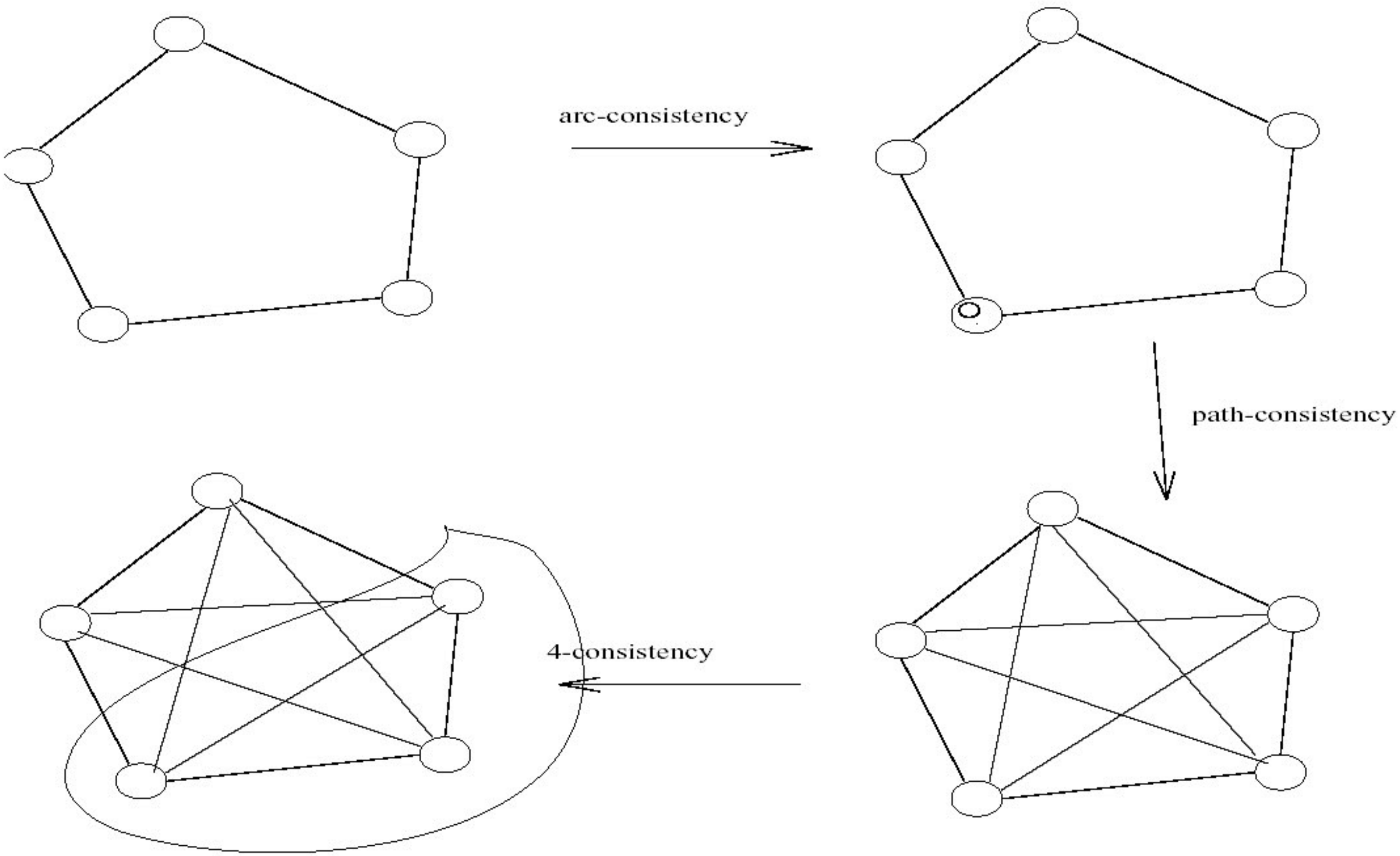
**Theorem 3.6.1** *Algorithm* UNIT-PROPAGATION *has a linear time complexity.*

# Consistency for numeric constraints (Gausian elimination)

Gausian elimination of

Gausian Elinination of:

# Impact on graphs of i-consistency



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# Distributed arc-consistency (Constraint propagation)

- Implement AC-1 distributedly.

