CompSci 275, Constraint Networks

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Consistency algorithms, part a
Chapter 3

Outline

- Arc-consistency algorithms
- Path-consistency and i-consistency
- Arc-consistency, Generalized arc-consistency, relation arc-consistency
- Global and bound consistency
- Distributed (generalized) arc-consistency
- Consistency operators: join, resolution, Gausian elimination

Consistency methods

- Approximation of inference:
  - Arc, path and i-consistecy
- Methods that transform the original network into tighter and tighter representations

Inference Algorithms
will help search

Definition 3.1.1 (partial solution) *Given a constraint network $\mathcal{R}$, we say that an assignment of values to a subset of the variables $S = \{x_1, ..., x_j\}$ given by $\bar{a} = (< x_1, a_1 >, < x_2, a_2 >, ..., < x_j, a_j >)$ is consistent relative to $\mathcal{R}$ iff it satisfies every constraint R_{S_i} such that $S_i \subseteq S$. The assignment $\bar{a}$ is also called a partial solution of $\mathcal{R}$. The set of all partial solutions of a subset of variables S is denoted by ρ_S or $\rho(S)$.*

Arc-consistency

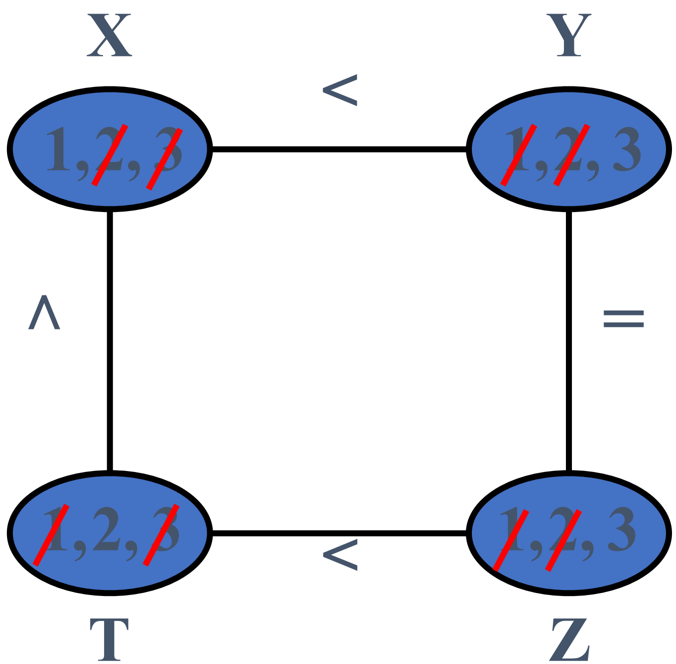
$1 \le X, Y, Z, T \le 3$

$X < Y$

$Y = Z$

$T < Z$

$X \le T$



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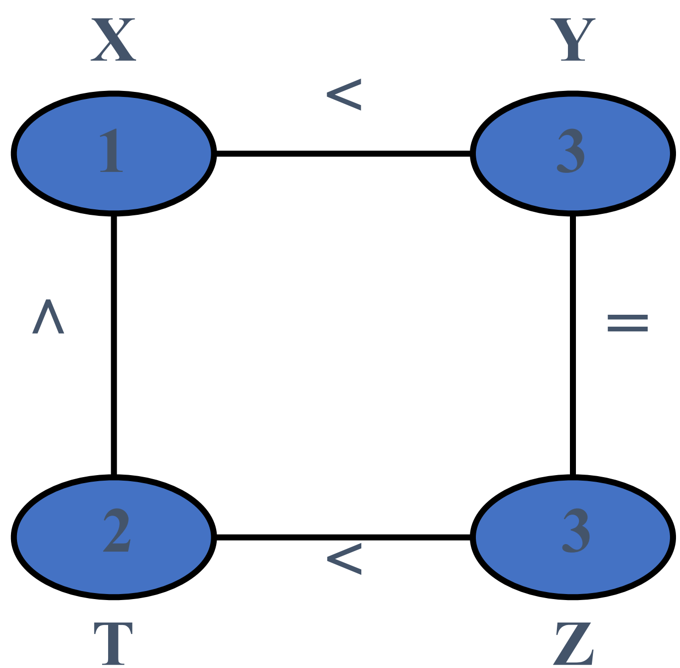


