Advanced consistency methods

Chapter 8

Relational consistency (Chapter 8)

- Relational arc-consistency
- Relational path-consistency
- Relational m-consistency
- Relational consistency for Boolean and linear constraints:
  - Unit-resolution is relational-arc-consistency
  - Pair-wise resolution is relational path-consistency

Example

- Consider a constraint network over five integer domains, where the constraints take the form of linear equations and the domains are integers bounded by
  - D_x in [-2,3]
  - D_y in [-5,7]
  - R_{xyz}:= x + y = z
  - R_{ztl}:= z + t = l
  - fromD_x and R_xyz infer z-y in [-2,3] from this and D_y we can infer z \in [-7,10]

Relational arc-consistency

Let R be a constraint network , X= {x_1,...,x_n}, D_1,...,D_n, R_S a relation.

R_S in *R* is *relational-arc-consistent* relative to x in S, iff any consistent instantiation of the variables in S- {x} has an extension to a value in D_x that satisfies R_S. Namely,

$$\rho(S-x) \subseteq \pi_{S-x} R_S \otimes D_S$$

Enforcing relational arc-consistency

- If arc-consistency is not satisfied add:

$$R_{S-x} \leftarrow R_{S-x} \cap \pi_{S-x} R_S \otimes D_S$$

Example

- $R_{xyz} = \{(a,a,a),(a,b,c),(b,b,c)\}$.
- This relation is not relational arc-consistent, but if we add the projection $R_{xy} = \{(a,a),(a,b),(b,b)\}$, then R_{xyz} will become relational arc-consistent relative to $\{z\}$.
- To make this network relational-arc-consistent, we would have to add all the projections of R_{xyz} with respect to all subsets of its variables.

Relational path-cosistency

- Let R_S and R_T be two constraints in a network.
- R_S and R_T are relational-path-consistent relative to a variable x in S U T iff any consistent instantiation of the variables in S U T - {x} has an extension to a value in the domain D_x, that satisfies R_S and R_T simultaneously;
- $$\rho(A) \subseteq \pi_A R_S \otimes R_T,$$
  $$A = S \cup T - x$$
- A pair of relations R_S and R_T is relational-path-consistent iff it is relational-path-consistent relative to every variable in S U T. A network is relational-path-consistent iff every pair of its relations is relational-path-consistent.

Example

- we can assign to x, y, l and t values that are consistent relative to the relational-arc-consistent network generated in earlier. For example, the assignment
- (<x,2>,<y,-5>,<t,3>,<l,15>) is consistent, since only domain restrictions are applicable, but there is no value of z that simultaneously satisfies x+y = z and z+t = l. To make the two constraints relational path-consistent relative to z we should deduce the constraint x+y+t = l and add it to the network.

Relational m-consistency

- let R_{S_1}, … , R_{S_m} be m distinct constraints.
- R_{S_1}, … , R_{S_m} are *relational-m-consistent* relative to x in U_{i=1}^m S_i iff any consistent instantiation of the variables in A = U_{i=1}^m S_i-{x} has an extension to x that satisfies R_{S_1}, … , R_{S_m} simultaneously;

$$\rho(A) \subseteq \pi_A \otimes_{i=1,m} R_{S_i} \otimes D_x$$

$$A = S_1 \cup \ldots S_m - x$$

- A set of relations { R_{S_1}, … , R_{S_m} } is relational-m-consistent} iff it is relational-m-consistent relative to every variable in their scopes. A network is relational-m-consistent iff every set of m relations is relational-m-consistent. A network is strongly relational-m-consistent if it is relational-i-consistent for every i <= m.

SPACE BOUND RELATIONAL CONSISTENCY

- A set of relations $R_{S_1}, \ldots, R_{S_m}$ is relationally (i,m)-consistent} iff for every subset of variables A of size i, A in $U_{j=1}^m S_j$, any consistent assignment to A can be extended to an assignment to $U_{i=1}^m S_i - A$ that satisfies all m constraints simultaneously.

- A network is relationally (i,m)-consistent iff every set of m relations is relationally (i,m)-consistent. A network is strong relational (i,m)-consistent iff it is relational (j,m)-consistent for every $j <= i$.

