STATS 8: Introduction to Biostatistics

Hypothesis Testing

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Hypothesis

- In general, many scientific investigations start by expressing a hypothesis.
- For example, Mackowiak et al (1992) hypothesized that the average normal (i.e., for healthy people) body temperature is less than the widely accepted value of $98.6F$.
- If we denote the population mean of normal body temperature as μ, then we can express this hypothesis as $\mu < 98.6$.

Null and alternative hypotheses

- The null hypothesis usually reflects the "status quo" or "nothing of interest".
- In contrast, we refer to our hypothesis (i.e., the hypothesis we are investigating through a scientific study) as the **alternative hypothesis** and denote it as H_A.
- For hypothesis testing, we focus on the null hypothesis since it tends to be simpler.

Null and alternative hypotheses

- Consider the body temperature example, where we want to examine the null hypothesis $H_0 : \mu = 98.6$ against the alternative hypothesis $H_A : \mu < 98.6$.

- To start, suppose that $\sigma^2 = 1$ is known.

- Further, suppose that we have randomly selected a sample of 25 healthy people from the population and measured their body temperature.

Hypothesis testing for the population mean

- To decide whether we should reject the null hypothesis, we quantify the empirical support (provided by the observed data) against the null hypothesis using some statistics.

- We use statistics to evaluate our hypotheses.

- We refer to them as **test statistics**.

- For a statistic to be considered as a test statistic, its sampling distribution must be fully known (exactly or approximately) under the null hypothesis.

- We refer to the distribution of test statistics under the null hypothesis as the **null distribution**.

Hypothesis testing for the population mean

- To evaluate hypotheses regarding the population mean, we use the sample mean $\bar{X}$ as the test statistic.

$$\bar{X} \sim N(\mu, \sigma^2/n).$$

- For the above example,

$$\bar{X} \sim N(\mu, 1/25).$$

- If the null hypothesis is true, then

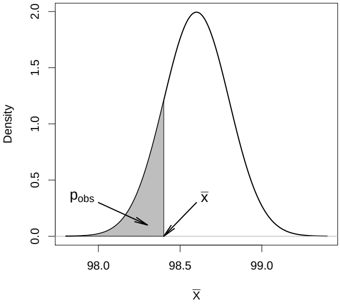
$$\bar{X} \sim N(98.6, 1/25).$$

Hypothesis testing for the population mean

- In reality, we have one value, $\bar{x}$, for the sample mean.
- We can use this value to quantify the evidence of departure from the null hypothesis.
- Suppose that from our sample of 25 people we find that the sample mean is $\bar{x} = 98.4$.

Hypothesis testing for the population mean

- To evaluate the null hypothesis $H_0 : \mu = 98.6$ versus the alternative $H_A : \mu < 98.6$, we use the lower tail probability of this value from the null distribution.



Observed significance level

- The *observed significance level* for a test is the probability of values as or more extreme than the observed value, based on the null distribution in the direction supporting the alternative hypothesis.

- This probability is also called the *p*-**value** and denoted p_{obs}.

- For the above example,

$$p_{\mathrm{obs}} = P(\bar{X} \leq \bar{x} | H_0),$$

z-score

- In practice, it is more common to use the standardized version of the sample mean as our test statistic.

- We know that if a random variable is normally distributed (as it is the case for $\bar{X}$), subtracting the mean and dividing by standard deviation creates a new random variable with standard normal distribution,

$$Z \sim N(0, 1).$$

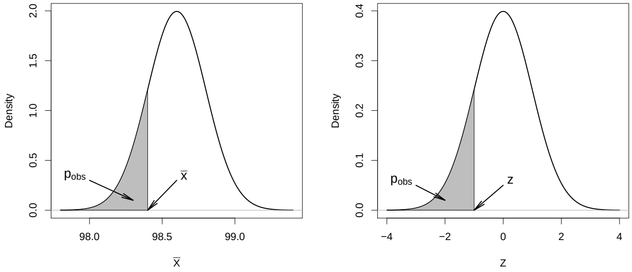
- We refer to the standardized value of the observed test statistic as the *z*-**score**,

$$\begin{aligned} z &= \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}, \\ &= \frac{98.4 - 98.6}{0.2} = -1. \end{aligned}$$

z-test

- We refer to the corresponding hypothesis test of the population mean as the z-**test**.

