ICS 6N Computational Linear Algebra
The Matrix Equation $Ax = b$

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Matrix equation $Ax = b$

- **Definition**: If A is an $m \times n$ matrix, with columns $a_1, \cdots, a_n$, and if x is in R^n , then the product of A and x, denoted by Ax, is the **linear combination** of the columns of A using the corresponding entries in x as weights; that is

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

- Ax is defined only if the number of columns of A equals the number of entries in x.

Example

Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}, x = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

What is Ax?

Examples

- Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}, x = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

  What is Ax?

- Solution:

$$\begin{aligned} Ax &= \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \\ &= 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{aligned}$$

Matrix equation $Ax = b$

- Consider the following system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

- Write it as a **matrix equation**

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Matrix equation $Ax = b$

- If A is an $m \times n$ matrix, with columns $a_1, \cdots, a_n$, and if b is in R^n , then the matrix equation
$$Ax = b$$
has the same solution set as the vector equation
$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b,$$
which, in turn, has the same solution set as the system of linear equations whose augmented matrix is
$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}$$
- The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

Computing Ax

- Let A be an $m \times n$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- Then

$$Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$$

Dot Product

Let x and y be two vectors in R^n. We define the dot product between two vectors as:

$$x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

Transpose

- Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$
- Define $y^T = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix}$ - turning a column vector into a row vector
- Then

$$x \cdot y = y^T x$$

Computing Ax

$$Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} x \cdot 1^{st} \text{ row} \\ x \cdot 2^{nd} \text{ row} \\ \vdots \\ x \cdot m^{th} \text{ row} \end{bmatrix}$$

Example

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, calculate Ax

- First way (Using definition):

$$Ax = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

- Second way:

$$Ax = \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Identity Matrix

- The matrix with 1s on the diagonal and 0s elsewhere is called an **identity matrix** and is denoted by I.

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

- For example:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $Ix = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x$, for any x.

Properties of the matrix vector product Ax

If A is an $m \times n$ matrix, u and v are vectors in R^n , and c is a scalar, then

- $A(u+v) = Au + Av$
- $A(cu) = c(Au)$

Matrix equation: $Ax = 0$

A system of linear equation is said to be **homogeneous** if it can be written in the form $Ax = 0$, where A is an $m \times n$ matrix and 0 is the zero vector in R^m.

- $x = 0$ is always a solution, called the trivial solution.
- $Ax = 0$ has a nontrivial solution (nonzero vector) if and only if the equation has at least one free variable.

Example

Determine if the following homogeneous system ha a nontrivial solution. Describe the solution set.

$$
\begin{aligned}
3x_1 + 5x_2 - 4x_3 &= 0 \\
-3x_1 - 2x_2 + 4x_3 &= 0 \\
6x_1 + x_2 - 8x_3 &= 0
\end{aligned}
$$

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- Reduce the augmented matrix [A 0] to echelon form

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Solution: $x_1 = \frac{4}{3}x_3, x_2 = 0$ with x_3 free.
- Written down in vector form: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$

Linear Independence

- **Definition**: An indexed set of vectors $\{v_1, v_2, \ldots, v_p\}$ in R^n is called **linearly independent** if

$$x_1 v_1 + x_2 v_2 + \ldots + x_p v_p = 0$$

  has only the trivial solution.
- Otherwise, the set is called **linearly dependent**.

Example

Determine if $\{v_1, v_2\}$ is linearly independent.

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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Properties

Given an indexed set of vectors $\{v_1, v_2, \dots, v_p\}$ in R^n,

- If the set contains the zero vector, then the set is linearly dependent.
- If the set has only one vector, it is linearly independent
- If $p > n$, then the set is linearly dependent.
- If the set with $p \geq 2$ are linearly dependent, then at least one of the vectors is a linear combination of the others.
  - Suppose x_j is not zero, then $v_j = -\frac{x_1}{x_j} v_1 - \frac{x_2}{x_j} v_2 - \dots - \frac{x_p}{x_j} v_p$
  - $\{v_1, v_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.

Summary statements

Let $\{v_1, v_2, \cdots, v_p\}$ be a set of vectors in R^n, and $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$, the following statements are equivalent:

a) The set is linearly dependent.

b) $Ax = 0$ has nontrivial solutions.

c) A has at least one free variable.

d) The number of pivots in A is less than p.

e) rank(A) $< p$. Define **rank(A) = number of pivots in A**.

Summary statements

Let $\{v_1, v_2, \cdots, v_n\}$ be a set of vectors in R^n, and $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, the following statements are equivalent:

a) The set is linearly independent.

b) $Ax = 0$ has only trivial solutions.

c) A has no free variables.

d) rank(A) $= n$. Such a matrix is called **non-singular**.

e) $Ax = b$ has exactly one solution.