Arc-consistency

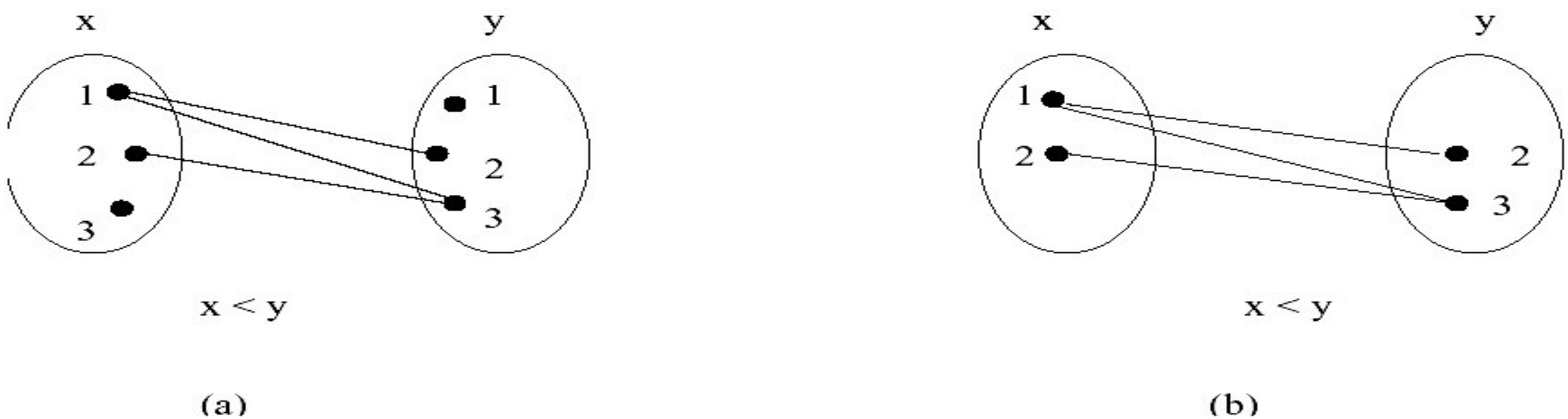


Figure 3.1: A matching diagram describing the arc-consistency of two variables x and y. In (a) the variables are not arc-consistent. In (b) the domains have been reduced, and the variables are now arc-consistent.

Definition 3.2.2 (arc-consistency) *Given a constraint network* $\mathcal{R} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$, *with* $R_{ij} \in C$, *a variable* x_i *is* arc-consistent *relative to* x_j *if and only if for every value* $a_i \in D_i$ *there exists a value* $a_j \in D_j$ *such that* $(a_i, a_j) \in R_{ij}$. *The subnetwork (alternatively, the arc) defined by* $\{x_i, x_j\}$ *is arc-consistent if and only if* x_i *is arc-consistent relative to* x_j *and* x_j *is arc-consistent relative to* x_i. *A network of constraints is called* arc-consistent *iff all of its arcs (e.g., subnetworks of size 2) are arc-consistent.*

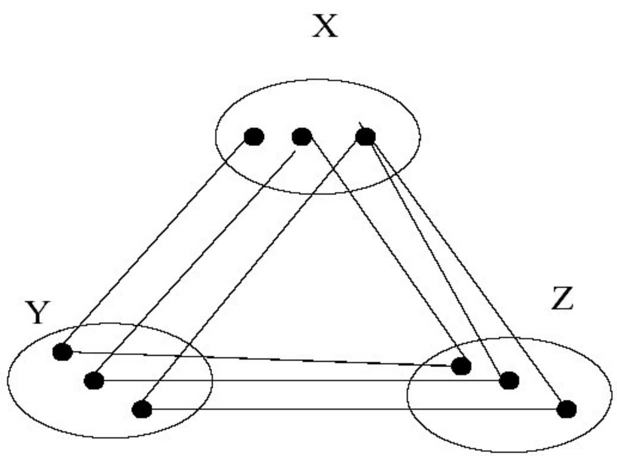
Revise for arc-consistency

```
REVISE((x_i), x_j)
input: a subnetwork defined by two variables X = {x_i, x_j}, a distinguished variable x_i,
    domains: D_i and D_j, and constraint R_ij
output: D_i, such that, x_i arc-consistent relative to x_j
for each a_i ∈ D_i
        if there is no a_j ∈ D_j such that (a_i, a_j) ∈ R_ij
                then  delete a_i from D_i
        endif
endfor
```

