# Stabbing Faces By a Convex Curve

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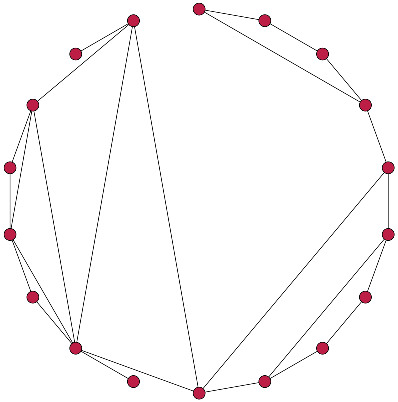
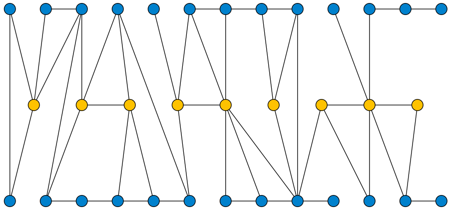
# Motivating question

How does the shape of a point set restrict the graphs with drawings of that shape?

(planar straight line drawings with the points as vertices)



Convex ⇒ outerplanar



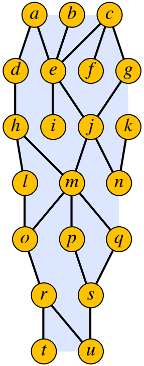
$O(1)$ parallel lines ⇒ $O(1)$ pathwidth

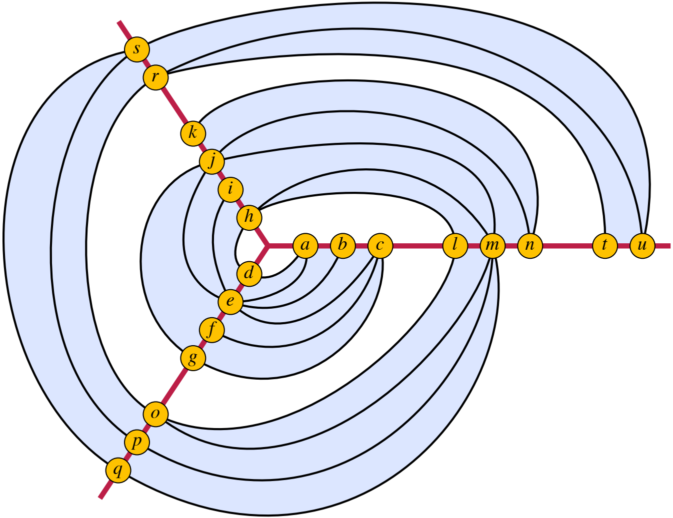
[Dujmović et al. 2008]

If not all planar graphs ⇒ cannot form universal point sets

# Non-parallel lines I

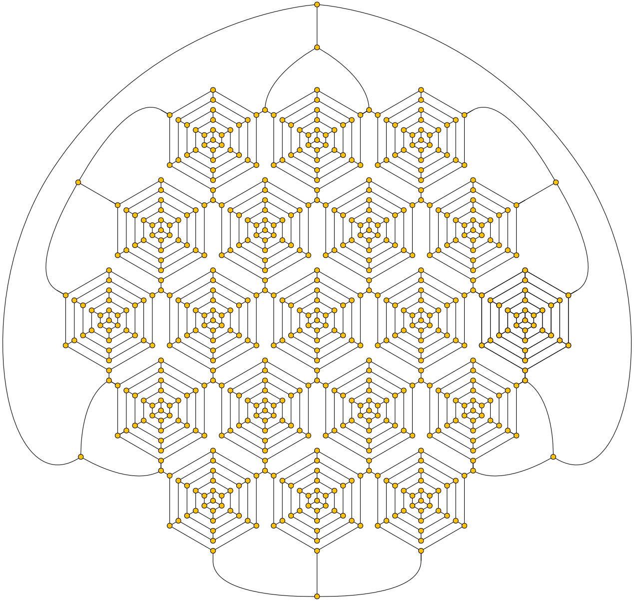
Every leveled planar graph can be drawn on three non-crossing rays or two crossing lines
(And leveled planar = 3-ray drawing, both NP-complete) [Bannister et al. 2019]



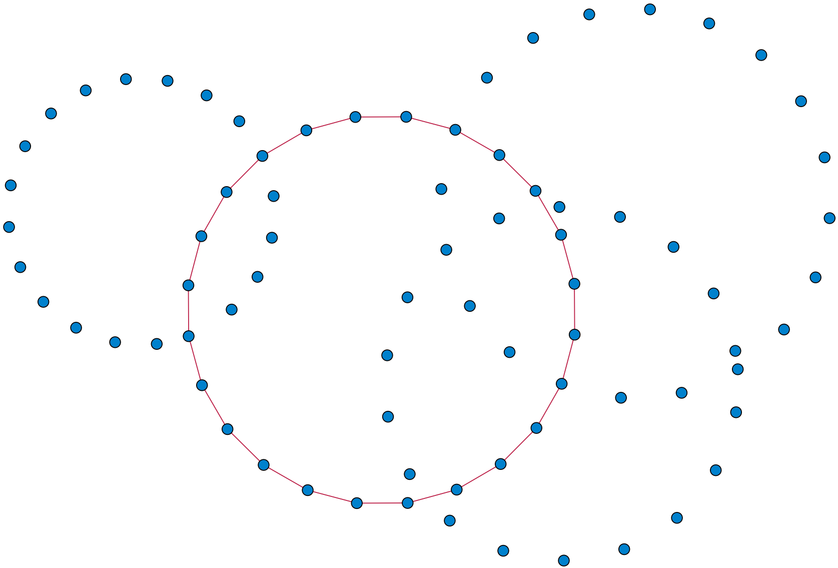



# Non-parallel lines II

Some cubic planar graphs require $\Omega(n^{1/3})$ lines [Eppstein 2021]