- Node $x_j$ sends the message to node $x_i$

- Node $x_i$ updates its domain:

- Relational and generalized arc-consistency can be implemented distributedly: sending messages between constraints over the dual graph

# Relational Arc-consistency

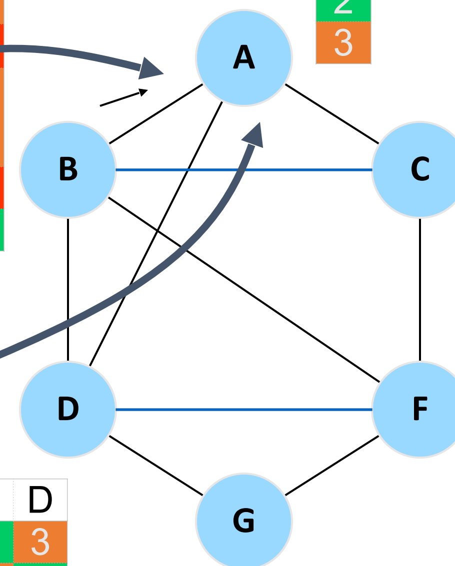
The message that R2 sends to R1 is

R1 updates its relation and domains and sends messages to neighbors

| A | B |
|---|---|
| 1 | 2 |
| 1 | 3 |
| 2 | 1 |
| 2 | 3 |
| 3 | 1 |
| 3 | 2 |

| A |
|---|
| 1 |
| 2 |
| 3 |

| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |




| B | C | F |
|---|---|---|
| 1 | 2 | 3 |
| 3 | 2 | 1 |

| A | B | D |
|---|---|---|
| 1 | 2 | 3 |
| 1 | 3 | 2 |
| 2 | 1 | 3 |
| 2 | 3 | 1 |
| 3 | 1 | 2 |
| 3 | 2 | 1 |

| D | F | G |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 1 | 3 |

# Distributed Relational Arc-Consistency

- DRAC can be applied to the dual problem of any constraint network:

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k \in ne(i)} h_k^i)) \qquad (1)$$

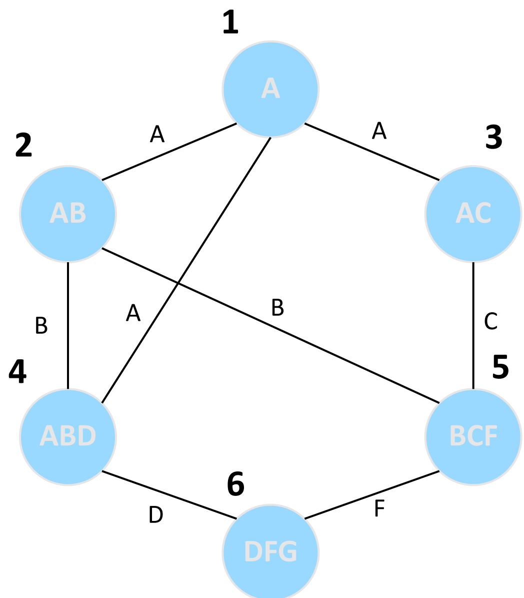
$$R_i \leftarrow R_i \cap (\bowtie_{k \in ne(i)} h_k^i) \qquad (2)$$