- In a z-test, instead of comparing the observed sample mean $\bar{x}$ to the population mean according to the null hypothesis, we compare the z-score to 0.



Interpretation of p-value

- The p-value is the conditional probability of extreme values (as or more extreme than what has been observed) of the test statistic assuming that the null hypothesis is true.
- When the p-value is small, say 0.01 for example, it is rare to find values as extreme as what we have observed (or more so).
- As the p-value increases, it indicates that there is a good chance to find more extreme values (for the test statistic) than what has been observed.
- Then, we would be more reluctant to reject the null hypothesis.
- A common **mistake** is to regard the p-value as the probability of null given the observed test statistic: $P(H_0|\bar{x})$.

One-sided vs. two-sided hypothesis testing

- The alternative hypothesis $H_A : \mu < 98.6$ or $H_A : \mu > 98.6$ are called *one-sided* alternatives.

- For these hypotheses, $p_{\text{obs}} = P(Z \leq z)$ and $p_{\text{obs}} = P(Z \geq z)$ respectively.

- In contrast, the alternative hypothesis $H_A : \mu \neq 98.6$ is *two-sided*.

- For the above three alternatives, the null hypothesis is the same, $H_0 : \mu = 98.6$

- In this case, $p_{\text{obs}} = 2 \times P(Z \geq |z|)$.

Hypothesis testing using t-tests

- So far, we have assumed that the population variance σ^2 is known.

- In reality, σ^2 is almost always unknown, and we need to estimate it from the data.

- As before, we estimate σ^2 using the sample variance S^2.

- Similar to our approach for finding confidence intervals, we account for this additional source of uncertainty by using the t-distribution with $n - 1$ degrees of freedom instead of the standard normal distribution.

- The hypothesis testing procedure is then called the **t-test**.

Hypothesis testing using t-tests

- Using the observed values of $\bar{X}$ and S, the observed value of the test statistic is obtained as follows:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}.$$

- We refer to t as the t-**score**.

- Then,

$$\begin{aligned} &\text{if } H_A : \mu < \mu_0, \quad p_{\text{obs}} = P(T \leq t), \\ &\text{if } H_A : \mu > \mu_0, \quad p_{\text{obs}} = P(T \geq t), \\ &\text{if } H_A : \mu \neq \mu_0, \quad p_{\text{obs}} = 2 \times P\big(T \geq |t|\big), \end{aligned}$$

- Here, T has a t-distribution with $n-1$ degrees of freedom, and t is our observed t-score.

Hypothesis testing for population proportion

- For a binary random variable X with possible values 0 and 1, we are typically interested in evaluating hypotheses regarding the population proportion of the outcome of interest, denoted as $X = 1$.
- As discussed before, the population proportion is the same as the population mean for such binary variables.
- So we follow the same procedure as described above.
- More specifically, we use the z-test for hypothesis testing.

Hypothesis testing for population proportion

- Note that we do not use *t*-test, because for binary random variable, population variance is $\sigma^2 = \mu(1-\mu)$.

- Therefore, by setting $\mu = \mu_0$ according to the null hypothesis, we also specify the population variance as $\sigma^2 = \mu_0(1-\mu_0)$.

Hypothesis testing for population proportion

- If we assume that the null hypothesis is true, we have

$$\bar{X}|H_0 \sim N\big(\mu_0, \mu_0(1-\mu_0)/n\big).$$

- This means that

$$Z = \frac{\bar{X} - \mu_0}{\sqrt{\mu_0(1-\mu_0)/n}} \sim N(0,1).$$

- As a result, we obtain the z-score as follows:

$$z = \frac{p - \mu_0}{\sqrt{\mu_0(1-\mu_0)/n}},$$

where p is the sample proportion (mean).