# What about: $O(1)$ convex polygons / curves?



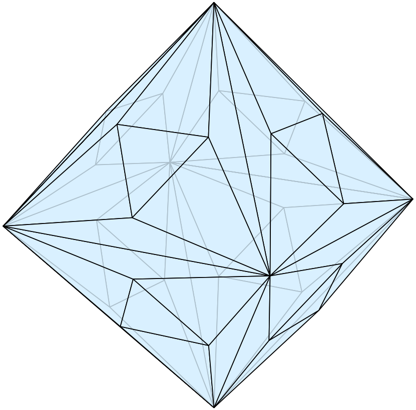
# Hope for proving not all planar graphs possible

Find (somehow) a triangulation $G^?$ with the property that:

no drawing of $G^?$ can have all faces crossed by a single convex curve

Build a family of graphs by recursively expanding faces by copies of $G^?$

Any convex curve could only touch $O(n^{1-\varepsilon})$ faces for some $\varepsilon > 0 \Rightarrow$ covering all vertices would require $\Omega(n^{\varepsilon})$ convex curves

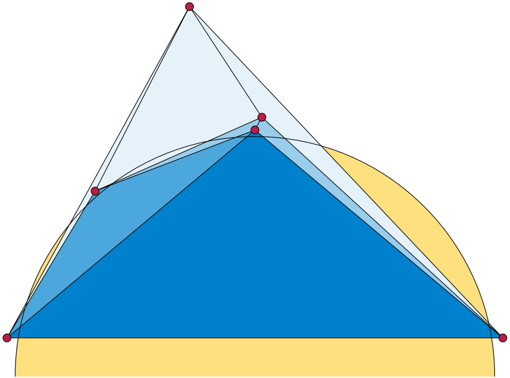


Example: replacement of the faces of an octahedron by copies of the graph of an octahedron

# Lost hope

Our main result: $G^?$ does not exist

For every plane graph $G$ and smooth convex curve $C$ (with $C \not\subset$ line)
$G$ has a straight-line drawing with all faces crossed by $C$

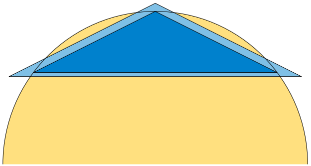


Example: The graph of an octahedron drawn with all faces crossed by a semicircle

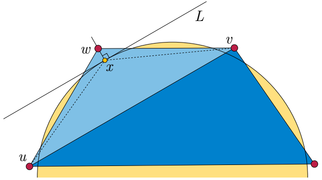
# Proof sketch

Without loss of generality we can assume $G$ is a triangulation

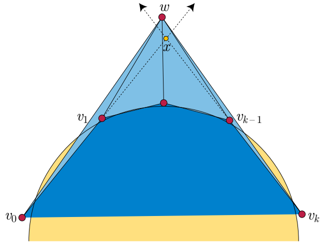
Induction using canonical ordering: add vertices one at a time to a triangulated disk, preserving the property that $C$ bulges through each boundary edge of the disk



Base case:
$G$ is a triangle




Inductive case I:
new vertex is an ear




Inductive case II:
new vertex is not an ear

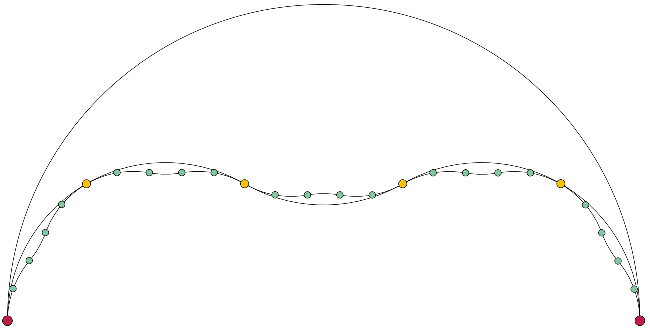
# Conclusions

Every graph can be drawn with all faces crossed by any smooth convex curve

The drawings are not nice (mostly useful as counterexamples)

Still open: Can a universal point set lie on $O(1)$ convex curves?

Likely improvement (not in proceedings): $C^1$-smooth curves might not have any convex arcs (see fractal construction below) but the same proof appears to work for them



# Image credits and references

Three-ray leveled planar drawing from [Bannister et al. 2019]
Graph that cannot be drawn on few lines from [Eppstein 2021]
Point set on $O(1)$ convex curves from [Eppstein 2025]

Michael J. Bannister, William E. Devanny, Vida Dujmović, David Eppstein, and David R. Wood. Track layouts, layered path decompositions, and leveled planarity. *Algorithmica*, 81 (4):1561–1583, 2019. doi: 10.1007/s00453-018-0487-5. Preliminary version in *24th Int. Symp. Graph Drawing*, 2016, LNCS 9801, pp. 499–510.

Vida Dujmović, Michael R. Fellows, Matthew Kitching, Giuseppe Liotta, Catherine McCartin, Naomi Nishimura, Prabhakar Ragde, Frances Rosamond, Sue Whitesides, and David R. Wood. On the parameterized complexity of layered graph drawing. *Algorithmica*, 52(2): 267–292, 2008. doi: 10.1007/s00453-007-9151-1. Preliminary version in *9th European Symp. Algorithms*, 2001, LNCS 2161, pp. 488–499.

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David Eppstein. Decremental greedy polygons and polyhedra without sharp angles. In *Proc. 37th Canadian Conference on Computational Geometry (CCCG 2025)*, pages 85–91. York University, 2025. URL `https://cccg-wads-2025.eecs.yorku.ca/cccg-papers/33.pdf`.