ICS 6D **Due: Wednesday, April 13, 2016, 3:00PM**

Homework 2

Instructor: Sandy Irani Sections 8.1-8.5

1. Give the first six terms of the following sequences. You can assume that the sequences start with an index of 1.

   (a) The n^{th} term is $\lceil \sqrt{n} \rceil$.

   (b) The n^{th} term is 7.

   (c) The n^{th} term is the largest k such that $k! \leq n$.

   (d) The first term is 1 and the second term is 2. The rest of the terms are the product of the two preceding terms.

   (e) $g_1 = 1$ and $g_2 = 5$. $g_n = 2 \cdot g_{n-1} + g_{n-2}$.

   (f) A geometric sequence in which the first value is 5 and the common ratio is $1/2$.

   (g) An arithmetic sequence in which the first value is 4 and the common difference is $-1/2$.

2. Suppose someone takes out a home improvement loan for \$30,000. The annual interest on the loan is 6% and is compounded monthly. The monthly payment is \$600. Let a_n denote the amount owed at the end of the nth month. The payments start in the first month and are due the last day of every month. Give a recurrence relation for a_n. Don't forget the base case.

3. Express the following sums using summation notation:

   (a) $(-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5$

   (b) $2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$

   (c) $0^3 + 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \cdots + (15)^3$.

   (d) The sum of the squares of the even integers between 1 and 99.

4. Indicte whether the following equality is true and justify your answer:

$$\sum_{j=0}^{100} j^3 = \sum_{j=1}^{100} j^3$$

5. For each of the following expressions, write down an equivalent expression where the last term in the sum is outside the summation.

   (a)

$$\sum_{j=-2}^{18} 2^j$$

   (b)

$$\sum_{k=0}^{n} (k^2 - 4k + 1)$$

(c)

$$\sum_{k=0}^{m+2} (k^2 - 4k + 1)$$

6. Evaluate the following summations:

   (a) $\sum_{j=-2}^{2} j^3$

   (b) $\sum_{k=0}^{3} 2^k$

   (c) $\sum_{k=0}^{3} (2k + 1)$

   (d) $\sum_{k=0}^{3} 2k + 1$ (Note that your answer should be different than the previous answer).

7. Give an equivalent recerrence relation to the one given below but substituting $k + 1$ for n. Make sure and determine the correct lower bound for k. Your recurrence relation should express g_{k+1} as a function of earlier terms in the sequence.

$$g_n = 3g_{n-1} - 2g_{n-2} \qquad \text{for } n \geq 2.$$

8. Define the statement $P(n)$ to be: "4 evenly divides $3^{2n} - 1$".

   (a) Verify that $P(3)$ is true. (You can use a calculator to compute the exponent). In order to show formally that 4 evenly divides a number, you need to show that the number is equal to $4m$ for some integer m. For example, 4 evenly divides 32 because $32 = 4 \cdot 8$.

   (b) Express $P(k)$.

   (c) Express $P(k + 1)$.

   (d) In an inductive proof of the fact that for every $n \geq 1$, 4 evenly divides $3^{2n} - 1$, what must be proven in the base case?

   (e) In an inductive proof of the fact that for every $n \geq 1$, 4 evenly divides $3^{2n} - 1$, what must be proven in the inductive step?

   (f) What would the inductive hypothesis be in the inductive step from your previous answer.

In each of the inductive proofs below, you must first state that you are proving the theorem by induction. The base case and inductive step must be clearly labeled. At the beginning of the inductive step, you need to state clearly what you are assuming and what you are proving. You must also clearly indicate where you are using the inductive hypothesis.

9. Prove that for every positive integer n, $\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$.

10. Prove that for every positive integer n, $\sum_{j=1}^{n} j \cdot 2^j = (n - 1) \cdot 2^{n+1} + 2$.

11. Prove that for $n \geq 2$, $n^3 \geq 2n + 4$.

12. Prove that for $n \geq 1$, $3^n \geq 2^n + n^2$.

13. Prove that for $n \geq 4$, $n! \geq 2^n$.

14. Prove that for any $n \geq 1$, 4 evenly divides $3^{2n} - 1$.

15. Prove that for any $n \geq 1$, 6 evenly divides $7^n - 1$.

16. Define the sequence $\{c_n\}$ as follows:

- $c_0 = 5$
- $c_n = (c_{n-1})^2$, for $n \geq 1$.

Consider an inductive proof of the following theorem:

Theorem 1. *For $n \geq 0$, $c_n = 5^{2^n}$.*

(In the above expression, 2^n is the exponent of 5).

(a) What do you need to show in the base case of an inductive proof of the theorem?

(b) What do you need to show in the inductive step of the theorem?

(c) Now give a proof by induction of the theorem.

17. Define the sequence $\{b_n\}$ as follows:

- $b_0 = 1$
- $b_n = 2 \cdot b_{n-1} + 1$, for $n \geq 1$.

Prove that for $n \geq 0$, $b_n = 2^{n+1} - 1$.