# DRAC on the dual join-graph

| A |
|---|
| 1 |
| 2 |
| 3 |

| A | B |
|---|---|
| 1 | 2 |
| 1 | 3 |
| 2 | 1 |
| 2 | 3 |
| 3 | 1 |
| 3 | 2 |

| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |




| A | B | D |
|---|---|---|
| 1 | 2 | 3 |
| 1 | 3 | 2 |
| 2 | 1 | 3 |
| 2 | 3 | 1 |
| 3 | 1 | 2 |
| 3 | 2 | 1 |

| B | C | F |
|---|---|---|
| 1 | 2 | 3 |
| 3 | 2 | 1 |

| D | F | G |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 1 | 3 |

# Iteration 1

| A |
|---|
| 1 |
| 2 |
| 3 |

| A |
|---|
| 1 |
| 2 |
| 3 |

| A |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 2 |
| 3 |

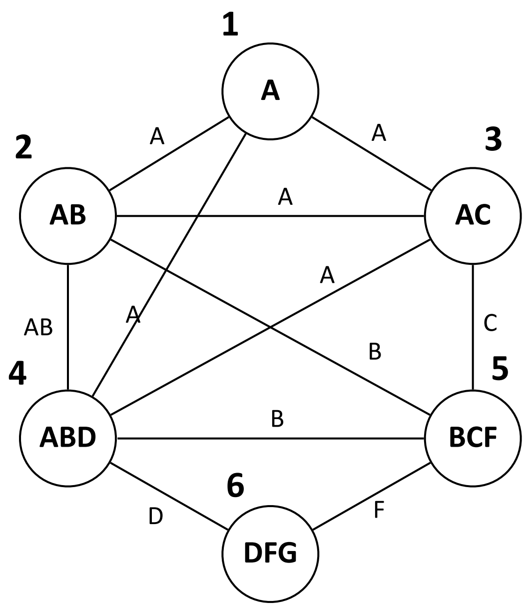
| B |
|---|
| 1 |
| 3 |

| A | B |
|---|---|
| 1 | 2 |
| 1 | 3 |
| 2 | 1 |
| 2 | 3 |
| 3 | 1 |
| 3 | 2 |

| A |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 2 |
| 3 |

| A | B |
|---|---|
| 1 | 2 |
| 1 | 3 |
| 2 | 1 |
| 2 | 3 |
| 3 | 1 |
| 3 | 2 |




| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |

| A |
|---|
| 1 |
| 2 |
| 3 |

| A |
|---|
| 1 |
| 2 |
| 3 |

| A |
|---|
| 1 |
| 2 |
| 3 |

| C |
|---|
| 2 |

| D |
|---|
| 1 |
| 2 |

| B |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 3 |

| A | B |
|---|---|
| 1 | 2 |
| 1 | 3 |
| 2 | 1 |
| 2 | 3 |
| 3 | 1 |
| 3 | 2 |

| A |
|---|
| 1 |
| 2 |
| 3 |

| A | B | D |
|---|---|---|
| 1 | 2 | 3 |
| 1 | 3 | 2 |
| 2 | 1 | 3 |
| 2 | 3 | 1 |
| 3 | 1 | 2 |
| 3 | 2 | 1 |

| B | C | F |
|---|---|---|
| 1 | 2 | 3 |
| 3 | 2 | 1 |

| B |
|---|
| 1 |
| 2 |
| 3 |

| C |
|---|
| 2 |

| B |
|---|
| 1 |
| 2 |
| 3 |

| F |
|---|
| 1 |
| 3 |

| D | F | G |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 1 | 3 |

| D |
|---|
| 1 |
| 2 |
| 3 |

| F |
|---|
| 1 |
| 3 |

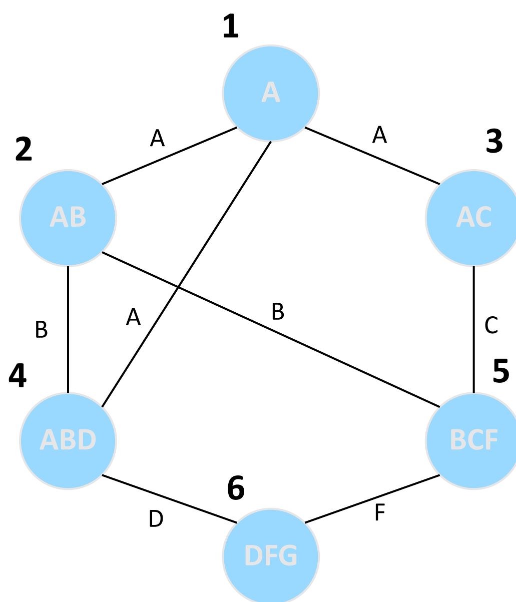
$$R_i \leftarrow R_i \cap (\bowtie_{k \in ne(i)} h_k^i) \qquad (2)$$

# Iteration 1

| A |
|---|
| 1 |
| 3 |

| A | B |
|---|---|
| 1 | 3 |
| 2 | 1 |
| 2 | 3 |
| 3 | 1 |

| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |




| A | B | D |
|---|---|---|
| 1 | 3 | 2 |
| 2 | 3 | 1 |
| 3 | 1 | 2 |
| 3 | 2 | 1 |

| B | C | F |
|---|---|---|
| 1 | 2 | 3 |
| 3 | 2 | 1 |

| D | F | G |
|---|---|---|
| 2 | 1 | 3 |

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k\in ne(i)} h_k^i)) \qquad (1)$$