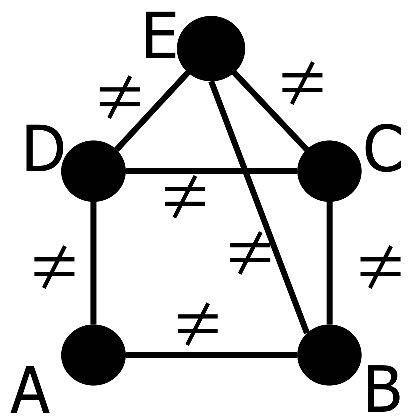
Extended composition

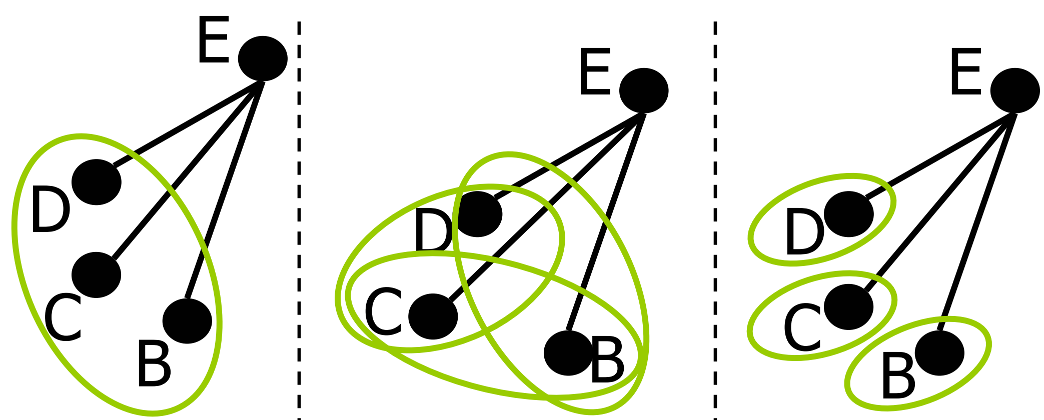
- The extended composition of relation $R_{S_1}, \ldots, R_{S_m}$ relative to A in $\bigcup_{i=1}^m S_i$, $EC_A(R_{S_1}, \ldots, R_{S_m})$, is defined by
- $EC_A(R_{S_1}, \ldots, R_{S_m}) = \pi_A(\Join_{i=1}^m R_{S_i})$
- If the projection operation is restricted to subsets of size i, it is called extended (i,m)-composition.
- Special casses: domain propagation and relational arc-consistency
- $D_x \leftarrow \pi_x (R_S \Join D_x)$
- $R_{S-x} \leftarrow \pi_{S-x} (R_S \Join D_x)$

Directional relational consistency

- Given an ordering $d = (x_1, \ldots x_n)$, *R* is m-directionally relationally consistent iff for every subset of constraints $R_{S_1}, \ldots, R_{S_m}$ where the latest variable is x_l, and for every A in $\{ x_1, \ldots, x_{l-1} \}$, every consistent assignment to A can be extended to x_l while simultaneously satisfying all these constraints.

Summary: directional i-consistency





E : $E \neq D, E \neq C, E \neq B$ → R_{DCB}

D : $D \neq C, D \neq A$

C : $C \neq B$

B : $A \neq B$

A :

Adaptive	d-path	d-arc
R_{DCB}	R_{DC}, R_{DB} R_{CB}	R_D R_C R_D

Example: crossword puzzle

$$R_{1,2,3,4,5} = \{(H, O, S, E, S), (L, A, S, E, R), (S, H, E, E, T), (S, N, A, I, L), (S, T, E, E, R)\}$$

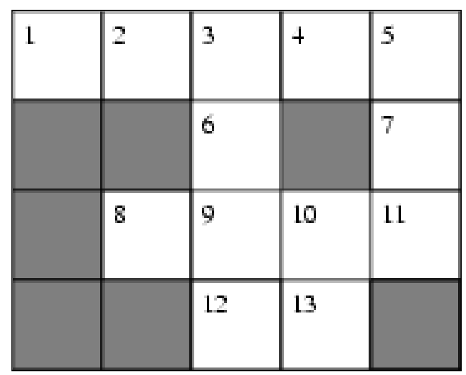
$$R_{3,6,9,12} = \{(H, I, K, E), (A, R, O, N), (K, E, E, T), (E, A, R, N), (S, A, M, E)\}$$