Figure 3.2: The Revise procedure

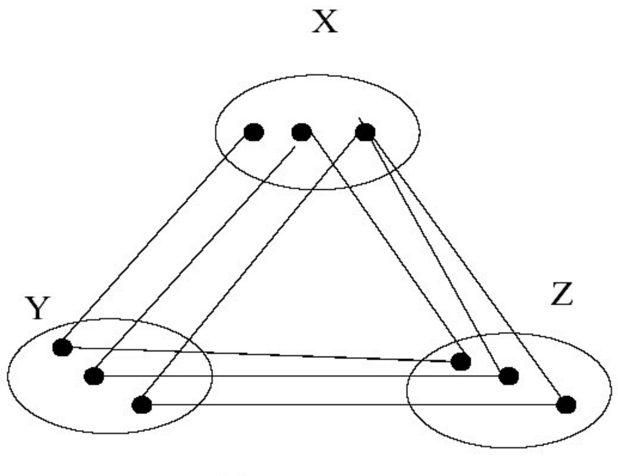
$\bowtie = \otimes$

A matching diagram describing a network of constraints that is not arc-consistent (b) An arc-consistent equivalent network.

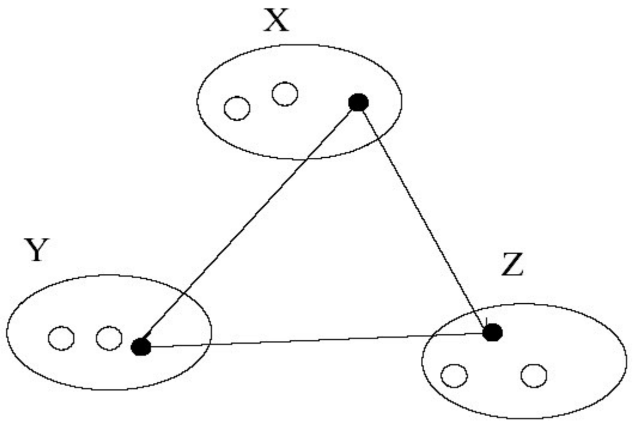


(a)

A matching diagram describing a network of constraints that is not arc-consistent (b) An arc-consistent equivalent network.



(a)



(b)

AC-1

AC-1($\mathcal{R}$)

input: a network of constraints $\mathcal{R} = (X, D, C)$
output: $\mathcal{R}'$ which is the loosest arc-consistent network equivalent to $\mathcal{R}$

```
repeat
    for every pair {x_i, x_j} that participates in a constraint
        Revise((x_i), x_j) (or D_i ← D_i ∩ π_i(R_ij ⋈ D_j))
        Revise((x_j), x_i) (or D_j ← D_j ∩ π_j(R_ij ⋈ D_i))
    endfor
until no domain is changed
```

Figure 3.4: Arc-consistency-1 (AC-1)

- Complexity (Mackworth and Freuder, 1986):
- *e* = number of arcs, *n* variables, *k* values
- (ek^2, each loop, *nk* number of loops), best-case = *ek*,
- Arc-consistency is:

AC-3

AC-3($\mathcal{R}$)

input: a network of constraints $\mathcal{R} = (X, D, C)$

output: $\mathcal{R}'$ which is the largest arc-consistent network equivalent to $\mathcal{R}$