# Iteration 2

| A |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 2 |
| 3 |

| A |
|---|
| 1 |
| 3 |

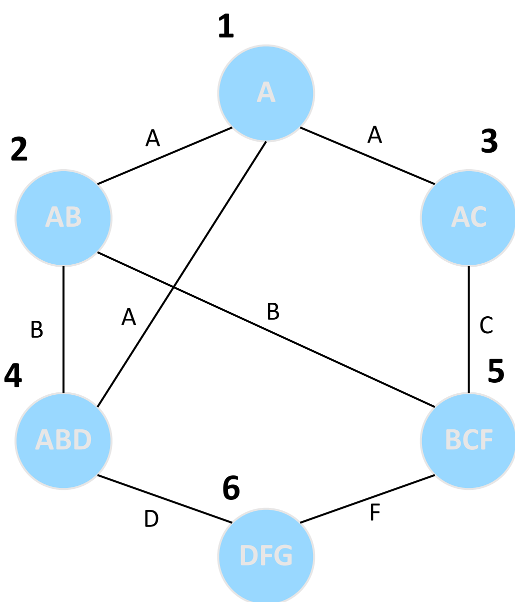
| A |
|---|
| 1 |
| 2 |
| 3 |

| B |
|---|
| 1 |
| 3 |

| B |
|---|
| 1 |
| 2 |
| 3 |

| A |
|---|
| 1 |
| 3 |

| A | B |
|---|---|
| 1 | 3 |
| 2 | 1 |
| 2 | 3 |
| 3 | 1 |




| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |

| A |
|---|
| 1 |
| 3 |

| C |
|---|
| 2 |

| D |
|---|
| 2 |

| B |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 3 |

| A | B | D |
|---|---|---|
| 1 | 3 | 2 |
| 2 | 3 | 1 |
| 3 | 1 | 2 |
| 3 | 2 | 1 |

| B | C | F |
|---|---|---|
| 1 | 2 | 3 |
| 3 | 2 | 1 |

| B |
|---|
| 1 |
| 3 |

| C |
|---|
| 2 |

| F |
|---|
| 1 |

| D | F | G |
|---|---|---|
| 2 | 1 | 3 |

| D |
|---|
| 1 |
| 2 |

| F |
|---|
| 1 |
| 3 |

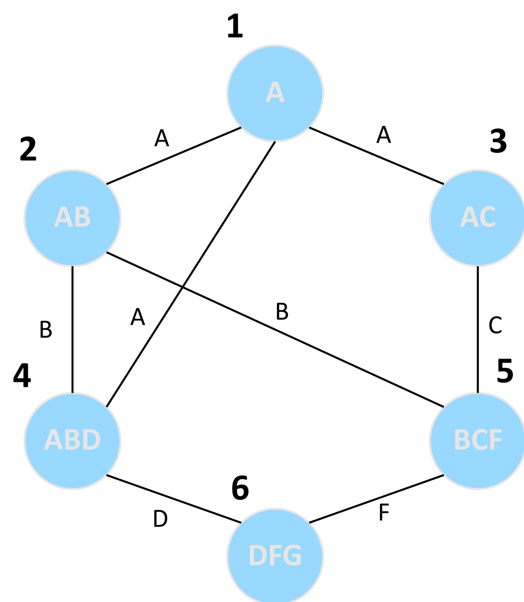
$$R_i \leftarrow R_i \cap (\bowtie_{k \in ne(i)} h_k^i) \qquad (2)$$

# Iteration 2

| A |
|---|
| 1 |
| 3 |

| A | B |
|---|---|
| 1 | 3 |
| 3 | 1 |

| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |




| A | B | D |
|---|---|---|
| 1 | 3 | 2 |
| 3 | 1 | 2 |

| B | C | F |
|---|---|---|
| 3 | 2 | 1 |

| D | F | G |
|---|---|---|
| 2 | 1 | 3 |

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k \in ne(i)} h_k^i)) \qquad (1)$$