$$R_{5,7,11} = \{(R, U, N), (S, U, N), (L, E, T), (Y, E, S), (E, A, T), (T, E, N)\}$$

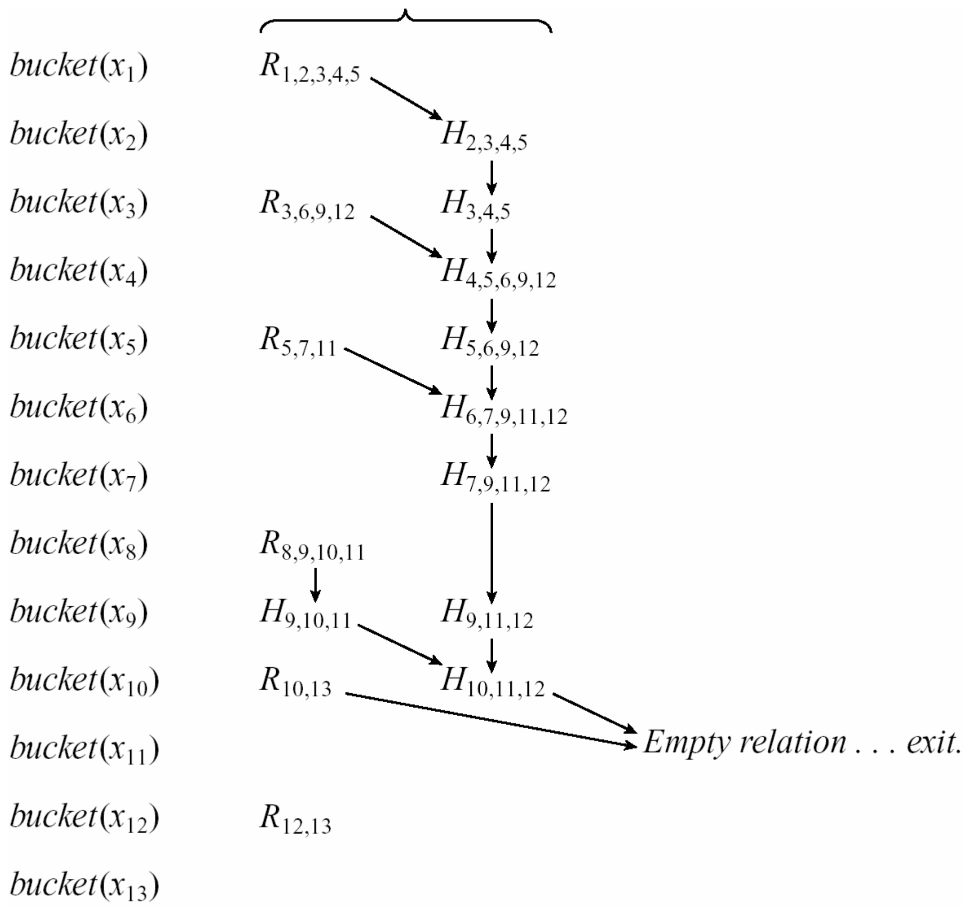
$$R_{8,9,10,11} = R_{3,6,9,12}$$

$$R_{10,13} = \{(N, O), (B, E), (U, S), (I, T)\}$$

$$R_{12,13} = R_{10,13}$$



Example: crossword puzzle, DRC_2



Complexity

- Even DRC_2 is exponential in the induced-width.
- Crossword puzzles can be made directional backtrack-free by DRC_2