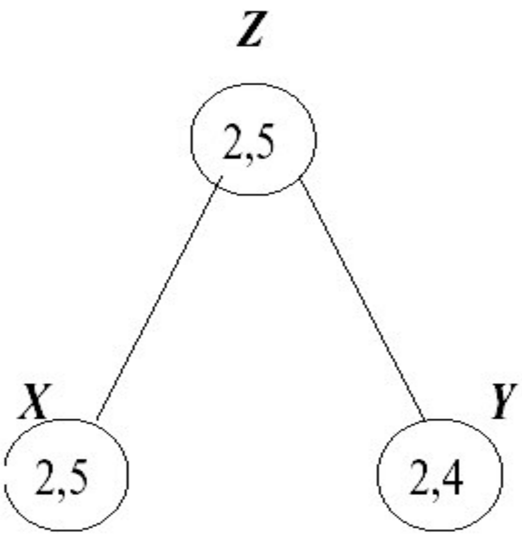
for every pair $\{x_i, x_j\}$ that participates in a constraint $R_{ij} \in \mathcal{R}$
    $queue \leftarrow queue \cup \{(x_i, x_j), (x_j, x_i)\}$
endfor
while $queue \neq \{\}$
    select and delete (x_i, x_j) from $queue$
    $Revise((x_i), x_j)$
    if $Revise((x_i), x_j)$ causes a change in D_i
        then $queue \leftarrow queue \cup \{(x_k, x_i), i \neq k\}$
    endif
endwhile

Figure 3.5: Arc-consistency-3 (AC-3)

- Complexity: since each arc may be processed in O(2k)
- Best case O(ek),

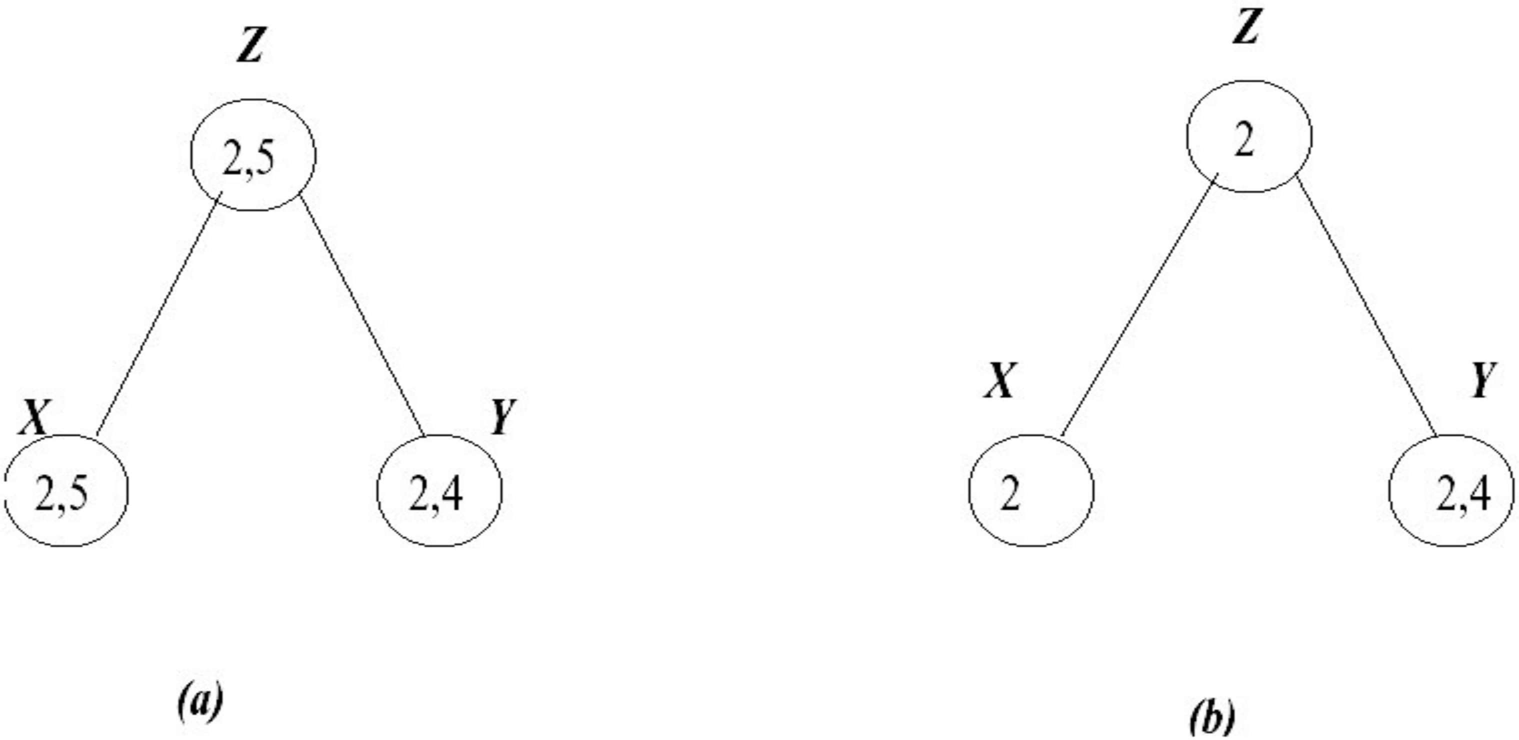
Example: a 3 variables network with 2 constraints: z divides x and z divides y