# Iteration 3

| A |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 3 |

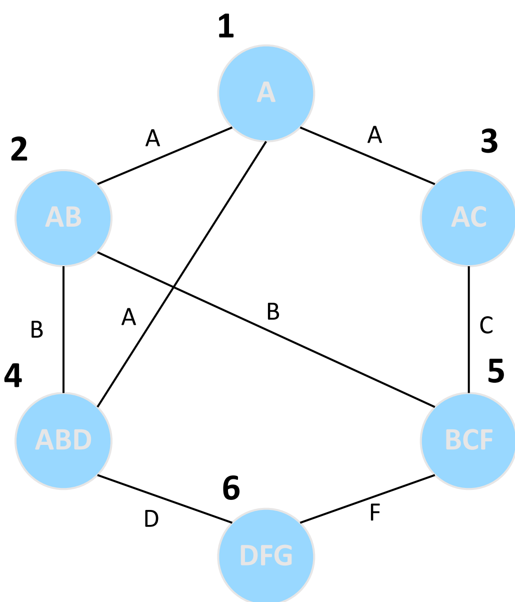
| A |
|---|
| 1 |
| 3 |

| B |
|---|
| 3 |

| B |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 3 |

| A | B |
|---|---|
| 1 | 3 |
| 3 | 1 |




| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |

| A |
|---|
| 1 |
| 3 |

| C |
|---|
| 2 |

| D |
|---|
| 2 |

| B |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 3 |

| A | B | D |
|---|---|---|
| 1 | 3 | 2 |
| 3 | 1 | 2 |

| B | C | F |
|---|---|---|
| 3 | 2 | 1 |

| B |
|---|
| 1 |
| 3 |

| C |
|---|
| 2 |

| F |
|---|
| 1 |

| D | F | G |
|---|---|---|
| 2 | 1 | 3 |

| D |
|---|
| 2 |

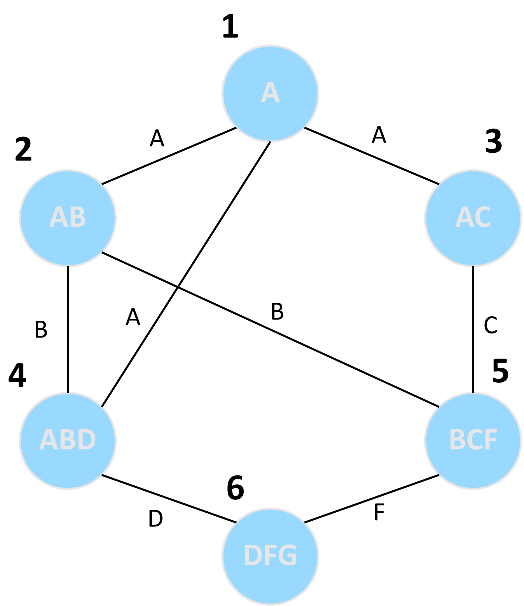
| F |
|---|
| 1 |

$$R_i \leftarrow R_i \cap \left(\bowtie_{k \in ne(i)} h_k^i\right) \tag{2}$$

# Iteration 3

| A |
|---|
| 1 |
| 3 |

| A | B |
|---|---|
| 1 | 3 |




| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |

| A | B | D |
|---|---|---|
| 1 | 3 | 2 |
| 3 | 1 | 2 |

| B | C | F |
|---|---|---|
| 3 | 2 | 1 |

| D | F | G |
|---|---|---|
| 2 | 1 | 3 |

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k \in ne(i)} h_k^i)) \qquad (1)$$