Domain and constraint tightness

- **Theorem:** a strong relational 2-consistent constraint network over bi-valued domains is globally consistent.
- **m-tightness**: R_S of arity r is m-tight if, for any variable x_i \in S and any instantiation of the remaining r-1 variables in S - x_i, either there are at most m extensions of to x_i that satisfy R_S, or there are exactly | D_i | such extensions.
- **Theorem**: A strong relational k-consistent constraint network with at most k values is globally consistent.
- Example: D_i = {a,b,c},
- R_{x1,x2,x3} = { (aaa),(aac),(abc),(acb)(bac)(bbb)(bca)(cab)(cba)(ccc)}

Inference for Boolean theories

- Resolution is identical to Extended 2 decomposition
- Boolean theories are 2-tight
- Therefore DRC_2 makes a cnf globally consistent.
- DRC_2 expressed on cnfs is directional resolution

Directional resolution

DIRECTIONAL-RESOLUTION

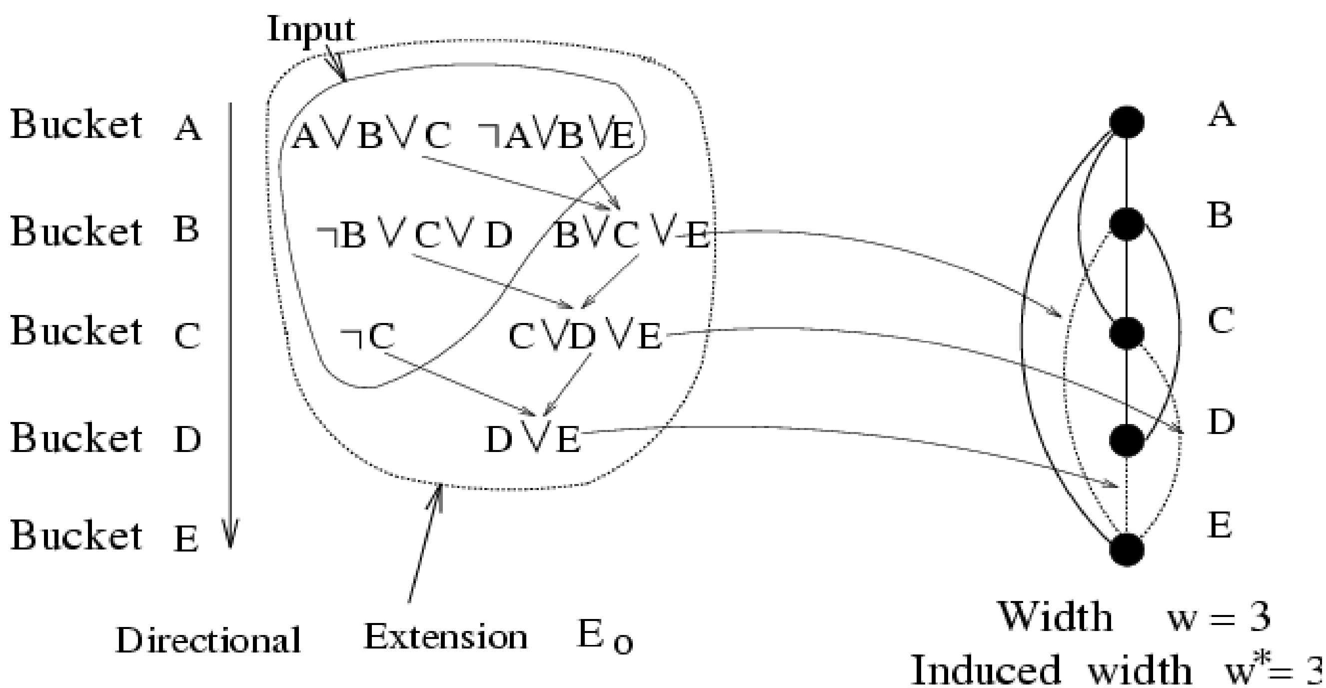
Input: A *CNF* theory φ, an ordering $d = Q_1, \ldots, Q_n$ of its variables.

Output: A decision of whether φ is satisfiable. If it is, a theory $E_d(\varphi)$, equivalent to φ, else an empty directional extension.

