CS 261: Data Structures

Week 6–7: Binary search

Lecture 7b: Multi-level structures and fractional cascading

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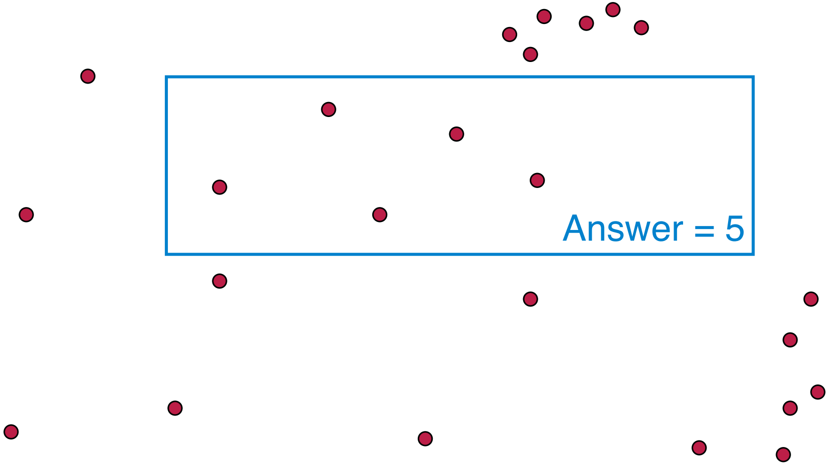
Multi-level range search

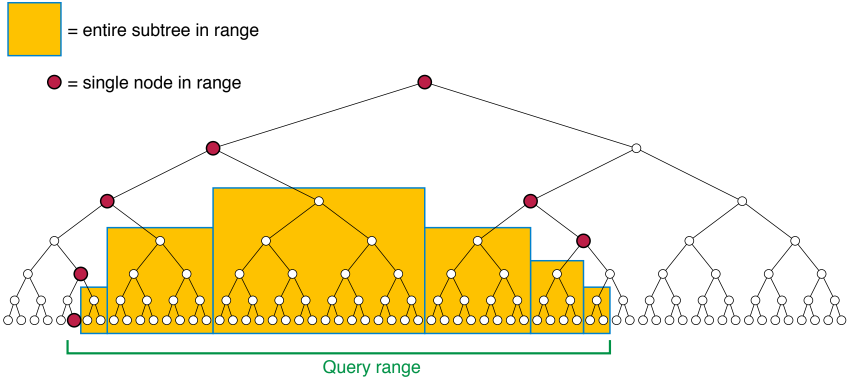
Example: Rectangular range counting

Data: 2d points represented as (x, y) coordinate pairs

Query: How many points are inside a given rectangle?



Binary search tree on x-coordinates



Query range: left and right x-coordinates of rectangle

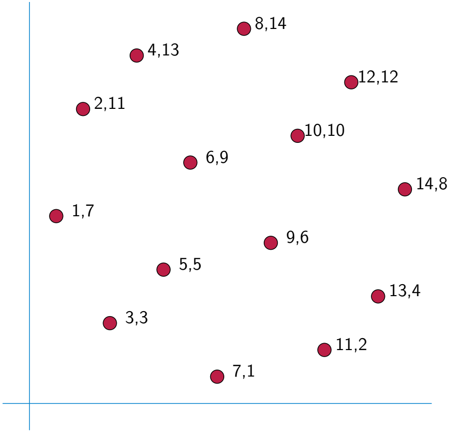
Decomposes the points whose x-coordinate is in range into

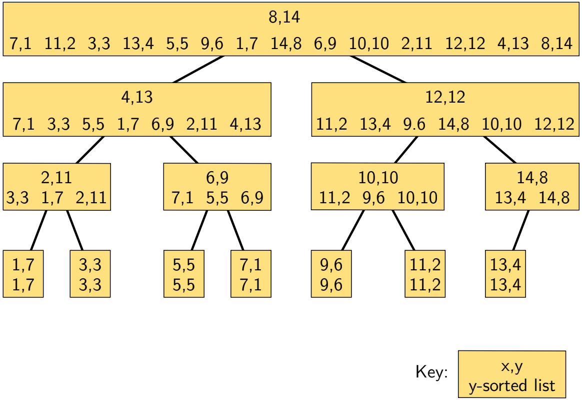
- $O(\log n)$ individual points
- $O(\log n)$ larger sets of points