(a) before and (b) after AC-3 is applied.

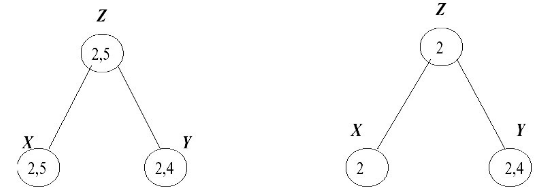


(a)

Example: A 3 variables network with 2 constraints: z divides x and z divides y
(a) before and (b) after AC-3 is applied.



AC-4



(a) (b)

AC-4($\mathcal{R}$)

input: a network of constraints $\mathcal{R}$

output: An arc-consistent network equivalent to $\mathcal{R}$

Initialization: $M \leftarrow \emptyset$,
    initialize $S_{(x_i,c_i)}$, $counter(i, a_i, j)$ for all R_{ij}
    for all counters
        if $counter(x_i, a_i, x_j) = 0$ (if $< x_i, a_i >$ is unsupported by x_j)
            then add $< x_i, a_i >$ to $LIST$
        endif
    endfor
while $LIST$ is not empty
    choose $< x_i, a_i >$ from LIST, remove it, and add it to M
    for each $< x_j, a_j >$ in $S_{(x_i,a_i)}$
        decrement $counter(x_j, a_j, x_i)$
        if $counter(x_j, a_j, x_i) = 0$
            then add $< x_j, a_j >$ to $LIST$
        endif
    endfor
endwhile

Figure 3.7: Arc-consistency-4 (AC-4)

- Complexity:
- (Counter is the number of supports to a_i in x_i from x_j. $S_{(xi,ai)}$ is the set of pairs that (x_i, a_i) supports)

Example applying AC-4

Example 3.2.9 Consider the problem in Figure 3.6. Initializing the $S_{(x,a)}$ arrays (indicating all the variable-value pairs that each $< x, a >$ supports), we have :
$S_{(z,2)} = \{< x, 2 >, < y, 2 >, < y, 4 >\}$, $S_{(z,5)} = \{< x, 5 >\}$, $S_{(x,2)} = \{< z, 2 >\}$, $S_{(x,5)} = \{< z, 5 >\}$, $S_{(y,2)} = \{< z, 2 >\}$, $S_{(y,4)} = \{< z, 2 >\}$.
For counters we have: $counter(x, 2, z) = 1$, $counter(x, 5, z) = 1$, $counter(z, 2, x) = 1$, $counter(z, 5, x) = 1$, $counter(z, 2, y) = 2$, $counter(z, 5, y) = 0$, $counter(y, 2, z) = 1$, $counter(y, 4, z) = 1$. (Note that we do not need to add counters between variables that are not directly constrained, such as x and y.) Finally, $List = \{< z, 5 >\}$, $M = \emptyset$. Once $< z, 5 >$ is removed from $List$ and placed in M, the counter of $< x, 5 >$ is updated to $counter(x, 5, z) = 0$, and $< x, 5 >$ is placed in $List$. Then, $< x, 5 >$ is removed from $List$ and placed in M. Since the only value it supports is $< z, 5 >$ and since $< z, 5 >$ is already in M, the $List$ remains empty and the process stops. $\square$