1. **Initialize:** generate an ordered partition of clauses into buckets. $bucket_1, \ldots, bucket_n$, where $bucket_i$ contains all clauses whose highest literal is Q_i.
2. **for** $i \leftarrow n$ **downto** 1 process $bucket_i$:
3. **if** there is a unit clause **then** (the instantiation step)
   apply unit-resolution in $bucket_i$ and place the resolvents in their right buckets.
   if the empty clause was generated, theory is not satisfiable.
4. **else** resolve each pair $\{(\alpha \vee Q_i), (\beta \vee \neg Q_i)\} \subseteq bucket_i$.
   if $\gamma = \alpha \vee \beta$ is empty, return $E_d(\varphi) = \{\}$, theory is not satisfiable
   else determine the index of γ and add it to the appropriate bucket.
5. **return** $E_d(\varphi) \leftarrow \bigcup_i bucket_i$

Figure 4.20: Directional-resolution

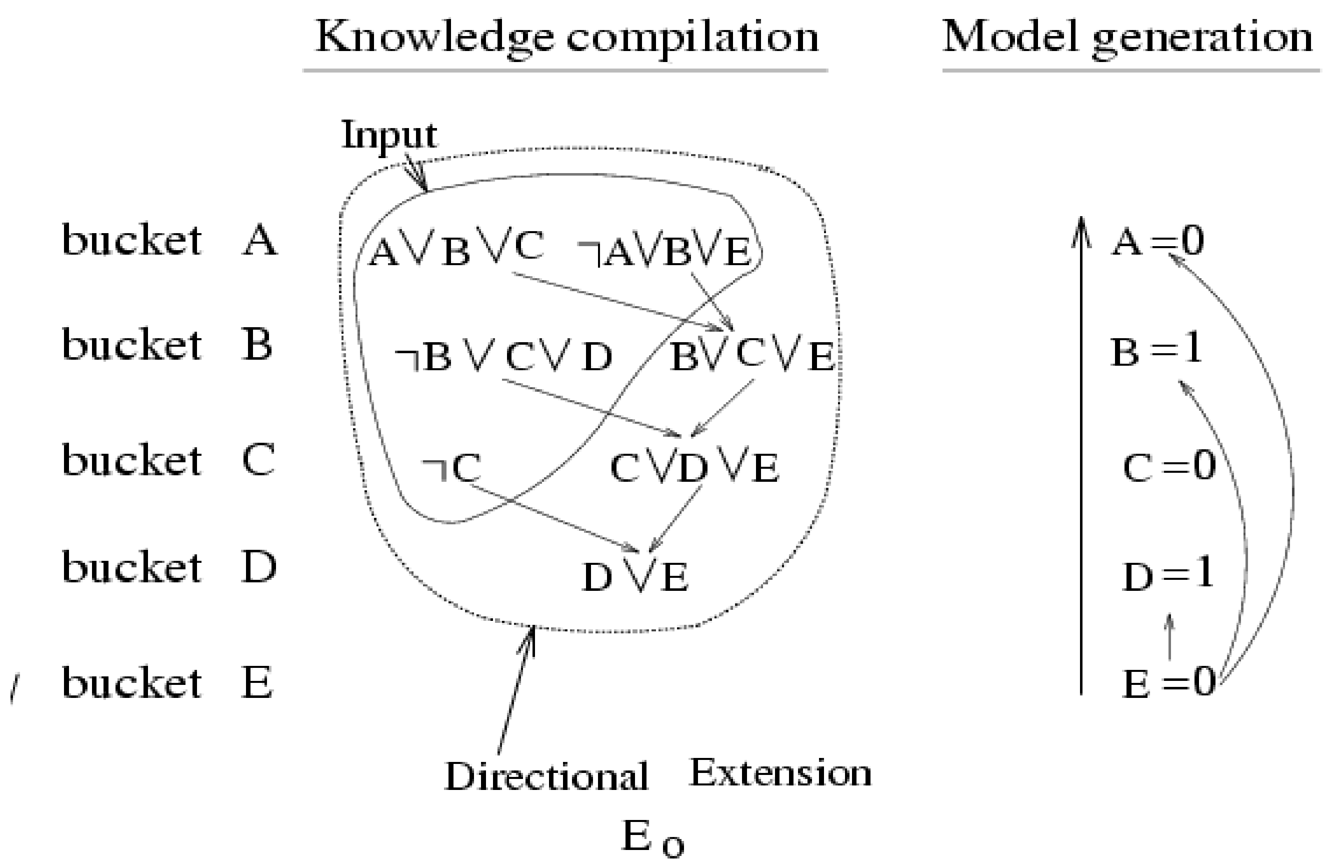
DR resolution = adaptive-consistency=directional relational path-consistency