Multi-level structure

Binary search tree of points sorted by x-coordinates

Each node stores a 1D range search structure for intervals of y-coordinates, for points in its subtree (e.g. a sorted array)





Using a multi-level structure

To count points in a query rectangle:

- Perform query on x-range of rectangle
- For each individual point (x, y) found by query:

  Test whether y is in range
- For each subtree identified by query:

  Use 1d structure at subtree root to count descendants whose y coordinate is in range
- Add the results and return the total

Multi-level analysis

If x-tree is balanced $\Rightarrow$ each point contributes to y-structures in $O(\log n)$ ancestors $\Rightarrow$ total space is $O(n \log n)$

Each rectangle query makes $O(\log n)$ calls to one-dimensional y-structures $\Rightarrow$ query time is $O(\log^2 n)$

Making it dynamic

Suppose we want to insert or delete points?

- Use a dynamic binary search tree on x-coordinates
- Replace 1D sorted arrays by dynamic binary search trees on y-coordinates

We cannot rotate quickly because that would cause big changes to the 1D structures

Instead, use weight-balanced binary search tree on x-coordinates, and when we rebuild a subtree we also rebuild the recursive structures stored in its nodes

Weight-balanced trees

Also called $BB[\alpha]$-trees

Jörg Nievergelt and Ed Reingold, 1973

Each node stores a number, the size of its subtree

Constraint: left and right subtrees at each node have sizes within a factor of α of each other $\Rightarrow$ height $\leq \log_{1/(1-\alpha)} n = O(\log n)$

Original update scheme: rotations, works only for small α

Simpler: rebuild unbalanced subtrees, amortized $O(\log n)$/update
(potential function: sum of unbalance amounts at each node)

Fractional cascading

Related binary searches

In the multi-level structure for rectangular range counting, each query does $O(\log n)$ binary searches:

- In one-dimensional structures stored at certain tree nodes
- All searching for the same y-coordinates
  (top and bottom coordinates of query rectangle)
- In a related sequence of nodes
  (children of the nodes on a tree path)

Goal of fractional cascading: Speed up multiple related binary searches without paying too big a penalty in space

A simpler multi-binary-search problem

Data: k sorted lists of numbers $S_0, S_1, \ldots S_{k-1}$

Total length: $n = |S_0| + |S_1| + \cdots + |S_{k-1}|$

No repeated values, even in different lists

Query: find the successors of a given number q in each list

(s_i = successor of q in list S_i)

Example

Data:

- $S_0 = [0, 10, 20, 30, 40, 50, 60, 70]$
- $S_1 = [1, 2, 13, 25, 27, 51, 57]$
- $S_2 = [21, 22, 31, 32, 33, 41, 99]$
- $S_3 = [67, 68, 69]$

Total length $n = 8 + 7 + 7 + 3 = 25$

Query for $q = 24$ would find
$s_0 = 30 \quad s_1 = 25 \quad s_2 = 31 \quad s_3 = 67$

Naïve solutions

Do the binary searches separately

Space $= O(n)$ for storing each S_i as a sorted list

Query time $= O(k \log n)$ for k binary searches

Merge into one list

For each value x, store k-tuple of successors
for queries that return x as their smallest value

0:(0,1,21,67), 1:(10,1,21,67), 2:(10,2,21,67), 10:(10,13,21,67), 13:(20,13,21,67),
20:(20,25,21,67), 21:(30,25,21,67), ...