# Iteration 4

| A |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |

| A |
|---|
| 1 |
| 3 |

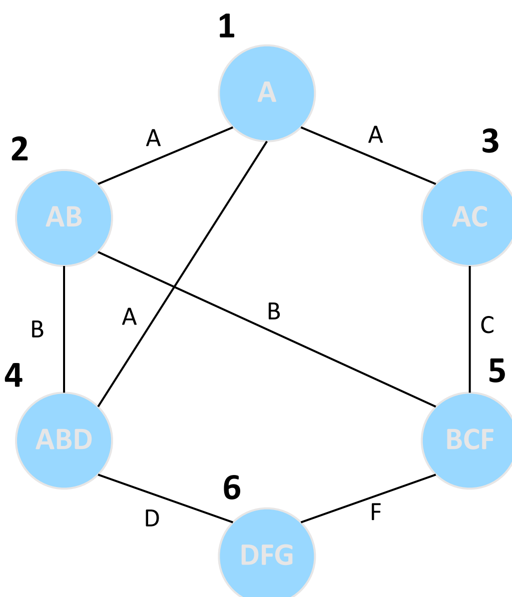
| A |
|---|
| 1 |
| 3 |

| B |
|---|
| 3 |

| B |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 3 |

| A | B |
|---|---|
| 1 | 3 |




| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |

| A |
|---|
| 1 |
| 3 |

| C |
|---|
| 2 |

| D |
|---|
| 2 |

| B |
|---|
| 1 |
| 3 |

| A |
|---|
| 1 |
| 3 |

| A | B | D |
|---|---|---|
| 1 | 3 | 2 |
| 3 | 1 | 2 |

| B | C | F |
|---|---|---|
| 3 | 2 | 1 |

| B |
|---|
| 3 |

| C |
|---|
| 2 |

| F |
|---|
| 1 |

| D | F | G |
|---|---|---|
| 2 | 1 | 3 |

| D |
|---|
| 2 |

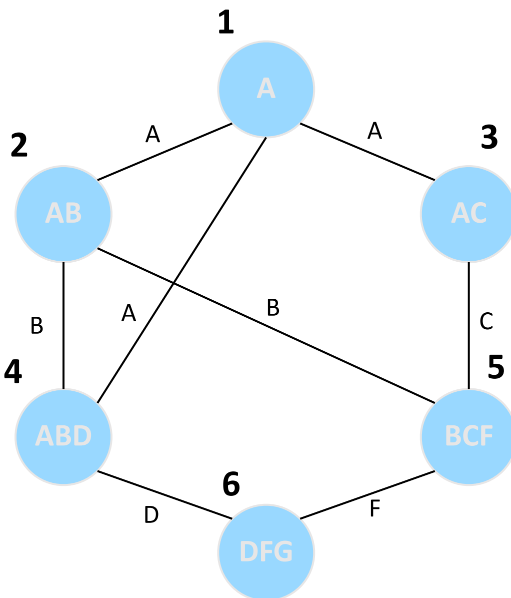
| F |
|---|
| 1 |

$$R_i \leftarrow R_i \cap \left(\bowtie_{k \in ne(i)} h_k^i\right) \tag{2}$$

# Iteration 4

| A |
|---|
| 1 |

| A | B |
|---|---|
| 1 | 3 |




| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |

| A | B | D |
|---|---|---|
| 1 | 3 | 2 |

| B | C | F |
|---|---|---|
| 3 | 2 | 1 |

| D | F | G |
|---|---|---|
| 2 | 1 | 3 |

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k \in ne(i)} h_k^i)) \qquad (1)$$