$| \mathit{bucket}_i | = O(\exp(w^*))$

DR time and space : $O(n \exp(w^*))$

Directional Resolution ⇔ Adaptive Consistency



History

- 1960 – resolution-based Davis-Putnam algorithm
- 1962 – resolution step replaced by conditioning (Davis, Logemann and Loveland, 1962) to avoid memory explosion, resulting into a backtracking search algorithm known as Davis-Putnam (DP), or DPLL procedure.
- The dependency on induced width was not known in 1960.
- 1994 – Directional Resolution (DR), a rediscovery of the original Davis-Putnam, identification of tractable classes (Dechter and Rish, 1994).

Complexity of DR

Theorem 4.7.6 *(complexity of DR)*
Given a theory φ and an ordering of its variables o, the time complexity of algorithm DR along o is $O(n \cdot 9^{w_o^*})$, and $E_o(\varphi)$ contains at most $n \cdot 3^{w_o^*+1}$ clauses, where w_o^* is the induced width of φ's interaction graph along o. □

- 2-cnfs and Horn theories

Theorem 4.7.7 *Given a 2-cnf theory φ, its directional extension $E_o(\varphi)$ along any ordering o is of size $O(n \cdot {w_o^*}^2)$, and can be generated in $O(n \cdot {w_o^*}^2)$ time.*

Theorem 4.7.8 *The consistency of Horn theories can be determined by unit propagation. If the empty clause is not generated, the theory is satisfiable.* □

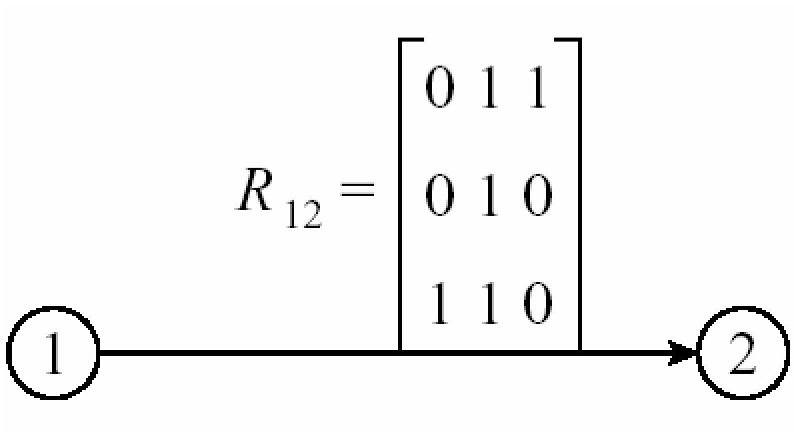
Row convexity

- **Functional constraints**: A binary relation R_{ij} expressed as a (0,1)-matrix is functional iff there is at most a single "1" in each row and in each column.

- **Monotone constraints**: Given ordered domain, a binary relation R_{ij} is monotone if (a,b) in R_{ij} and if c >= a, then (c,b) in R_{ij}, and if (a,b) in R_{ij} and c <= b, then (a,c) in R_{ij}.

- **Row convex constraints**: A binary relation R_{ij} represented as a (0,1)-matrix is row convex if in each row (column) all of the ones are consecutive}

Example of row convexity



- Lemma: Let F be a finite collection of (0,1)-row vectors that are row convex and of equal length. If every pair of rows have a non-zero intersection, then all of the rows have a non-zero entry in common.

Theorem:

- Theorem: Let R be a path consistent binary constraint network. If there exists an ordering of the domains $D_1, \ldots, D_n$ of R such that the relations of all constraints are row convex, the network is globally consistent and is therefore minimal.