Binary search in merged sorted array + look up k-tuple

Space $O(kn)$, query time $O(k + \log n)$

Fractional cascading

Working backwards through the sequence of lists S_i,
construct T_i: merged structure for (S_i + half the elements of T_{i+1})

Choosing the half of the elements that are in odd-numbered positions e.g. if $T = 1, 2, 3, 5, 7, 11, 20$ then $\frac{1}{2}T = 2, 5, 11$

So T_i consists of:

- A sorted array of the merged items from $S_i + \frac{1}{2}T_{i+1}$
- A dictionary mapping each merged item x to a pair (a, b) where one of a or b is x, and the other one is the successor of x in the other merged list
- When there is no successor in the other list, use $+\infty$

Example

- $S_3 = 67, 68, 69$ $T_3 = S_3$ (nothing to merge)   Half elements: 68
- $S_2 = 21, 22, 31, 32, 33, 41, 99$
- $T_2 =$ 21:(21,68), 22:(22,68), 31:(31,68), 32:(32,68), 33:(33,68), 41:(41,68), 68:(99,68), 99:(99,$+\infty$)
- Half the elements of T_2: 22, 32, 41, 99
- $S_1 = 1, 2, 13, 25, 27, 51, 57$
- $T_1 =$ 1:(1,22), 2:(2,22), 13:(13,22), 22:(25,22), 25:(25,32), 27:(27,32), 32:(51,32), 41:(51,41), 51:(51,99), 57:(57,99), 99:($+\infty$,99)
- Half the elements of T_1: 2, 22, 27, 41, 57
- $S_0 = 0, 10, 20, 30, 40, 50, 60, 70$
- $T_0 =$ 0:(0,2), 2:(10,2), 10:(10,22), 20:(20,22), 22:(30,22), 27:(30,27), 30:(30,41), 40:(40,41), 41:(50,41), 50:(50,57), 57:(60,57), 60:(60,$+\infty$), 70:(70,$+\infty$)

Searching fractionally cascaded lists

To find the successors of q:

- Binary search for successor t_0 in merged list T_0
- Set $i = 0$
- Then, repeat:
  - Use dictionary for T_i to find the pair (a, b) where $a = s_i =$ successor in S_i and b is successor in $\frac{1}{2}T_{i+1}$
  - Output s_i
  - Let c be the (skipped) element of T_{i+1} just before b
  - If $q < c$ then $t_{i+1} = c$ else $t_{i+1} = b$
  - Set $i = i + 1$

Example (continued)

To search for the successor of $q = 24$:

- Binary search in T_0 finds successor t_0: 27:(30,27)
- Output $s_0 = 30$, successor in S_0
- Successor in T_1 might be either 27 or previous item, 25
- Because $q < 25$, successor in T_1 is 25:(25,32)
- Output $s_1 = 25$, successor in S_1
- Successor in T_2 might be either 32 or previous item, 31
- Because $q < 31$, successor in T_2 is 31:(31,68)
- Output $s_2 = 31$, successor in S_2
- Successor in T_3 might be either 68 or previous item, 67
- Because $q < 67$, successor in T_3 is 67
- Output $s_3 = 67$, successor in S_3

Fractional cascading analysis

Query time

One binary search + $O(1)$ for each list after the first

Total $O(k + \log n)$

Space and set-up time

Each element of S_i contributes 1 to the length of T_i, $\frac{1}{2}$ to the length of T_{i-1}, $\frac{1}{4}$ to the length of T_{i-2}, ...

So the total space and total set-up time is $O(n)$

Best combination of time and space from naïve solutions

Also works for multi-level search trees, for example rectangular range counting with $O(n \log n)$ space and $O(\log n)$ query time

Summary

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- Ranking and unranking operations; efficient dynamic implementation by augmenting search tree with relative ranks
- Types of range searching problems including range counting, range reporting, range minimum, and range sum; decomposable problems using associative binary operation
- Dynamic range searching by augmenting search tree with value of its subtree and decomposing range into a logarithmic number of subtrees and individual nodes
- Cell probe model of computing and lower bound on dynamic prefix sums
- Multi-level range search and multi-level augmented binary search trees
- Fractional cascading