# Iteration 5

| A |
|---|
| 1 |

| A |
|---|
| 1 |

| A |
|---|
| 1 |

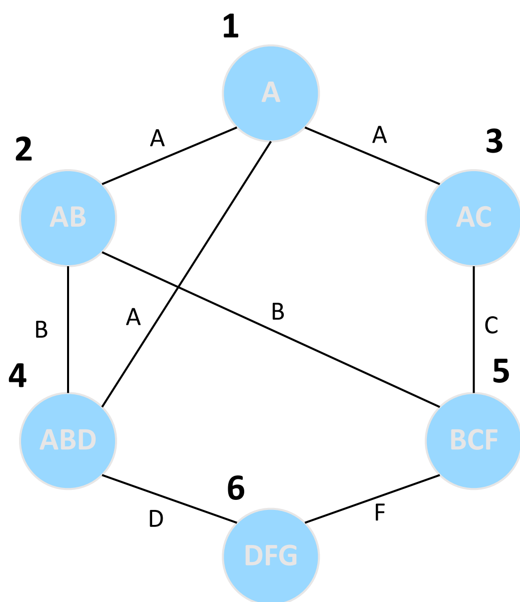
| A |
|---|
| 1 |

| B |
|---|
| 3 |

| B |
|---|
| 3 |

| A |
|---|
| 1 |

| A | B |
|---|---|
| 1 | 3 |




| A | C |
|---|---|
| 1 | 2 |
| 3 | 2 |

| A |
|---|
| 1 |

| C |
|---|
| 2 |

| D |
|---|
| 2 |

| B |
|---|
| 3 |

| A |
|---|
| 1 |

| A | B | D |
|---|---|---|
| 1 | 3 | 2 |

| B | C | F |
|---|---|---|
| 3 | 2 | 1 |

| B |
|---|
| 3 |

| C |
|---|
| 2 |

| F |
|---|
| 1 |

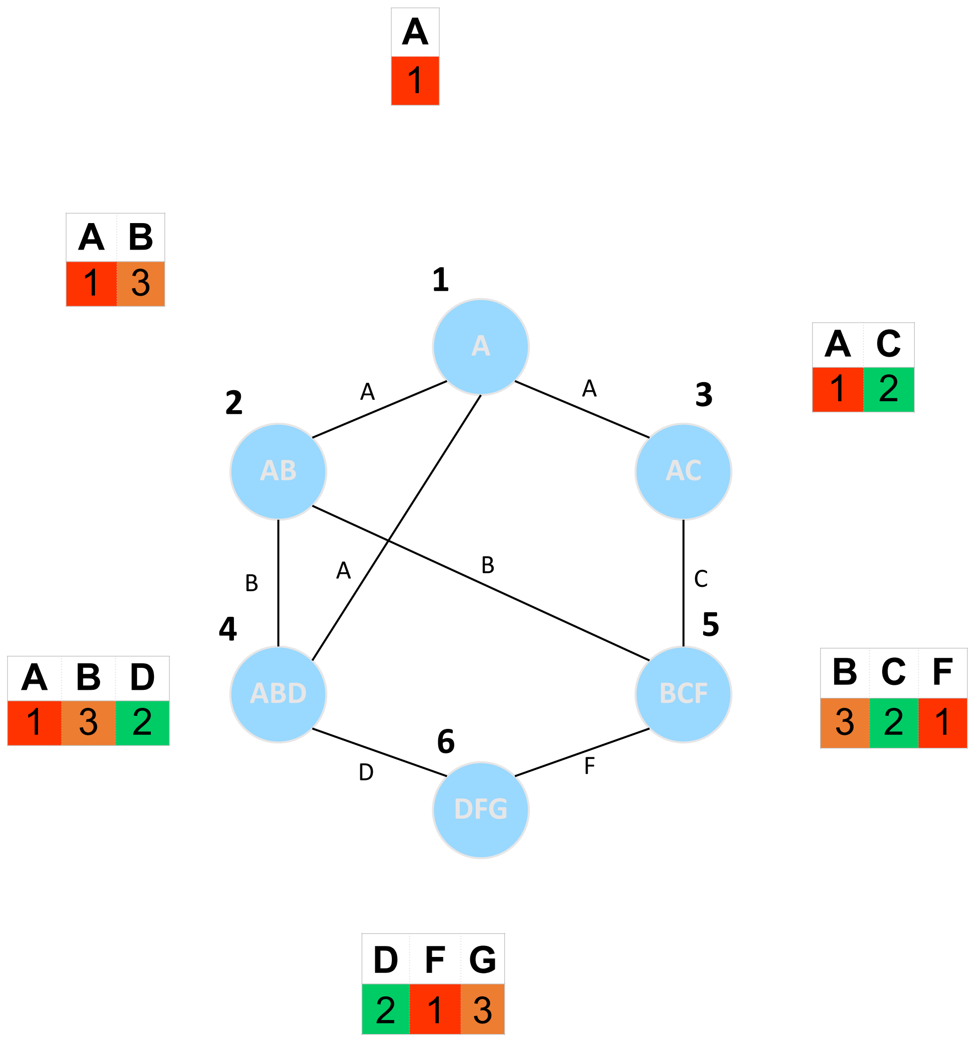
| D | F | G |
|---|---|---|
| 2 | 1 | 3 |

| D |
|---|
| 2 |

| F |
|---|
| 1 |

$$R_i \leftarrow R_i \cap (\bowtie_{k \in ne(i)} h_k^i) \qquad (2)$$

# Iteration 5



# Tractable classes

**Theorem 3.7.1** *1. The consistency binary constraint networks having no cycles can be decided by arc-consistent*

*2. The consistency of binary constraint networks with bi-valued domains can be decided by path-consistency,*

*3. The consistency of Horn cnf theories can be decided by unit propagation.*

# Outline

- Arc-consistency algorithms
- Path-consistency and i-consistency
- Arc-consistency, Generalized arc-consistency, relation arc-consistency
- Global and bound consistency
- Consistency operators: join, resolution, Gausian elimination
- Distributed (generalized) arc-consistency