Example:

- Cube 3-dimensional recognition
- Bi-valued binary constraints
- 2-colorability

Linear constraints

- inequalities of the form
- $a x_i - b x_j = c$,
- $a x_i - b x_j < c$,
- $a x_i - b x_j <= c$,
- a, b, and c are integer constants.

- However, it can be shown that each element in the closure under composition, intersection, and transposition of the resulting set of (0,1)-matrices is row convex, provided that when an element is removed from a domain by arc consistency, the associated (0,1)-matrices are ``condensed.''

- Hence, we can guarantee that the result of path consistency will be row-convex and therefore minimal, and that the network will be globally consistent for any binary linear equation over the integers.

Identifying row-convex constraints

- **Theorem**: [Booth and Lueker,1976]: An m x n (0,1)-matrix specified by its f nonzero entries can be tested for whether permutation of the columns exists such that the matrix is row convex in $O(m + n + f)$ steps.

Linear inequalities

- Consider r-ary constraints over a subset of variables $x_1, \dots x_r$ of the form
- $a_1 x_1 + \dots + a_r x_r \le c$, a_i are rational constants. The r-ary inequalities define corresponding r-ary relations that are row convex.
- Since r-ary linear inequalities that are closed under relational path-consistency are row-convex, relative to any set of integer domains (using the natural ordering).
- **Proposition:** A set of linear inequalities that is closed under RC_2 is globally consistent.

Linear inequalities

- Gausian elimination with domain constraint is relational-arc-consistency
- Gausian elimination of 2 inequalities isRelational path-consistency
- Theorem: directional path-consistency is complete for CNFs and for linear inequalities

DIRECTIONAL-LINEAR-ELIMINATION (φ, d)

Input: A *set of linear inequalities* φ, an ordering $d = x_1, \ldots, x_n$.

OutputA decision of whether φ is satisfiable. If it is, a backtrack-free theory $E_d(\varphi)$.

1. **Initialize:** Partition inequalities into ordered buckets.
2. **for** $i \leftarrow n$ **downto** 1 **do**
3. **if** x_i has one value in its domain **then**
. substitute the value into each inequality in the bucket and put the resulting inequality in the right bucket.
4. **else,for each** pair $\{\alpha, \beta\} \subseteq bucket_i$, compute $\gamma = elim_i(\alpha, \beta)$
   if γ has no solutions, return $E_d(\varphi) = \{\}$, "inconsistency"
   else add γ to the appropriate lower bucket.
5. **return** $E_d(\varphi) \leftarrow \bigcup_i bucket_i$

Figure 4.22: Fourier Elimination; DLE

Directional linear elimination, DLE : generates a backtrack-free representation

Theorem 4.8.3 *Given a set of linear inequalities φ, algorithm DLE (Fourier elimination) decides the consistency of φ over the Rationals and the Reals, and it generates an equivalent backtrack-free representation.* □

Example

$$
\begin{aligned}
bucket_4 &: \quad 5x_4 + 3x_2 - x_1 \leq 5,\ x_4 + x_1 \leq 2,\ \ - x_4 \leq 0, \\
bucket_3 &: \quad x_3 \leq 5,\ x_1 + x_2 - x_3 \leq -10 \\
bucket_2 &: \quad x_1 + 2x_2 \leq 0. \\
bucket_1 &:
\end{aligned}
$$

Figure 4.23: initial buckets

$$
\begin{aligned}
bucket_4 &: \quad 5x_4 + 3x_2 - x_1 \leq 5,\ x_4 + x_1 \leq 2,\ \ - x_4 \leq 0, \\
bucket_3 &: \quad x_3 \leq 5,\ x_1 + x_2 - x_3 \leq -10 \\
bucket_2 &: \quad x_1 + 2x_2 \leq 0 \ || \ 3x_2 - x_1 \leq 5, x_1 + x_2 \leq -5 \\
bucket_1 &: \quad || \ x_1 \leq 2.
\end{aligned}
$$

Figure 4.24: final buckets