Distributed arc-consistency (Constraint propagation)

- Implement AC-1 distributedly.
- $h_{j \to i}$ node x_j sends the message to node x_i
- Node x_i updates its domain:
- Messages can be sent asynchronously or scheduled in a topological order

Exercise: make the following network arc-consistent

- Draw the network's primal and dual constraint graph
- Network =
  - Domains {1,2,3,4}
  - Constraints: y < x, z < y, t < z, f<t, x<=t+1, Y<f+2

Arc-consistency Algorithms

- **AC-1**: brute-force, distributed
- **AC-3**, queue-based
- **AC-4**, context-based, optimal
- **AC-5,6,7**,…. Good in special cases
- **Important:** applied at every node of search
- (*n* number of variables, *e*=#constraints, *k*=domain size)
- Mackworth and Freuder (1977,1983), Mohr and Anderson, (1985)…

Constraint tightness analysis

t = number of tuples bounding a constraint

- **AC-1**: brute-force,
- **AC-3**, queue-based
- **AC-4**, context-based, optimal
- **AC-5,6,7**,…. Good in special cases
- **Important:** applied at every node of search
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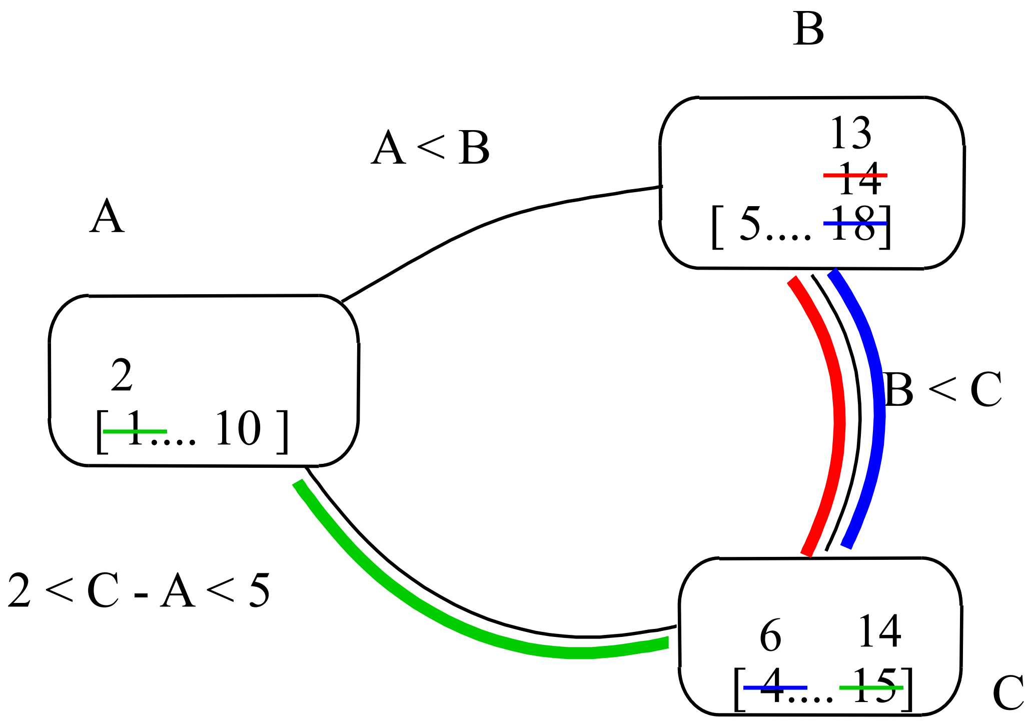
Constraint checking

→Arc-consistency



1- B: [5 .. 14]

C: [6 .. 15]

2- A: [2 .. 10]

C: [6 .. 14]

3- B: [5 .. 13]

Is arc-consistency enough?

- Example: a triangle graph-coloring with 2 values.
  - Is it arc-consistent?
  - Is it consistent?
- It is not path, or 3-consistent.

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